Heat and Mass Transfer
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Lecture No. 05
Heat Conduction-2

Towards the end of the last lecture, we were considering the case of a long hollow composite cylinder.

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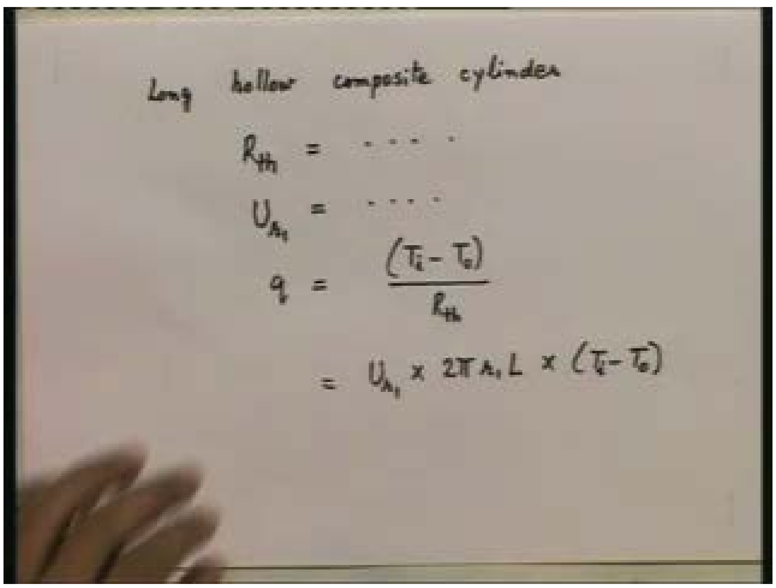

We were considering this case and you will recall that we had derived expressions for the thermal resistance of the long hollow composite cylinder which consisted of two materials with conductivities is K one and K two. We had derived an expression for the overall heat transfer coefficient for this cylinder and we had an expression for the heat flow rate. So, we had an expression for R_{th} - the total thermal resistance of the long composite cylinder; we had an expression for the overall heat transfer coefficient and you will also recall that since the area through which the heat is flowing goes on increasing as you go radially outwards, we have to specify on what area we are basing the overall transfer coefficient. So we had derived expressions say for the radius r_1 which is the inner radius of the tube of the inner cylinder or we could have based U even on the outer radius r_3, etcetera. So, we had expressions for R_{th} and for U_{r1} and once we have these expressions, you can calculate q - the heat flow rate - as being equal to; if you want to use R_{th}, you will say simply q is equal to T_i minus T_o, the temperature difference at the two extremes from inside to outside divided by the R_{th} or we can also say q is nothing but U_{r1} multiplied by the area 2 pi r_1 L multiplied by the temperature difference T_i minus T_o.

So, once we have an expression either for R_{th} or U_{r1}, we can write down the heat flow rate through that long composite cylinder; this is where we stopped last time. Now, we are therefore in a position to consider, we are therefore now in a position to take up composite cases that is cases where a number of thermal resistances occur in series. Now, I want to make an important statement which is written down here and I will read it slowly.

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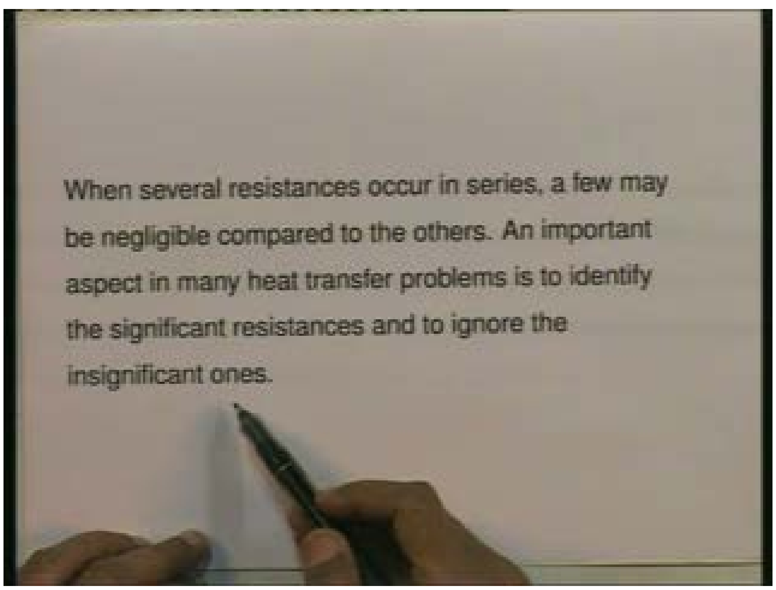

A statement is as follows; it says - when several resistances occur in series as in a long composite cylinder with two solid materials; we had four thermal resistances - one associated with the H_i, one associated with the thermal conductivity K_1, one associated with the thermal conductivity K_2 and finally the one associated with the outside in terms of coefficient H_0 or with the infinity slab where we took 3 solid materials, we had 5 thermal resistances. So, for any such situation when several resistances occur in series, a few may be negligible compared to the others. An important aspect in many heat transfer problems is to identify the significant resistances and to ignore the insignificant ones. When you have a number of thermal resistances in series, very often one or two or a few may be negligible. It is important if you are a good engineer that you should be able to identify which are the significant resistances and which are the insignificant resistances. So this is an important statement which I am making. Now in order to understand the significance of the statement, let us do the following problem. We are going to do a problem - the following. Let us say, let us go back to the earlier problem that we solved.

