Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 08
Algebras and polynomial algebras

So, welcome back to the this next half, in the last half I have been talking about modules, modules homomorphisms, generating sets, finitely generated modules, etc. Now, I want to introduce what are called A algebras. Algebras over a ring. So, to motivate this first I will take a simple case that of a field.

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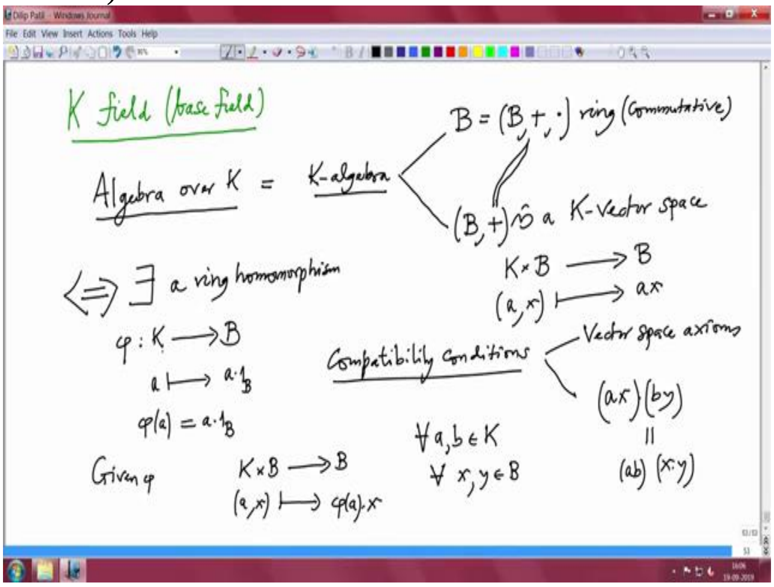

So, my base ring now is a field, K is a field. So, this is if you like I will call it a base field it is not necessary but just for the sake of motivation. So, now what I am trying to define is algebra over K also this is sometimes abbreviated as K algebra. So, same as this, so what is a K algebra? K algebra is a ring, so there are two things B and this B is a ring.

So I will write B equal to B, plus and dot ring, we will assume it is commutative. If this is not commutative we will do modify the definition little bit it is worth it I will do it after this. So, B first it is a ring, and also this same time it is a K vector space with respective to same addition and B, plus is a K vector space, with the same, this is same addition because it is abelian group and this is also abelian group. And the, this, this means what? This means that this abelian group has a K scalar multiplication by K that means we have such a map.

K cross B to B and this scalar multiplication map is denoted by A times x this goes to ax such as scalar multiplication is there and this scalar multiplication and these of course when you say vector space this scalar multiplication and this addition is compatible that is the vector space axioms and this scalar multiplication and this multiplication in the ring that is also compatible, that means what? That means, so compatibility conditions.

So, vector space conditions I will not write, vector space axioms, K vector space axioms on B that is this addition, this addition and this field axiom and also now this scalar multiplication of K scalar multiplication on B and multiplication in the ring B. So, now for example if a have scalar a multiply by element in B this is scalar multiplication another one. Now, there elements in B, so now I am multiply them in the ring.

On the other hand, I could multiply x and y in the ring and a and b in the field and multiply scalar. So, these two should be same for all a and b scalars and for all elements x and y in B then you can call it an algebra. Instead of writing so much this is equivalent to saying means all this data is equivalent to saying there exist a ring homomorphism phi from K to B. Once I have ring homomorphism then I can define a scalar multiplication which satisfies these properties and once I have a scalar (mul), this data I can define phi.

For example, let me just write down a definition of phi using this. So, that is any a now I want a element in the b so what can I do? So, take a times 1, 1b see I have given a scalar multiplication that means I have given this map ax a, x is going to ax. So, in particular I can take x equal to 1 b and then take this a equal to this is my phi a, phi a is by definition a times 1 times b and the check that this is a ring homomorphism.

Conversely by varying homomorphism I can define a scalar multiplication and how do I define that? So, given phi how do I define this map? K cross B to B. So, take any a, take any x and where can I send it? I have given this phi. So, obviously I can apply phi to a. So, phi a I get an element in a and then multiply it by x this makes sense, because we have given this and this is a multiplication in the ring, this is scalar multiplication.

