Nonlinear Control Design

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Week 3 : Lecture 14 : Lyapunov stability Theorems- Part 2

So, you have these definitions of positive definiteness and I want to connect it to this matrix yeah I want to sort of claim in some sense that the matrix function X transpose AX yeah or basically if I just add a t plus one does not change much. This is also positive definite function in the sense of you know how we talk about positive definiteness ok. So, I want both definitions to align. So, I want to claim that X transpose AX given that A is positive definite is a positive definite function ok. How do I do that? It is pretty straight forward. I can decompose A in this form because A is a symmetric matrix because I cannot talk about positive definiteness.

So, A is symmetric therefore, it is you can decompose it in this form this M transpose lambda M where lambda contains a diagonal matrix of eigenvalues and M is just the eigenvectors alright. So, I can do this sort of characterization. So, I can do this sort of derivation here right and then if I write my if I write X transpose AX in this form right basically I am just in a sense doing a similarity transformation if you think in terms of coordinates yeah. So, if I write my X transpose AX like this and I choose and so I can write this is MX transpose lambda MX and I choose Y as MX this is a new state right this is the similarity transformation right.

So, Y is constructed out of the eigenvectors right alright. Then this quadratic form can be written as this Y transpose lambda Y all of this is very simple for symmetric matrices right real eigenvalues diagonalizable naturally and all that. If it is not then you have more problem, but we are dealing with this nice symmetric matrices right. So, this is essentially sum of lambda i y i square and all the lambda i's are obviously strictly positive and so this has to be greater than equal to some lambda min norm y square right because I take this the smallest lambda i it is a finite number of lambda i. So, I can take the min.

So, I can just write this as lambda min sum of y i square. So, lambda min sum of y i square is just lambda min times norm of y square right and so basically what I have is a class k function right it is lambda min times class k function of norm y right because the norm itself is a class k function right. It is like saying x square is a class k function right similarly norm of y square is a class k function. In fact, it is a class k R function ok it is a class k R function not just a class k function ok. So, therefore, this is and this is just a scaling right it is a positive scaling it does not affect the class k nature of the function right.

So, this is just a positive scaling multiplied by phi norm y square. In fact, I can take this with the norm y square and just say that it is yeah it is a class k function. So, what have we shown? We have just shown that x transpose Ax is dominates a class k function. In fact, it dominates a class k R function. Therefore, x transpose Ax itself is a positive definite function ok.

That is all we need we need it has to be 0 at 0 and all that ok it is 0 at 0 right. x transpose Ax is obviously 0 at 0 right and it has to be continuous which is also obviously the case x transpose Ax is continuous in x no problem smooth in x in fact yeah. So, this is a x transpose Ax has been proven to be a class sorry a positive definite function ok and that is what we needed we wanted to reconcile yeah. So, even if I take my Vtx with a time argument here multiplied by x transpose Ax in this form no problem it is still greater than equal to x transpose Ax for all non-negative t. So, therefore, this is also a positive definite function ok.

Does that make sense? Alright? Ok great alright. So, now that we have this sort of characterization for class k function I will again say that it is not easy to verify this in general yeah because I do it all this I sort of cheated right because I first constructed a phi and then constructed a V. In reality for most systems you have to construct a V first and you do not think about constructing a class k function to dominate and all that alright. But once you have a V trying to find a class k function that it dominates is not very easy alright. And so, here we have this nice and easier conditions which obviously have been sort of collected here from Vidya Sagar's book ok.

And the characterization is very straight forward. If you have a function of only the state ok therefore, I use a different notation it is still the same V or whatever you want to call it. I use Wx because there is no time argument. I am using just a different notation to distinguish these cases. So, if you want to discuss positive definiteness of Wx which is a function of the state only then you require to check only two things.