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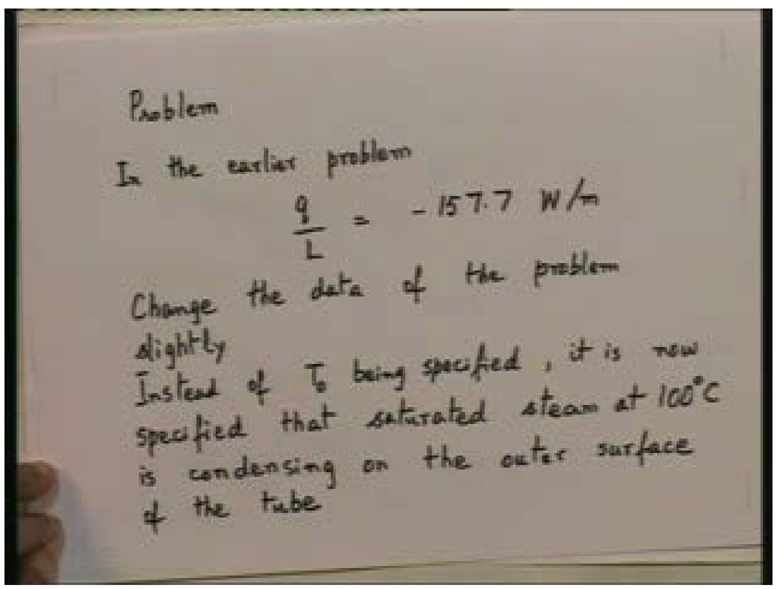

You recall the earlier problem in which we had a tube with an inner diameter of 2 centimeters and an outer diameter of 4 centimeters. One surface was at 70 degree centigrade, one was at 100 degrees centigrade and for that tube we found out the rate at which heat was flowing and we got an answer. In that problem, recall in the earlier problem, we got the answer q by L is equal to minus 157.7 watts per meter; you recall this problem.

Now, I am going to change the data of this problem very slightly; we are going to change the data of the problem slightly. Instead of the outer surface being at a specified temperature T_o, it is now specified; instead of T_o being specified - T_o was specified to be 100 remember - it is now specified that saturated steam at 100 degree centigrade is condensing on the outer surface. That is the change in the input data that i am making; saturated steam at 100 degree centigrade is condensing on the outer surface of the tube. That is the change that I am making; all other data is the same. That means what is the other data?

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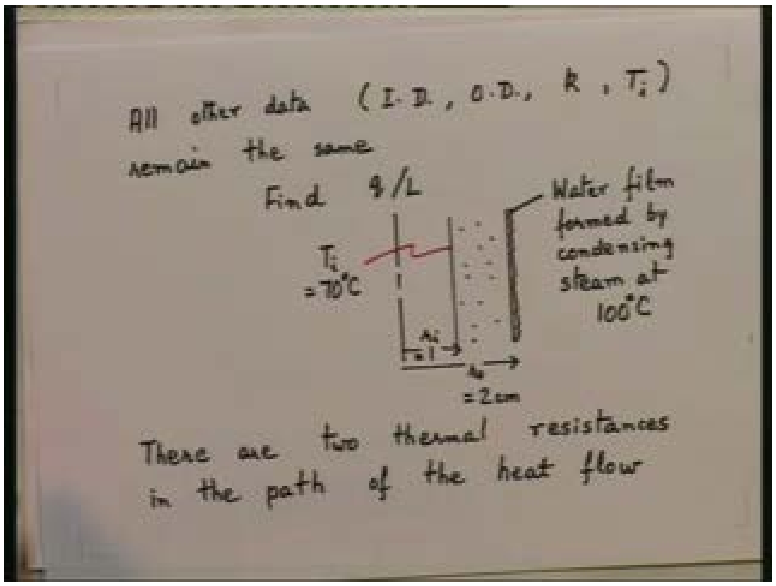

The other data is for instance the I D of the tube, all other data, I D of the tube, O D of the tube and thermal conductivity k of the material of the tube. The temperature T_i also remains the same; find q by L - that is the change. Now, how do we proceed now? First of all, let me just draw a rough sketch so that we get the picture in front of us. Here is the tube, the center line; let us say this is one - the inner diameter, this is the outer diameter and this would be our outer radius, inner radius and outer radius. We are given the values R_o and R_i, this is R_o equal to I think 2 centimeters; this is R_i equal to 1 centimeter; this is the solid material of the tube; k is given to be .58 and now on the outer surface we have steam condensing. So the steam is going to condense and form a water film on the outside; so on the outside here, you are going to have a water film. On the outer surface, you are going to have a water film formed by condensing steam at 100 degree centigrade; that is what you are going to have. The inner surface remains at 70 degree centigrade; the inner surface is this that remains T_i equal to 70 degrees centigrade. That is the position now; now what is the change compare to earlier?

Earlier from T_i to T_o, that is from 70 to 100, there was only one thermal resistance and that thermal resistance was the thermal resistance of the tube. Now, we have got two thermal resistances in the path of the heat flow when it is flowing between the temperature differences 70 from inside to 100 on the outside and what are the two thermal resistances? First, there are now two thermal resistances in the path of the heat flow; there are two thermal resistances in the path of the heat flow and what are the thermal resistances in the path of the heat flow? The thermal resistances in the path of the heat flow are: let us write them down.

