Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 3 : Lecture 15 : Lyapunov stability Theorems- Part 3

Positive definiteness done, radial unboundedness done, easier test done. Now we go to decrescents, this is the third property ok. A function is said to be decrescent again you see this local domain and all that mapping to real numbers. This cannot change because the same function is what is being used for analysis of this stability analysis of the system. So the domain range and image sets cannot change ok. So for decrescents you require that the function is 0 at 0 again cannot change because no Lyapunov function is allowed to be non-zero at the equilibrium and we will assume 0 to be the equilibrium ok for all time.

And there exists a class K function such that in absolute value this function V is dominated by the class K function ok. Just digest it that you know we have it is in a mention in a slightly different way you have put in the absolute value here yeah we are not which means we are not restricting somehow the V from being negative ok. Earlier definitions like class K and so on more or less guarantee V is non-negative because it is 0 at 0 and then it is dominating a class K function which is strictly increasing it means V can never be negative. Here somehow that is being allowed yeah so again of course it is for all time in R plus and X in V are these things are standard.

So the picture looks something like this when you draw the phi norm of X yeah you also draw the negative of that and your V has to lie between that. This is what the picture looks like because I mean I am just translating this into this image right here ok alright. Now if you go back if we go back and look at which we did not use at class L function ok. At a class L function and do you think we can actually instead of using a class K function there ok that is phi is class K do you think we can characterize this decrescence in terms of class L functions? Yeah this is a question I want you to think about so we will put it as an exercise. Can you characterize decrescence using class L function? Because there it is like because this is also somehow doing a upper bounding right instead of a lower bounding seems like it is doing a upper bounding instead of a lower bounding right.

So the question is can you use class L functions in some sense ok. Can you use class L functions to characterize decrescence functions ok. I want you to think about it yeah alright ok. Let us look at some of the functions we have already considered yeah. This was one of the functions we considered yeah.

This is a function which is positive definite right because it dominates this which is a class K function right. So but does this is this a decrescent function? Is this a decrescent function? Can this be dominated by a class K function? So the flip question right. Until now for positive definiteness you wanted this to be greater than some class K function which it is because if T is non-negative then if T is non-negative then this is greater than equal to norm X square divided by 1 plus norm X square right and that is a class K function. So great it is positive definite right or you can even verify positive definiteness via the easier test the easy test right same deal no difference ok. Now for decrescence what do we need? We need it to have some upper bound with some class K function ok.

So usually it is not very easy to claim something like this ok. In the sense you would be very hard pressed to prove that there does not exist or there exists a class K function ok which dominates this function not easy because you have to test every possible class K function right. But look this is only one possible class K function with which we use we prove positive definiteness ok and so I am asking a simple question does there exist some R such that this happens ok. It is rather obvious that this does not happen for any R right because if you choose give me any value of R I can push up time so that you know this condition is violated yeah because this these conditions have to hold for all time right you can see that this condition has to hold for all time right. So, I can always push up time such that for this particular choice of class K function things go wrong ok.

But now how can you guarantee that for every possible choice of class K function things will go wrong. I have just shown that for one choice of class K function things do not work out, but can I just categorize this as being not decrescent or how would you claim that this is not decrescent how would you claim that this is not a decrescent function. I have shown that for one choice of phi x this kind of upper bounding will not work. Can you say the same for every possible choice of phi x? Yes, no think about it. So, will this fail? Will this fail for this example for arbitrary phi x now not this specific choice because I cannot say that it did not work for one choice I made so it is not decrescent no that is not true there cannot if the negative of this is that there does not exist any phi x ok does not exist any phi x can you claim that this is indeed the case.