One that it is 0 at 0 this was anyway one of the conditions same condition does not change anything fine. And the second one is this guy. Yeah, the Wx has to be strictly positive for all x which is not 0 in this domain yeah. So, this domain that is ball of radius r is fixed we are somehow assuming that r states evolve in this ball ok. So, we have to only verify two things one that it is 0 at 0 value of the state and two that it is strictly positive whenever the states are non-zero ok.

Again something that should remind you of the norm right. Norm also has such a property right. It is 0 when the argument is 0 and it is strictly positive when argument is non-zero ok. You can see that there are these you know similarities between the two ok great. Now the easy check does not remain so easy when you have a function of both time and state.

In this case the only way to check is obviously you want the first condition which you cannot do away with in any case alright. The only way to verify this is that you have to find a positive definite W to dominate which is only a function of the state norm ok. So, instead of hunting for a class K function you are hunting for a positive definite Wx ok. So, I would not say this is significantly easier or anything like that but still it is another characterization ok. So, what this first characterization more often than not we are dealing with time invariant or autonomous systems.

We hardly talk about or we hardly see a lot of real examples where there is time varying quantities in the system. Usually we do not yeah and even if we do more often than not the Lyapunov or the Lyapunov function means that we look at do not contain a time invariant ok. We use a time invariant in V to analyze even systems that are time varying at times ok. So, this second one being not so useful does not impact us in a lot of scenarios but it can also yeah. So, these are the easy characterization you can see all you have to check is that it is 0 for especially for this case where there is only the state.

You just have to check that it is 0 at 0 and then it is positive for all non-zero values of the state very easy yeah and the proof of this is an obviously in Vidya Sagar's book yeah please take a look at it it is very interesting. What it basically says is that if you have this kind of a condition then you can always find a class K function to dominate ok. If this condition is satisfied you can always find a class K function to dominate ok. So, this is rather nice right rather powerful. It is not a it is not exactly constructive in the sense that the book is not actually showing you a construction of the class K function but it just shows that there exists such a class K function ok.

So, this is pretty cool ok alright. The only thing is for V to be negative definite minus V needs to be positive definite yeah this guy alright and the notation we use is just a flipped over here alright ok great. I do not know why I have repeated this because it is we have already looked at this idea ok. We have already looked at this example yeah. If you take V t x as t plus 1 x transpose A x ok then it is then in this case I know that it is greater than equal to W x defined in this form for all non-negative t right and once I have W x equal to this I just need to verify this positive definiteness of W x right.

It is not difficult at all it is 0 at 0 no problem yeah and since in fact I do not have to even look at a lot of arguments because it is a positive definite matrix in between the quadratic form is always positive right by virtue of it being a positive definite matrix x transpose A x is always positive for non-zero x yeah that is what it means for matrix to be positive definite. So, even without looking at this eigenvalue decomposition I can directly say this right because A is positive definite for all non-zero values of x, x transpose A x has to be strictly positive and that is all we need here strictly positive for all non-zero states ok which exactly is satisfied in this case ok. So, pretty straight forward. So, the idea is positive definite matrices lead to positive definite functions ok. So, and please do not think of this as a trivial result the point is we use in a lot of scenarios we do use quadratic eigen functions ok especially when you have systems which are to a large extent feedback linearizable ok.

We have not talked about this obviously we will come to this in the second half of the course, but lot of systems can be linearized via feedback yeah lot of aero mechanical system ok can be linearized via feedback ok. And for most of those systems we can use quadratic Lyapunov function yeah because once you have some kind something linear appearing in your dynamics then you can use the linear Lyapunov candidate right which is x transpose A x or P x or whatever you want to call ok all right excellent. What about this guy? This is this function ok earlier we try to verify that everything is a class K function and all that, but I know now that this is greater than equal to this guy and I know that this is 0 at 0 and I know that for all non-zero values of the state which is norm x is non-zero if norm x is non-zero this is positive right. So, therefore, V T x is dominating a positive definite function all right. I did not have to find any class K function to dominate again yeah of course, in this case this is also a class K function, but even if it is not the case we do not have to worry about it all right ok.