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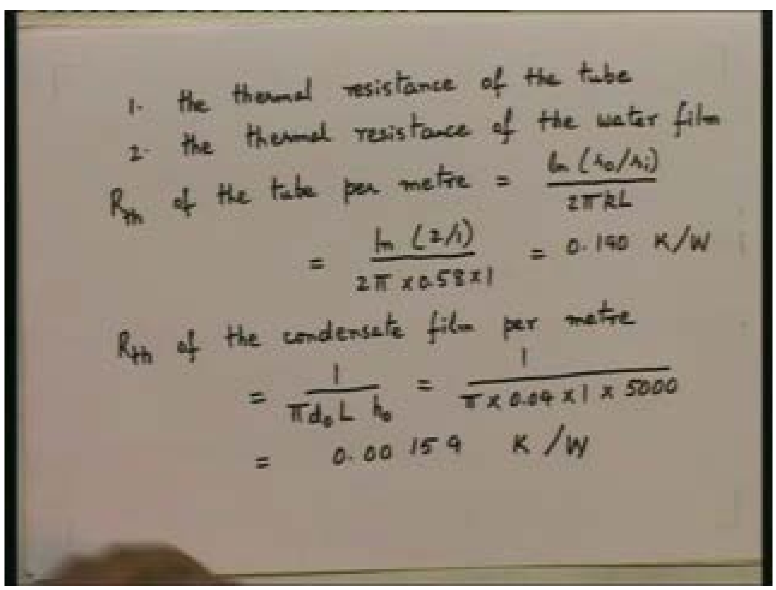

Number one; the two thermal resistances in the path of the heat flow are - number one: thermal resistance of the tube and secondly the thermal resistance of the water film; the thermal resistance of the water film on the outside. Let us estimate the two; let us put down values for them. R_{th} - thermal resistance of the tube, the first one of the tube per meter that is equal to log to the base e R_o by R_i log to the base e R_o by R_i divided by 2 pi kL. So if you substitute the data - the given data - that is log to the base e 2 by 1 R_o by R_i divided by 2 pi into .58 - the same thermal conductivity as earlier - into a meter length 1 that comes to .190 Kelvin per watt; thermal resistance of the condensate film per meter; thermal resistance of the condensate film - the water film - of the condensate film per meter.

What is that, that is equal to 1 upon HA, 1 upon A. What is A? pi d_o L that is multiplied by H_o that is equal to 1 upon pi into d_o is .04 centimeters into length, we will take as 1 meter, 1 meter length into H_o. Now, H_o is the heat transfer coefficient associated with the change of phase - that is condensation taking place on the outside of the tube. Let us take a typical value of Ho; I have already told you, it would be in thousands, 5000, 10000 typical values. Let us take 5000. Let us take value of 5000; if I take a value of 5000 and calculate this R_{th}, I will get this. This number will come out to be .00159 Kelvin per watt. So, what do we see? We have got two thermal resistances - the thermal resistance of the condensate film is insignificant compared to the thermal resistance of the tube; that is what we are seeing. The thermal resistance of the condensate film is insignificant compared to the thermal resistance of the tube. We see, let me write that down

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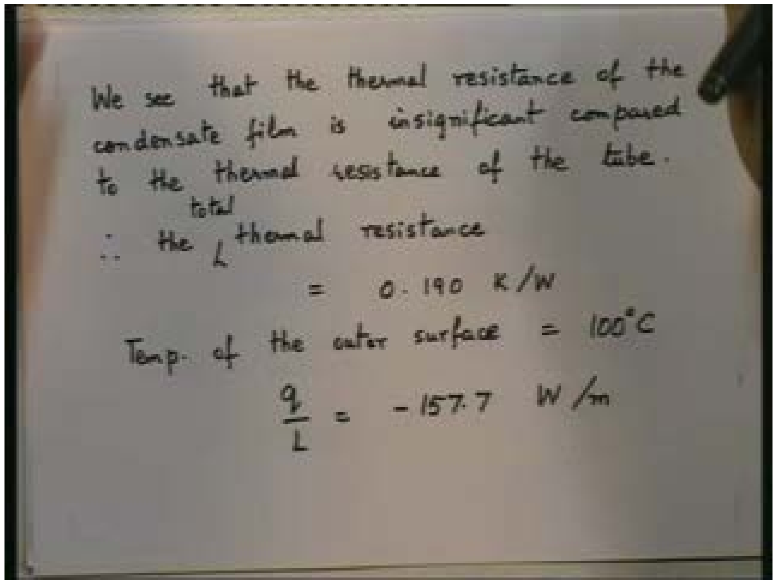

thermal resistance of the condensate film is insignificant compared to the thermal resistance of the tube. Now you may say - look the thermal resistance of the condensate film has only been estimated because we have taken H_o to be five thousand. Suppose we have taken it as 10000, it would be still smaller; it is in thousands whatever you do whether you take 5000, 10000, 7000, this value .00159, we would get a value which may be slightly different but it will always be insignificant compared to the value of the thermal condensate of the tube which is .190 Kelvin per watt. Therefore, the total thermal resistance is really .190; the condensation doesn't make a difference and the outer surface therefore is effectively at 100 degree centigrade and the flow rate remains the same.

Therefore, the conclusions are the thermal resistance - this total thermal resistance I should say - that is the sum of the 2 is equal to .190 Kelvin per watt. Temperature of the outer surface is effectively 100 degrees centigrade and the answer q by L is minus 157.7 watts per meter; the answer remains unchanged, that is what effectively we are saying. So although we have got a situation which seemingly looks different, the thermal resistance introduced because of this new situation is insignificant. The total thermal resistance remains that which we had when there is only one present; it remains .190, so effectively the outer surface of the tube remains at 100 degree centigrade and therefore the heat flow rate obviously is still the same – 157.7.

Now, suppose you haven't recognized this; then you would try, you would waste your time perhaps trying to calculate the value of the H_o more accurately from some formula. You may get a value like 6342 for H_o, would it make difference to this answer of this problem? No, it wouldn't. Whether you take 6342 which is a more precise value of H_o or whether you use 5000 just doesn't matter in this problem. In some other problem it may matter but right now it doesn't matter what is the value of H_o because the thermal resistance of the condensate film is negligible compared to the thermal resistance of the tube. You follow what I am saying? Always compare the value which is and say something is insignificant relative to something else; that is the point which I want to make.