Now, so this is in one to one correspondence. So, it is easier to think ring homomorphism mean and so difference scalar multiplications here, different K vector space structure on the same B plus will give different ring homomorphism and different ring homomorphism will give different algebra structure.

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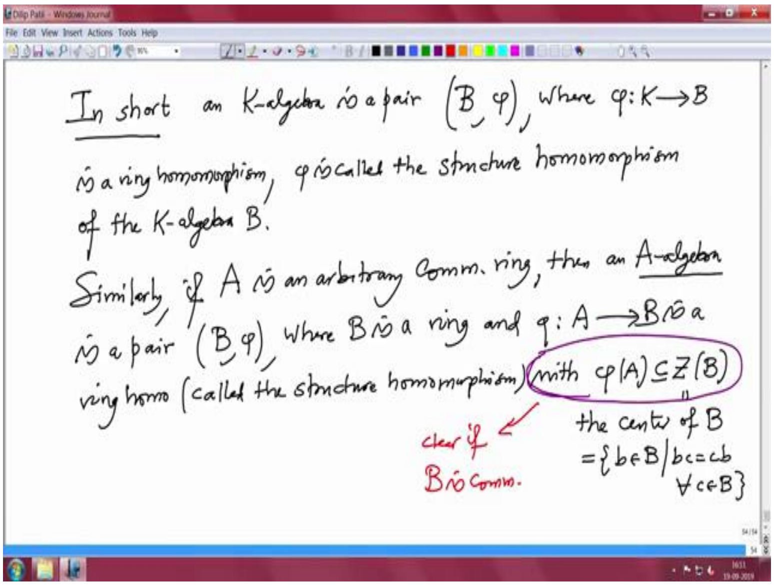

So this one the, so in short let me now be in short, in short, an a algebra, an K algebra is a pair, which pair? There is two, B, comma phi where phi is a ring homomorphism from K to B. And this phi, phi is called the structure homomorphism of the K algebra B. If I have different homomorphism then I will have a different algebra structure. So, when you say algebra that means this phi is fixed so that is a pair.

Now, just in this definition and in all this discussion we never really use a fact that K is a field. So, you could have taken arbitrary commutatively. So, similarly, if A is an arbitrary commutative ring, then an A algebra is a pair B, comma phi where B is a ring and phi from A to B is a ring homomorphism called the structure homomorphism. Now, just one comment here and we will see some examples as I said.

As I here I wrote B is just a ring. So, normally we assume commutative in this course whenever I write but in this particular case one can also allow non commutative rings but then you have to put a condition on this ring homomorphism. So, with this ring homomorphism should have property that the image of this phi of A should be contained in the center of B, this is the center of B. This is the center of B. What is a center? That is by definition all those elements b in B which commute with every other element of B.

So, b times c equals to c times b for every c in B that is called the center of B. If B is commutative then this condition is automatic so we do not have to write this condition explicitly this is automatic, this condition is immediate clear if B is commutative this is what usually people who work in specially representation theory of groups they will need to put his condition because the rings may not be commutative.

I will also give one example where the ring method may not be commutative but it is very useful to study. Now, we should see some examples of algebras. The most important example.

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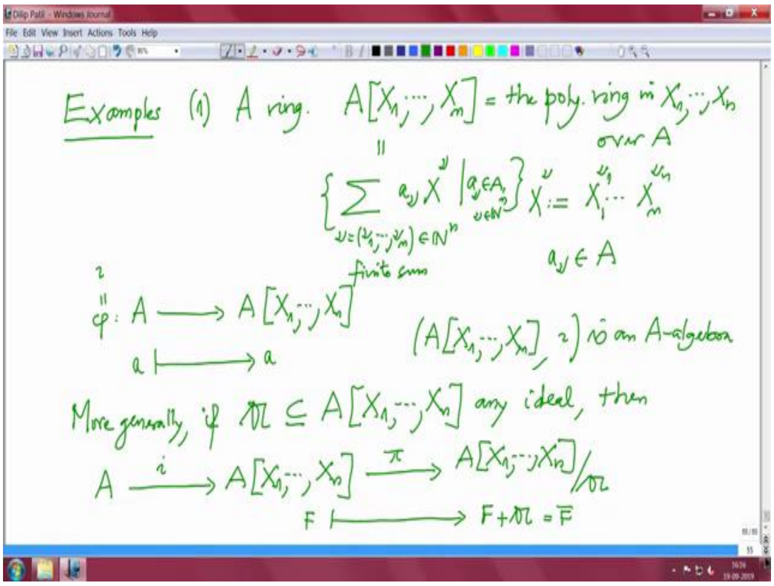