I am not giving you any structure of phi x or anything like that. Yes, absolutely it is pretty straight forward right I mean think about it you give me any phi x yeah it does not matter what you give me you know give me this phi x or this phi x or this phi x or whatever I mean some kind of increasing function yeah you give me any phi x or phi norm of x yeah once you your problems begin and I am talking about now v being below this what do I want? v has to be below this guy right but your problems begin once you freeze the x yeah once you freeze the x you are frozen here so for this guy you are frozen here, here, here wherever I don't care where you are frozen but my v this guy is not just a function of states in this case there is another x I mean another element that I can play with its time right so this picture is not very representative whatever value of this guys you give me your k class k function is frozen that's it it's a constant as good as a constant you gave me a constant it doesn't matter how big a constant you gave me I will choose my t to be really large because this has to hold for all t the left hand side has t right hand side has no t so I will bump up my t as much as I want because I can go all the way to infinity right and then this will fail my v will actually you know if this was for t equal to t1 I will make for this guy for v t2 same x or v t10 same x right I just bump up t and I will dominate there is no way any class k function will stand right why because the class k function by the nature of the definition is only a function of x once x is frozen the class k function evaluates to a constant however large this constant I have time still to play with on the left hand side and in this case it is an increasing function of time so this is where your class l should somehow think about you should think about whenever I have some increasing function of time here some strictly increasing function of time which is in fact going to infinity I have a problem you cannot claim that it is class k sorry class sorry it is a you cannot claim decrescence in this case ok because this will always dominate any class k function no problem ok so absolutely correct I am going to delete this sorry right ok is that clear ok this is not decrescent ok that is what I have said and this is fine I mean does not matter I mean what I have said here the basic point is right hand side only function of x left hand side increasing function of time even if x is constant I am done cannot this domination is not possible alright ok again decrescence connect to uniform stability alright suppose I flip this thing suppose I flip the example that is now it is not an increasing function of time but a decreasing function of time life is good why now it is definitely greater than equal to norm x square by 2 for all t because as t has to be greater than equal to 0 negative t's are not allowed so I cannot make this really large then all that ok so for non-negative t this is of course dominating norm x square by 2 sorry this is of course being dominated by norm x square by 2 right I am done ok decrescent now obviously it is not positive definite I hope you understand by the same argument if you give me any class k function to compare with I note that this is a decreasing function of time I will keep bumping up time and I will go below your class k function ok so this can never dominate any class k function therefore it is not positive definite ok so somehow one might get an impression that both seem mutually exclusive right if you have a positive definite function you did not get decrescence if you get got decrescence you did not get positive definite net ok seems like there is some mutual exclusivity here which would but then that would mean that if you get stability you cannot get uniformity if you got uniformity you cannot get stability that seems a bit ridiculous right so are there functions which are both positive definite and decrescent because that is what we need for uniform stability right we said that decrescence is connected to uniformity positive definiteness connected to stability so unless I have both positive definiteness and decrescence I cannot claim uniform stability so what are we saying there is no uniformity stable system in the universe so can you tell me if there is a function which is both positive definite and decrescent how do you think we can get a function which is both positive definite and decrescent I gave you an example of positive definite I gave you an example of decrescent now I want both time bounded function yeah little bit more I will say I will say it is to be bounded on both sides ok it is a function that is bounded the time is corresponding to time bounded on both sides which is why I had introduced this example earlier 1 plus sine square t by 2 x square norm x square so what do I know it is greater than equal to norm x square by 2 I already proved it for the purpose of positive definiteness right because sine square is lower bounded by 1 but then this is also upper bounded by 2 right so there is both a lower bound greater than 0 lower bound by the way that is important you can't have a equal to 0 lower bound or less than 0 lower bound has to be a greater than 0 lower bound so the lower bound was 1 upper bound is 2 so the lower bound is this guy upper bound is this guy yeah both sides satisfied ok not so complicated actually yeah until you have the example seems complicated but not so complicated right it is both decrescent and positive definite has both the properties unfortunately we don't have any short hand for decrescence we actually write it just like greater than 0 less than 0 unfortunately we don't have any short hand here you have to actually write it with decrescent ok. So now this is a good function to have this lets you sort of evaluate both properties yeah ok if there was no such function very funny ok obvious just like we said that uniformity for time invariant systems is free just like that if your v is purely a function of states then again uniformity is sort of free yeah there is nothing to no decrescence to evaluate it is decrescent and it is positive definite because there is no time argument in this itself ok so decrescence and positive definiteness anyway sorry decrescence is free yeah alright. Finally we have the property of semi-definiteness which is a very nice and weak property yeah again similar arguments 0 at 0 scalar value continuous and it just has to be greater than 0 as a greater than equal to 0 as a function ok for all time for all x non-negative that's it so all the examples we considered they were all semi-definite at least yeah T x1 square plus x2 square obviously this is the in fact positive definite also x1 square by 2 x1 square by 4 semi-definite ok x1 plus x2 whole square semi-definite because as functions yeah so this is also gives you a big distinction between semi-definiteness and definiteness ok semi-definiteness is just a property of that function it is like how you when you are plotting these functions how you look at it it is above 0 below 0 that's it ok as a function it is positive or negative it is just so a non-negative valued function which is 0 at 0 is semi-definite ok but when you talk about positive definiteness and negative definiteness there is certain definiteness alright you are you are basically looking at rather special properties ok these are not just basic properties yeah again why one might ask why we don't like these semi-definite functions ok look at look at this again think about inverse of V all our analysis when we look at proofs and so on and so forth they rely on V inverse ok now if I talk about V inverse of 0 ok in a positive definite function when I say V inverse of 0 what comes to your mind 0 states but if I look at a semi-definite function V inverse of 0 is what straight line all possible infinitely many states again this guy again the y axis right all possible infinitely many states come to mind ridiculous right I mean the question is what is it that even I mean you can't even like talk about an equilibrium in these scenarios right how do you even talk about equilibrium if you the V inverse 0 is not a single point then it's the V is irrelevant or useless for as far as Lyapunov theorems are concerned not to say they are useless in every context they are actually also methods which can let you conclude stability using just semi-definiteness type properties but not of V of V dot ok so those are basically what are these Lassalle invariance and Barbell art, Slema and things like that those are methods that let you talk about global or asymptotic stability when V dot is only semi-definite ok not V, V being semi-definite still tough again still not impossible you can do some analysis ok alright any questions? why are you looking at the V inverse? so we have not gone to the proof of the Lyapunov theorem we are actually going to state that now but when you look at the proof all the proof for the Lyapunov theorem are based on V inverse taking the inverse of the V and looking at what you get which is why I made this picture yeah where I made this picture and said that you know if you take V inverse of this set you get exactly this set so this bounded set gives a bounded inverse but here this bounded set doesn't give a bounded inverse so that kind of problem ok they rely the proof rely on inverse of these functions ok. So, what is the setup for the Lyapunov theorems? Non-linear system, function of time and state time from t0 to infinity usually t0 has to be greater than 0 greater than equal to 0 and states in a ball of radius r mapping to Rn is this function f with some initial condition without loss of generality we assume that 0 is an isolated equilibrium ok f is assumed to be locally Lipschitz continuous this is for existence of unique solutions we already spoke about this in the first class itself and finally we define the notion of V derivatives ok or directional derivatives nothing complicated what you think of as V dot what comes naturally to you as the derivative of V is actually the directional derivative why we use this different notation and different sort of method of talking about it is because when you write a V for example I wrote some function V like this x1 square plus x2 square it is such a common function alright I can use this to analyze hundreds of dynamical systems ok hundreds of dynamical systems could possibly be analyzed by the same V ok.