So, now we have looked at this situation in which we may have thermal resistances in series and as I said it is important to be able to decide which resistance is significant and which resistance is not. Now, we want to turn our attention to another composite situation; a problem associated with another composite situation. We want to look at what is called the critical radius of insulation; that is what we want to look at.

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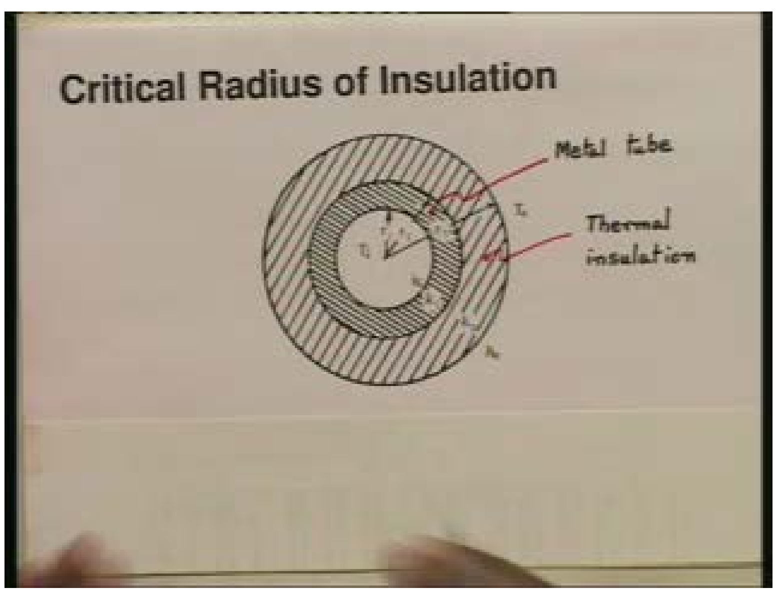

Consider again this is the long hollow cylinder composite situation. Let us say that the inner material here radius R_1 R_2, let us say that this is a metal tube; that is a metal tube and let us say the outer material thermal conductivity k_2, is thermal insulation which is put on the tube. So, it is composite situation in which one of the materials is the metal tube, the other is the thermal insulation which is put on the outside. In the inside of the tube, there is some fluid with the heat transfer coefficient H_i and the temperature T_i and on the outside there is some other fluid to which heat is being lost; it may be the surrounding air in the room in which this tube is located.

The outside heat transfer coefficient is H_o and the temperature outside is T_o; so there are four thermal resistances, two associated with the heat transfer coefficients on the inside and outside and two associated with the tube and with the thermal insulation. Now this is a composite case for which we know the value of the heat flow rate from inside to outside; if i want to put down an expression for the heat flow rate, then we will get the following. We have derived it earlier; i am just repeating what we have done earlier.

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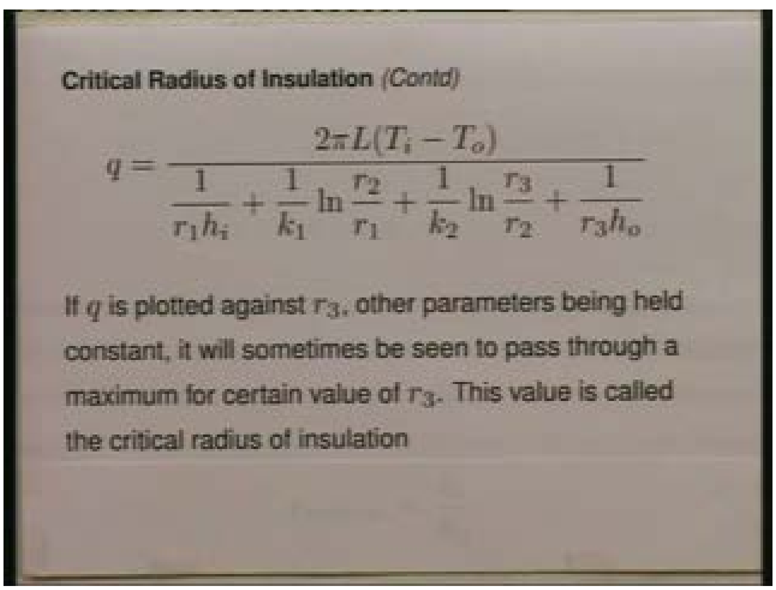

For such a situation the q is given by this expression 2 pi L into T_i minus T_o divided by 1 upon r_1h_i plus 1 upon k_1 log to the base r_2 by r_1 plus 1 upon k_2 log to the base e r_3 by r_2

plus 1 upon r_3 h_o; we know this. These are the four thermal resistances - the first one associated with h_i, the second one associated with the metal tube, the third one associated with the thermal insulation and the fourth one with the heat transfer coefficient on the outside. The 2 pi L has to be taken down in order to call it the thermal resistance so it will be 1upon 2 pi L into all this; then they actually become the thermal resistances. Now suppose I plot q against r_3, I hold all other parameters constant. What are the other parameters? r_1, r_2, k_1, k_2, T_i, T_o, L everything else constant excepting r_3; plot q against r_3 other parameters being held constant. It will sometimes be seen, the value of q will sometimes be seen to pass through a maximum for certain value of r_3. This value is called the critical radius of insulation. So if I plot q against r_3, sometimes we will see, go through a maximum for a certain value of r_3. This value of r_3, we call the critical radius of insulation.