Some examples, so I will write for, so if A is any ring commutative always then the polynomial ring O or A in several variables AK X1 to Xn, A X1 to Xn this is the polynomial ring the polynomial ring in X1 to Xn over A that means the coefficient are involves. So, in symbols this is summation the polynomials in this variables look like this nu equal to nu into nu n in N power n a nu X nu and what is this notation X nu this is a calculus notation this is X1 nu1 Xn nu n. This is a monomial corresponding to this n tuple nu in the variables X1 to Xn and these are the coefficients.

And these a nu are in the ring A and this is a finite sum any polynomial look like this. So, A nu are element in a. One say that for almost all tuple this is 0, nu are 0. So, that the sum is really are a finite sum and it make sense and that is the polynomial in a typical polynomial in this polynomial ring. And now to say that this is an A algebra what do I have to give? I have to give a algebra homomorphism from I have to give a ring homomorphism from a to this.

So, the phi this one is an natural map phi there. This is a natural map so I will call it as iota this is any a going to a itself think of a as a polynomial. So, this is a natural inclusion map. So, this is the therefore this A X1 to Xn, this iota is an a algebra more generally if you have any ideal if A is an ideal in this polynomial ring X1 to Xn any ideal then A to A X1 to Xn followed by that pie this is A X1 to Xn module of the ideal A this is a quotient ring and then we have this is natural map pie which maps any polynomial F to we have written F plus A.

But we will write this as F bar. So, this composition iota composition pie is a new homomorphism this is both these maps are natural maps, they are economical maps. So, therefore, this quotient ring also can think as an algebra over A with this as a structure homomorphism.

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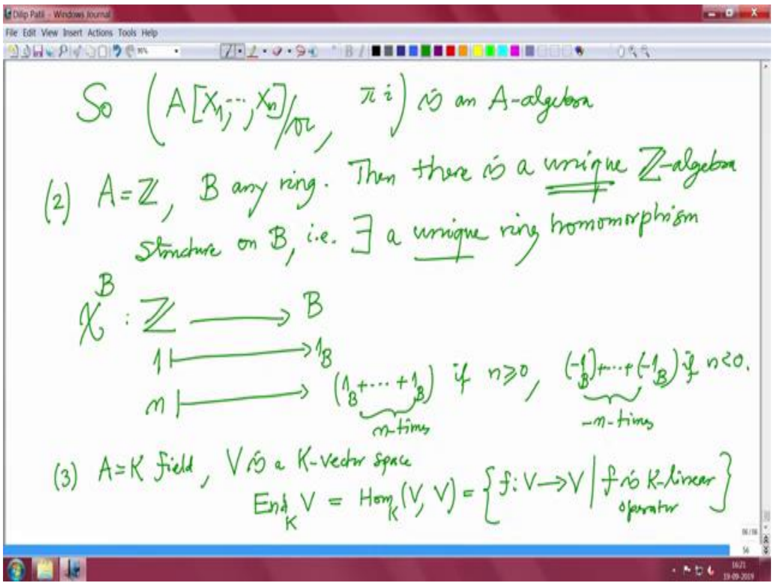

So, A X1 to Xn module of the ideal A, comma iota compose pi this is a ring homomorphism so this is an A algebra and these are the algebras, we want to study more these are the algebras which will come in over algebraic geometry very often for different A and therefore, we have to study this algebra. Now, the next will be when do you say an algebra is finitely generated algebra for example.

So, but before that I have to give this little more examples. So, the second example any ring B is a Z algebra in a natural way or maybe I will formulate little bit better. So, we are taking the base ring to be Z and B is any ring, any commutative ring, commutative I will tell you when the ring is not commutative otherwise all our rings are commutative then there is a unique Z algebra structure on B.

So, that means so that is in other words there is, there exist a unique ring homomorphism from Z to B. This I will denote by chi B and what is that? So, if I tell you we know under ring homomorphism 1 has to go to 1B that is the part of a definition of a ring homomorphism, once we know 1 goes to 1B then arbitrary n where will it go? It has to go to 1B plus, plus, plus, plus 1B n times if n is non negative if n is non negative this is n times otherwise it is minus 1B plus, plus, plus, plus, minus 1B n times, minus n times if n is negative.