The next more stringent or more or smaller class of functions is the radially unbounded function ok. In this case we can no longer take as argument states in a ball, but it has to take arguments which are all of from all of R m ok. So, no more B R ball of radius R and all that yeah because we are talking about radial unboundedness which is a global property in some sense ok. And again the everything else is the same time and states maps to some real number right. You require that the function is 0 at 0.

So, the only difference now is that it has to dominate a class K R function ok. The function has the function V T x has to dominate a class K R function ok for all T in R plus and for all x in R n ok. Now slightly different picture V is allowed to be oscillating no problem oscillating V is fine, but it is fine again I should say yeah fine yeah. And oscillating V is fine, but it has to be above this class K R function. Therefore, as you can imagine as x goes to infinity V T x also goes to infinity right because it is always above this guy.

So, if this guy is going to infinity this also has to go to infinity ok alright. So, that is the difference. So, the property of going to infinity is inherited by the radially unbounded function also. Therefore, the word radially unbounded ok. Why radially because you can think of somehow norm of x as some kind of you know radial direction ok.

So, it basically as norm of x goes unbounded which is the same as saying state goes unbounded V also has to go unbounded ok. And as I mentioned this is connected to global stability ok yeah. We will talk about y in some sense I mean may be later, but the idea is not that complicated. If your function sort of I mean if you have a function say if your function looks like this yeah I mean I am at 0 yeah going to infinity on both sides this is V and this is x forget the time argument ok. If the function looks like this then great I mean if you say that if you somehow say that in the y argument that is in if you somehow can claim that my function lies below this guy yeah if I can claim that my function all value always remains below this right.

Then I can claim that my x lies within this yes no problem this is a radially unbounded function ok. This is a radially unbounded function yeah it goes all the way to infinity. But now if I have a different scenario I have to make a different picture sorry I cannot fit it here. If I have a different scenario which is that you have a function which now does this yeah very much a class I mean positive definite function by the way right. Because I am only concerned about you know say some domain whatever yeah you can think of this as increasing even there yeah I mean you can sort of imagine that this is also increasing I mean not actually flat ok.

But the point is now if I say that my y just look at this now if I say that my y is restricted to this level ok. Again this level is not whatever I mean this level is close to the upper bound in some sense. I cannot say much about x right x can be really large the bound on x could be really really very large. Of course if I give you smaller bound then ok but if I am actually giving you bound right on the boundary then I cannot say anything much about x, x could be very large ok. Therefore you can see this is not the one on the left this is not a radially unbounded function ok.

Even let's assume that in both cases the domain is Rn let's not worry about Br and all that there is no Br let's assume that the domain in both cases is Rn but this function is this structure and this function is this structure. This function at this guy let's you actually conclude something about yeah for all possible values of V you can claim some bound on x here you can't. So, this invertibility sort of property is what makes radially unbounded functions amenable to global results yeah and here it will only give you local results. Why local only until some x you can some levels you can work with from this level ok, this level ok, this level ok here not ok beyond that forget it you cannot say anything beyond this level though nothing obviously cannot say anything about x, x could be anything because we can never reach that level alright. So, that keep that in mind that is the idea we can we will of course, prove some things.

So, that these ideas are not just ideas and you see that it works in the math also, but it all depends it is always using invertibility property you will always think of using V inverse whenever you look at the proof you will see entire proof goes by using V inverse and so on and so forth and that is what is this is this is V inverse right ok alright great ok. So, examples this function obviously class kr just the norm in fact this is the Euclidean norm yeah this itself is a class kr function if I take V as that V has to be radially unbounded because it is equal to a class kr function alright simple. This example again obviously class kr why because this dominates this way right this dominates this for all non-negative time right. So, again class kr sorry dominates class kr function or you know positive definite or a radially unbounded function right.