Now, first of all let us find out mathematically by differentiation, what is this value of r_3 at which the q acquires a maximum value; that is very easy to do, it is just matter of differentiating. So let us do that, let us just put down an expression, derive an expression for the value of r_3 at which q goes through a maximum. Now in the formula that we have put down, let me repeat that, we want to know where this goes through a maximum-this expression; the numerator here is constant so if i want to know where this goes through a maximum, effectively I want to know for what value of r_3 the denominator goes through a minimum. So, let us differentiate the denominator only with respect to r_3 and equate that to zero; differentiate the denominator with respect to r_3 and equate that to 0.
The first term - if I differentiate – is going to give me nothing because there is no r_3 in its all constants; the second term will also give me nothing, it also doesn't have an r_3 which is our variable, it is going to give zero. Only the third term and the fourth term will give me something because they have an r_3 occurring in them.

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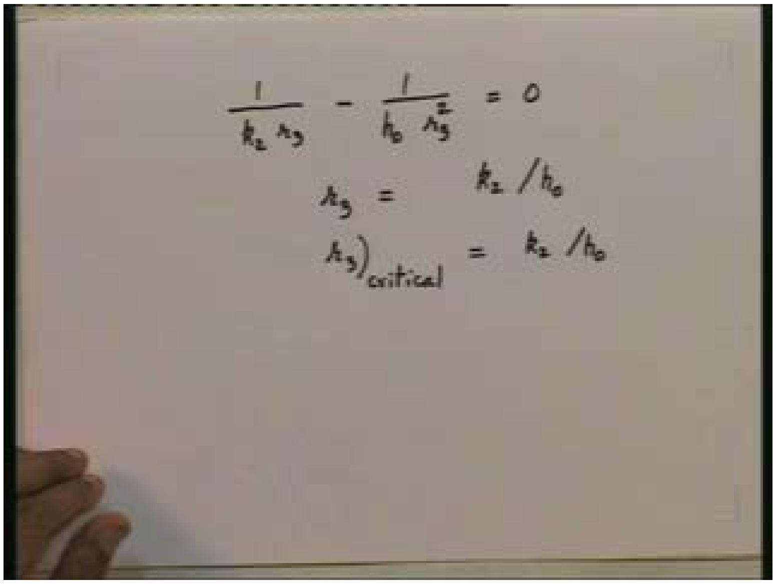

So if I differentiate the third term and the fourth term and equate to zero, I will get 1 upon - I am now differentiating the denominator third term -1 upon k_2 r_3 that is the third term differentiated minus 1 upon h_o r_3 squared is, on differentiation gives me that and I equate that to zero, I will get r_3 is equal to k_2 by h_0; so it is this value of r_3 given by k_2 by h_0 that the denominator goes through a minimum and therefore the whole expression for q goes through a maximum. We will call this value of r_3 as the critical radius of insulation; this is the critical radius of insulation. So we will denote it as $r_{3critical}$; this is the critical radius of insulation. Now in effect what I am saying is the following.

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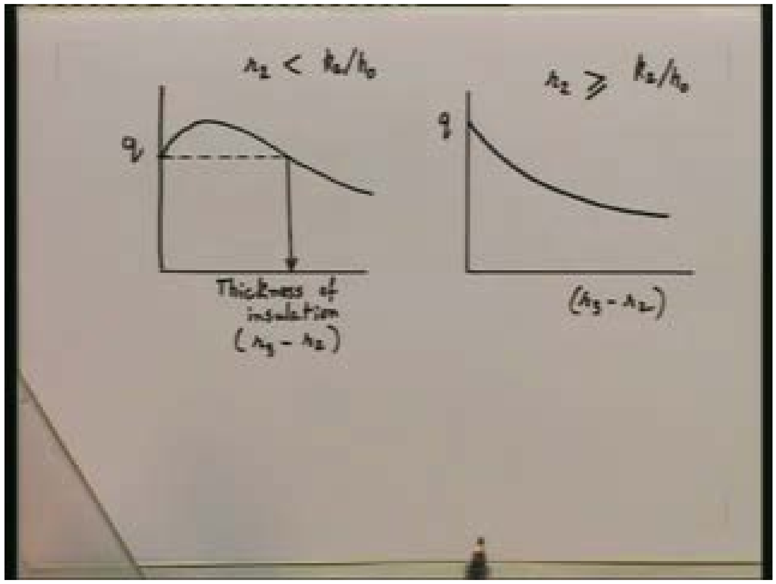

I am saying if I were to plot q against r3, let us do that, if I were to plot q against r_3 or instead of r3, if I were to plot it against r_3 minus r_2 that is the thickness of the insulation, if I want to plot q against thickness of insulation that is r_3 minus r_2 - r_2 is a constant so whether I plot against r_3 or r_3 minus r_2 I will get the same variation - then I will sometimes get a behavior which will look like this and sometimes i will have behavior which will look like this. When do I get the first type of behavior? When does q go through a maximum? q will go through a maximum and then go on decreasing if r_2 that is the outer radius of the metal pipe is less than the critical value which is given by k_2 by h_o. If to start with, I have metal pipe and the r_2 of that metal pipe is less than k_2 by h_0, k_2 is the thermal conductivity of the insulation and h_o is the heat transfer coefficient on the outer surface of the insulation. If r_2 is less than this, then we are going to get this type of situation of q going through a maximum then decreasing.

On the other hand, if the radius r_2 of the metal pipe to start with is greater than or equal to k_2 by h_0 - k_2 by h_0 is the critical radius of insulation - then straight away if I put insulation on that pipe I will start getting a reduction in q. So keep in mind the addition of the insulation on a pipe will help in reducing the heat flow rate if to start with r_2 is greater than or equal to k_2 by h_0; if it is less, if r_2 is less than k_2 by h_0, the addition of insulation will first cause the value of q_2 go up a little, then it will start coming down as we go on putting more and more insulation.