So, this is the only ring homomorphism and that will give you a unique structure of Z algebra on B. Now, the third one, third one is noncommutative and that is only example I would like to study for noncommutative case. So, my base ring is a field now A is K, is a field and V is a K vector space then to each vector space I have these endomorphism of end KV what is that? This is simply, see this V is a K module.

So, this is hom K V V so what is this mean? This is, these are all linear operators on V F V to V, V to V f is K linear they are also called operators. This is obviously a ring because I can compose, composition is a multiplication. So, and addition is standard how do you add two linear maps.

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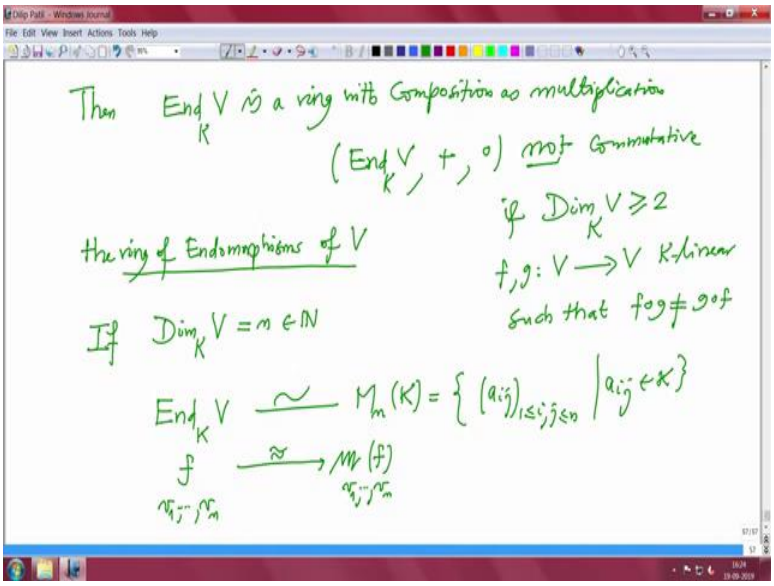

So, then end K V is a ring with composition as multiplication. So, this is end K V plus end composition and this ring is not commutative if dimension of V as a K vector space is at least 2. In that case you can find two linear operator f and g from V to V K linear operators. Such that f compose g and g compose f they are not equal. Dimension at least 2 unit because we need play little bit. So, that I will leave it for you to check.

So, this is also called ring of, the ring of endomorphisms of V. Well, if V is finite dimensional, if dimension of V equal to n in N is a finite dimension of dimension n then this ring end Kv is isomorphic to Mn K what is Mn K? Mn K this is they are matrices a. So, I will not use that gothic letters. So, a, i, j 1 less equal to i, comma j less equal to n and this a, i, j are in the field and you know how to add matrices, etc.

So, this is an isomorphism any linear map that is if you choose a bases V1 to Vn, this is linear map is uniquely determined by the matrix of F with respective to this bases V1 to Vn. So, this is the isomorphism. Give a matrix we can define a linear operator and given a linear operator we can this. But this is a with a fix bases. So, and, so that is this is therefore, commutative ring and there is a natural map. So, let me go to the next page.

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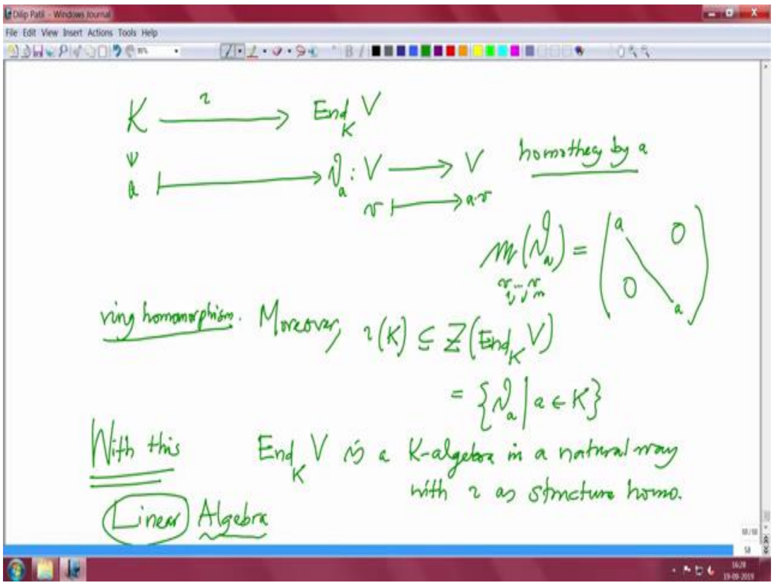