So, the V itself has no connection to any dynamical system V is just a function of some states not a function of nothing to do with any dynamical system but we want to study how V evolves along the solutions of a dynamical system ok. So, we take its derivatives along the trajectories of a particular system and when we identify this V with trajectories of a particular system what you compute as V dot is just partial of V with respect to time plus partial of V with respect to states times f t x is the L f V is the V derivative ok it is called a L E derivative ok it is the directional derivative of V ok. So, this is the notation but as far as you are concerned you are just computing V dot ok alright great. If there is a nice C1 function V mapping time and states in a domain to real numbers such that for some are positive such that it is positive definite this much assumption the highlighted assumption if you have this highlighted assumption to be satisfied then this function V is already called a candidate Lyapunov function ok this is the terminology it is a candidate Lyapunov function if it is a C1 function of time and states and positive definite then it is a candidate Lyapunov function then in the sense of Lyapunov if V dot is negative semi definite origin is stable if V dot is negative semi definite and V is decrescent then origin is uniformly stable ok. So, these are the first two Lyapunov theorem very simple statements I have written them in a very simple way if you go to the text book of course there is a little bit more buff in the you know statement itself because it writes a lot of things and so on writes the system and so on and so forth but the basic statement is pretty simple you take the V evaluate the V dot of course you verify that V is positive definite otherwise it is not even a candidate Lyapunov function right.

So, that is clearly specified here ok if it is not this then it is not a candidate Lyapunov function you cannot use it for the Lyapunov theorem ok you can use it in other places we will talk about those examples later on but for candidate for Lyapunov stability theorems no this is essential ok and then we say that system is stable if the derivative of V the way we defined it is negative semi definite only semi definiteness required and further if V was also decrescent then you have uniform stability yeah you can already see the simplicity of this result once you have a V evaluating stability is just super easy of course it is not remember nothing is easy yeah having such a V is also it is something that you try to find yeah just because you took a random candidate Lyapunov function V and V dot turned out to be not negative semi definite does not mean your system is not stable it just means that you did not find a good enough V ok. So, this is only sufficiency condition as you can see not necessary condition yeah there is an existence necessary condition but that will not help you find the Lyapunov function ok. So, Lyapunov functions are constructed primarily by experience previous literature energy of the system gives you some motivation but it is still a hunt ok. So, this is one of the complaints most people have about nonlinear system but this is why I love nonlinear system because it is not just you know for everything I take X transpose P X and it works no it does not work you have to put some effort into constructing this V and more often than not it captures some fundamental property of the system ok just like energy of the system captures the fundamental property of the system ok. So, if you have a Lagrange system then energy remains constant.

So, you know that V dot is if you take energy as your V then V dot is exactly 0 therefore, it is a stable system Lagrange system is a stable system ok. So, this kind of a simple confusion can be offered alright. Thank you.