Now, why does this happen? It happens, it is very easy to see if you look at the expression for q; let us go back to that. The expression for q has two thermal resistances right here - one is the thermal resistance associated with the insulation, the other is the thermal resistance associated with the heat transfer coefficient on the outer surface. The first one which contains the log to the base e r_3 by r_2- this increases as r_3 increases; the second one - this one decreases as r_3 increases, as you can see it is 1 upon r_3 here.

So, up to the value of k_2 by h_0, that is the critical radius; the rate at which this increases is less than the rate at which this decreases and therefore the overall thermal resistance is decreasing and therefore the value of q goes up. Once we have got the critical radius, the reverse happens and the overall thermal resistance goes on increasing and therefore q goes on decreasing. That is really mathematically why it is happening in this fashion. So when we put insulation on a pipe the lesson to learn is, when we put insulation on a pipe, you want q to go down so make sure to start with that; your value of r_2 is greater than or equal to k_2 by h_0. If this is so no problem, whatever the thickness of insulation you put given r_3 minus r_2, it will help.

If on the other hand, you have a situation in which to start with r_2 is less than k_2 by h_o, then what should you do? You must put enough insulation to see that you cross this hump and you have reached a point where the value of q is less than the value you get to start with when there is no insulation. So if I were to take a horizontal line from here; what I am saying is, if you have a situation in which r_2 is less than k_2 by h_0, then you should always put, you should put a thickness of insulation greater than the value shown by this arrow. If you put less, you will in fact get a value of q which is more; so adding insulation will hurt you, will not help you. That is the meaning and that is the sum and substance of what we are trying to do when we talk about the critical radius problem.

So now we can look at a problem and I want to solve a numerical problem just to illustrate ideas. I am going to take the following situation, I am going to say let us say there is a pipe - just going to draw a sketch.

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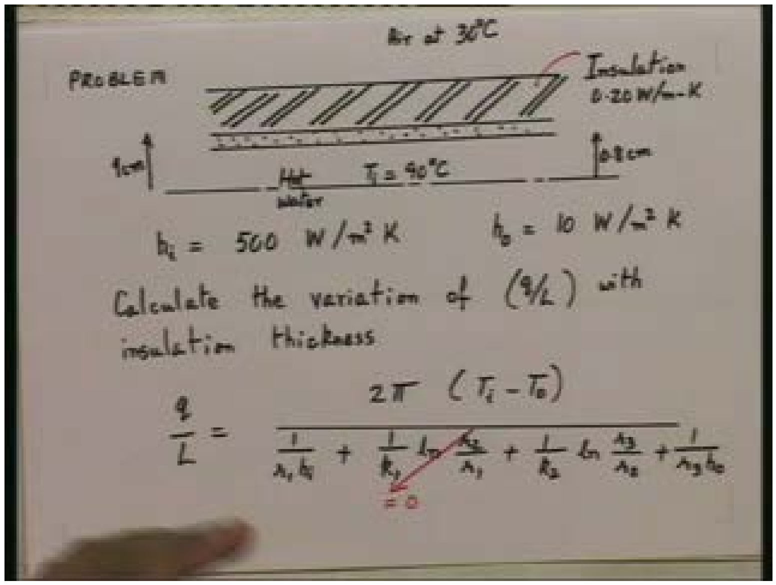

Let us say there is a pipe like this, some pipe whose inner, I am drawing only half of it, some pipe; the inner radius is .8 centimeters and the outer radius is 1 centimeter; a pipe with an inner radius of .8 centimeters and outer radius of one centimeter. Let us say it is a metal pipe like this, some metal pipe like this and we put insulation on the outside of this pipe, we you put thermal insulation on the outside of this pipe; this is the thermal insulation on the outside of this pipe and let us say the insulation - thermal insulation - conductivity is .20 watts per meter k that is the thermal conductivity of the insulation. You are told that on the inside hot water flows and the temperature T_i is 90, hot water at 90 degree centigrade; hot water flows on the inside at 90 and on the outside there is air at 30; ambient air at 30 centigrade. Value of h_i on the inside of the tube given to be 500 watts per meter squared Kelvin and on the outside h_o, value of heat transfer coefficient given to be 10 watts per meter square Kelvin.

500 is a typical value for low velocity water flowing at a low velocity through a tube and 10 is a typical value for heat being lost by natural convection from an outer surface. Calculate the variation of q by L with the - q by L that is the heat loss rate per meter length - with insulation thickness. This is the numerical problem we want to do; it is a straightforward substitution problem. There is no real complication; I just want to illustrate the ideas that we have been talking about.

Now, a typical situation metal pipe is insulation rounded, hot water flowing inside; how does the q by L vary with the insulation thickness? Now for this case if I substitute in the expression; what is my expression? q by L is equal to 2 pi T_i minus T_o divided by, the expression is 1 by r_i $r_1 h_i$ plus 1 by k_1 log to the base e r_2 by r_1 plus 1 by k_2 log to the base e r_3 by r_2 plus 1 upon r_3 h_o; that is my expression for the q by L. Now the first thing we want to say is we have got a metal pipe made of say steel or stainless steel; what will be the k value? If it is say steel, it will be 10, 12 watts per meter Kelvin; if it is steel not stainless steel but steel it may be 40, 30 or 40 or 50 watts per meter Kelvin. In any case, it is just a pipe, a tube rather with a thickness 2 millimeters.

r_1 is .8 centimeter then r_2 is 1 centimeter; whatever the value, the thermal resistance of this metal tube is going to be insignificant compared to the thermal resistance of the insulation. So straight away let us drop this out and say this we are going, not even going to calculate it. We are just going to take it equal to 0; going to make that approximation which is a very accurate approximation to make. So let us forget the thermal resistance; neglect the thermal resistance of that metal tube compared to the thermal resistance of the insulation which is coming on the outside. It is quite justified so with this approximation what do we get now for the q by L?