So, you are fine. So, this is radially unbounded yeah in fact you can also think of it differently you can say that this is dominating a class k function and goes to infinity as x goes to infinity in any direction. So, we are fine ok alright I think that was until now. So, this is the definition of class kr function let us see like this yeah. Let us look at this guy what about this guy 1 plus sine squared t divided by 2 x1 square plus x2 square ok. I am again claiming that this is greater than equal to half x1 square plus x2 square for all non-negative t convinced yes because sinusoid smallest value is 0 because I took a square deliberately.

So, this is therefore, I get the half ok I have deliberately taken this example for one specific reason ok. Easier conditions ok unfortunately for all the examples I am taking easier conditions and the normal conditions do not look too different, but I can promise you these easier conditions are what most people use yeah they never try to find a lower bounding class k function and kr function alright. What are the easier conditions the first two conditions look exactly the same for radial unboundedness the only thing additional is this going to infinity condition right because obviously this verifies that w is a class k function and radial unboundedness is just a class k function with going it going to infinity that is it alright. Though again remember it is not that simple this is being verified for all rn other than 0 not just x in a ball ok not just x in a ball. So, the first two conditions in Vidya Sagar's book especially they tend to have additional notation they use local positive definite LPDF and PDF.

So, locally positive definite functions and positive definite functions. So, this is actually a PDF a positive definite function in Vidya Sagar's notation why because this is verifying this class k condition for all states not just states within some ball around the origin alright. So, that is the difference here. So, that is why I have written global positive definiteness plus the unboundedness condition ok great. Counter examples ok counter examples are very good because they tend to job you.

If you look at this function this is positive definite radial unbounded I have written an explanation the function is x1 plus x2 whole squared divided by 2 yeah what happens there are only two states x1 and x2 and I am giving you a function x1 plus x2 whole squared divided by 2 is irrelevant but whatever is this class k class kr what is it is this class kr is ok. So, this is you are talking about class k ok. So, what are those non zero points right as simple as that yeah this is not even class k why because if x2 and x1 are opposite signs x2 is minus x1 alright for all possible values of x2 equal to minus x1 basically this line that is what I have drawn here yeah along this line v is zero along this line in state space this is zero and this does not satisfy our easier test easier test requires for all possible non zero states yeah and these are obviously non zero states for all possible non zero states except for this guy v has to be strictly positive or w has to be strictly positive which is not ok. So, that is a problem not positive definite what about this guy x1 square plus x1 four yes yes. So, if I take points of the form this where x2 is zero.

So, basically along the sorry where x1 is zero yeah basically along the y axis I take anything this function is exactly zero yeah again zero for non zero values. So, not positive definite yeah state being zero means every x1 is zero. So, that is why I have taken this as a positive definite state. So, this is a positive definite state and this is a non zero state.

So, this is a non zero state. So, this is a non zero state element has to be zero yeah. So, this is a non zero state zero comma alpha and this is evaluating to be zero at that ok. So, these are counter. So, simplest thing to remember if all the states do not appear in your w or v whatever notation you want to use if all the states do not appear then it is immediately not positive definite ok and it is not a function then we can do Lyapunov analysis with ok. So, all states must appear simple this is the first key requirement alright next, but this we have already done right this function this is positive definite right you already done that because it is strictly positive yeah for non zero values of norm x right, but this is not readily unbounded that should also be very evident why it maxes out at 1 right maximum value this can take is 1 as x goes to infinity therefore, it is not unbounded ok.

So, this is some something like a I mean I mean I am maybe the shape will be different, but it is something like this ok tapers off does not go to infinity. So, this is again cannot be used for global stability analysis only for local results you can use ok alright. Similarly, for the easier test for radial unboundedness or not necessarily easier anymore if it is a function of time and state you still need the zero condition and you need a w x which is readily unbounded that you can dominate ok. So, this is not an easier condition like I said because it is almost the same as trying to find a class k r function ok. So, for the time varying cases as you can see the easier conditions are not too easy, but that is about all you have ok.

So, but for the time invariant v or w you have very easy results to verify ok. Thank you. .