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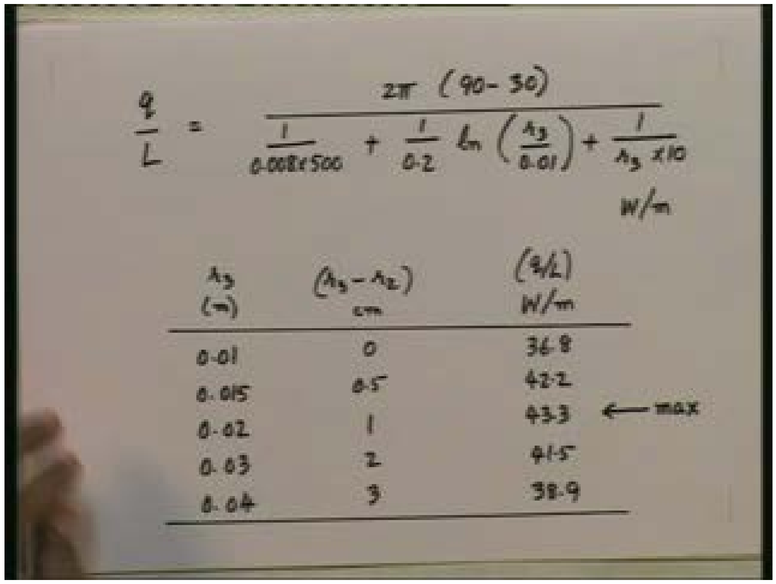

By putting the numbers, I will get q by L is equal to if I put in the given data, I will get q by L is equal to 2 pi into 90 minus 30 divided by - let us put down the data now - divided by I will get 1 upon r_i .008 multiplied by 500 plus neglect the second resistance. The resistance of the insulation .2 log to the base e r_3 upon .01 centimeter that is r_2 plus 1 upon r_3 multiplied by 10; 10 is the value of h_o. Now for different values of r_3, we can calculate q upon L in so many watts per meter. Calculate the value of the q by L against, with different values of r_3; if you do that and I want you to do this now; you should get the following answers.

Show that for different values of r_3, r_3 in meters; I would like you to get the following answers. Show that you will get for different values of r_3 .01, .015, 0.02, .03, .04 and what will be the thickness of insulation? r_3 minus r_2 which is the thickness of insulation in centimeter; so will come out to be here it is 0 because here r_3 is equal to r_2 then it will be .5 centimeters then 1 centimeter, 2 centimeters, 3 centimeters and you will get q by L if you substitute into the above expression. You will get q by L in watts per meter to be 36.8, 42.2, 43.3, 41.5 and 38.9 and with more thickness of insulation that value of q by L will go on decreasing.

So notice now that in this particular case with the data that we got the value of q by L goes through a maximum goes through a maximum at an insulation thickness of 1 centimeter and the value of r_3 equal to 2 centimeters. Do these calculations yourself so that you will be convinced of what we got. Now why is this happening?

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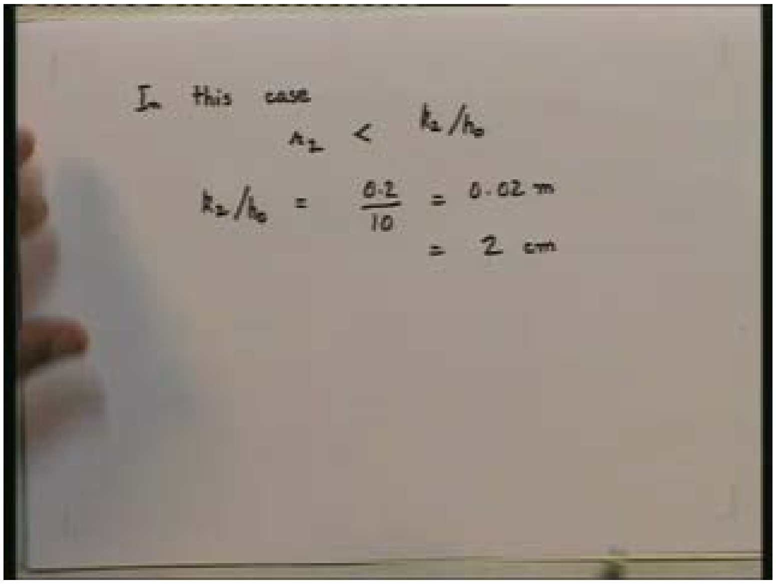

This is happening because, q by L is going through a maximum because in this case in the data that we have given in this case r_2 to start with is less than k_2 by h_0 - that is the critical radius of insulation and what we will get for k_2 by h_0? We get k_2 by h_0 is, it is k_2 by h_0; in this case, the critical radius is equal to .2 divided by 10 which is nothing but .02 meters which is equal to 2 centimeters. So, this is a case where r_2 was less than the critical value given by k_2 by h_0. Therefore in this case the value of q first increased, q by L first increased with thickness of insulation then decreased and started decreasing. So, this is just an illustration, a numerical example to illustrate all these ideas.

Now final comment on the critical radius; keep in mind this situation does not happen very often. When does it happen? It happens when k_2 is high, that is the thermal conductivity of the insulation is high; this situation will also occur when the value of h_o on the outside is low. Then the combination k_2 by h_0 will give me higher value of $r_{critical}$

and then only r_2 may be less and q by L will first go through a maximum. So it will occur only under these situations.

Now on your own, I would like you to do the following problem. Please do this on your own and I will readout the problem; problem is the following.

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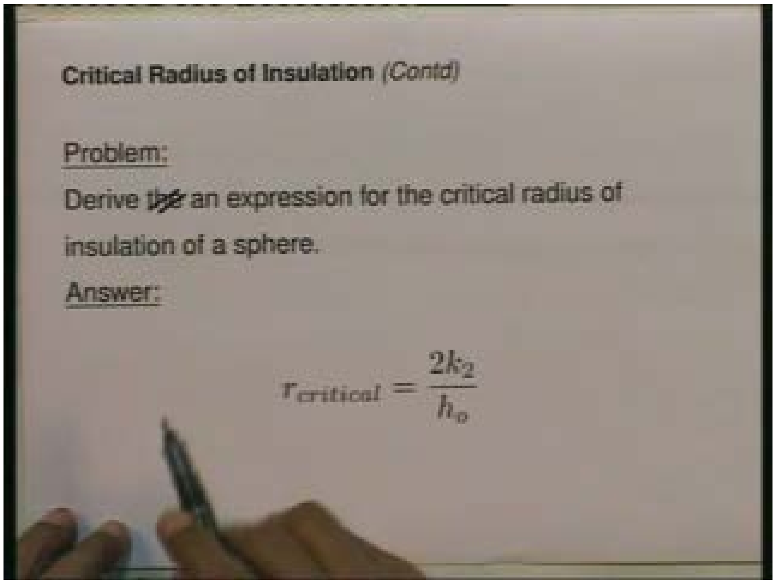

Derive an expression for - 'the' should not be there - derive an expression for the critical radius of insulation of a sphere. Suppose I have a sphere - a hollow sphere; inner radius r_1, outer radius r_2 and I put insulation up to a thickness, up to a radius r_3 around it. So what is the thickness of insulation r_3 minus r_2? Find an expression for the critical radius for this situation and the answer you should get is $r_{ritical}$ for the spherical k should be 2 k_2 by h_o. Please do this yourself in the same manner that we derived the value, this expression for the long composite cylinder for the cylindrical case.

Now, we will just briefly today start with the next topic. I just want to introduce you to the next but we will not really do it today that is so far we have taken up one dimensional cases, one dimensional steady state through a slab, one dimensional steady state through a cylinder, one dimensional steady state through a composite slab, one dimensional steady state through a composite cylinder; the sphere case I asked you to do on your own. Suppose all these cases remember temperature has only been a function of one dimension, that is why one dimensional cases, temperature has been a function of x or a function of r. Suppose now we want to take up situations in which the temperature varies in more than one direction, say it is a coordinate system - cartesian coordinate system - in which temperature is varying in the x and the y direction and the z direction; temperature is varying in space in more than one direction. Suppose also temperature is varying with time that means it is not a steady state situation. Whenever we have situations in which temperature is a function of more than one variable then we need to derive a general differential equation which will be a partial differential equation for solving heat conduction problems.

So, now our next job will be to derive the general differential equation for heat conduction for situations in which temperature may vary in space in more than one direction and temperature may vary with time also; that is what we are going to do. In order to do that, we need to consider: anytime you have to derive a general situation in differential equation, a partial differential equation or an ordinary differential equation for that matter, you need to consider some element, some arbitrary differential element in that solid and apply the first law and Fourier's law to that arbitrary element.

So, what we are going do now is we are going derive a general differential equation for heat conduction. We are going to derive it for an isotropic material that is a material in which k does not vary in direction but k may vary from point to point and we are going to derive it in the cartesian coordinate system, that is in the xyz system. Later on, we will generalize to another coordinate system; so we are going to now derive a general differential equation in the cartesian coordinate system and for an isotropic material, that is a material in which k does not vary with direction. So pick an element inside the solid.

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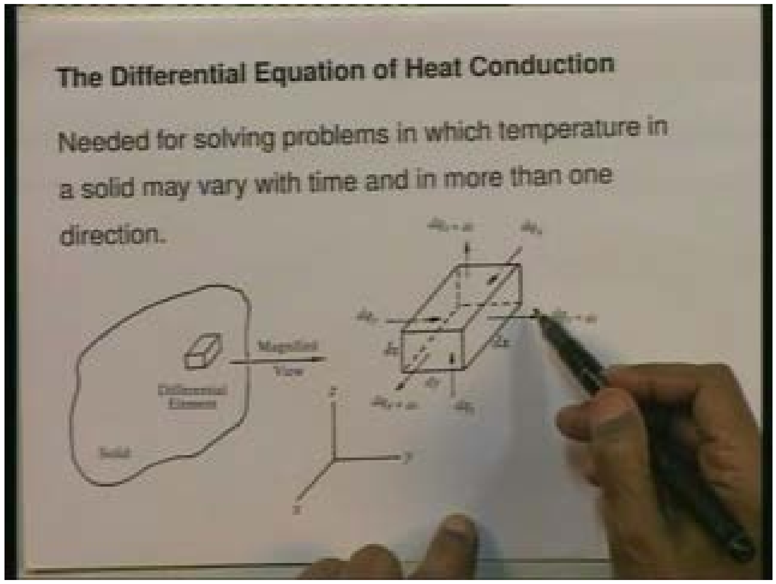

Let us say the element we pick is an element dx dy dz - a rectangular element somewhere within this solid; this is some arbitrary solid - some boundary conditions which we will take up later; this is the differential element. I take a magnified view of it and say this is the direction dx, this is the direction dy, this is the direction dz. And now on this element, now we are going to, through this element - not on this element - through this element, we are going to apply the first law of thermodynamics as for a closed system. In the process, we will also use Fourier's law of heat conduction and get our general differential equation; so that is what we will be doing next time.