There is a natural map from K to endomorphisma of A. What is this natural map? This is also inclusion map. So, namely any A in K where do you map it to that scalar. So, that means I have to define a linear map that is usually denoted by theta a from V to V this is V going to aV this is called homothecy by a. So, they correspond to what, they call it a scalar matrix, the matrix of any bases of the this homothecy with theta a with any bases V1 to Vn if it is finite dimensional then this is nothing but a on the diagonal zeros everywhere that is clear.

Because V1 equal to a V1 V2 equal to a V2 and so on, so this is, this is unique and this is obviously ring homomorphism. So, this is a ring homomorphism more over the image of this iota of K this goes to inside of the center of this endomorphism ring of V note that the center is nothing but scalar matrices, a matrix commit with every other matrix when it is a scalar matrix. So, this is precisely in this notation it is theta a, where a varies in K.

So, our second condition when we said that it is an algebra that is satisfied in this case. So, with this endomorphism ring of V is K algebra in a natural way this map iota or with iota as structure homomorphism and by now you would have realize that this endomorphism algebra is the object of study in linear algebra. So, you see linear algebra so this linear word corresponds to the every element in the linear map and this algebra corresponds to the fact that this endomorphism ring is actually K algebra that is why the subject usually called linear algebra if people are studying only matrices then it is usually called matrix algebra and the matrix, set of matrix is also an algebra.

So, these are the only noncommutative examples of K algebra’s are studied very extensively in a extensive way but other non-commutative algebra’s are rarely studied. So, now let us now go on. Now, I want to recall or define the concept which is analogous to the finite generation.

So, now we have seen when is a module finitely generated, say now remember the module has operation only addition which is underling abelian group and sacral multiplication from a ring where it is a module over which ring but in case of algebra now we have two structures. We have a sacral multiplication of the base ring and also we have a multiplication in the ring itself that algebra itself that is a ring multiplication.

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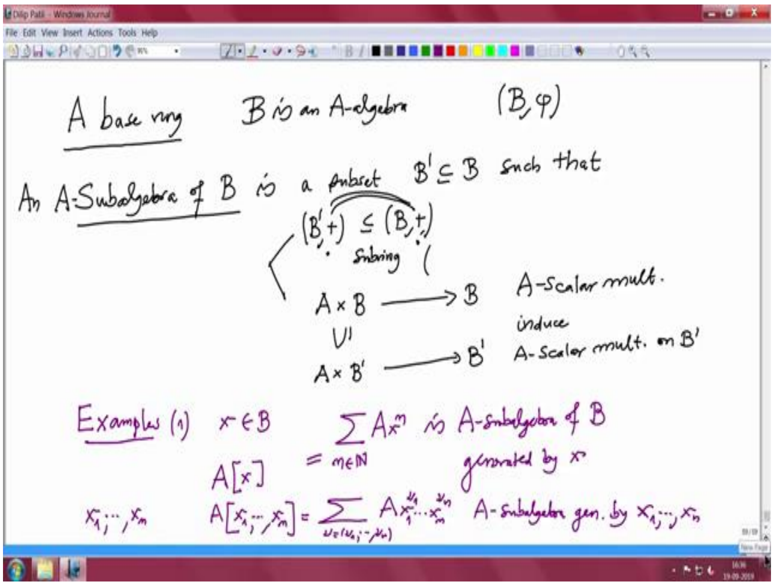

So, with this so A is a let us say A is a base ring whenever we have a difficulty or the question are not easily answerable, one should specialize this ring to a field and then ask what is, what is the answer and then you can go on in the general setup. A is a base ring and B is an A algebra strictly speaking one should write B, comma phi but I will drop this in the notation because it is understood there is only 1 ring homomorphism where we will make there are many ring homomorphism but each one of them will make a different algebraic structure.

So, when you say B is an algebra that means your algebra structure is fixed that, means this is fixed. So, therefore, we will not keep writing it in every time. Now, when do you say B algebra what do you mean by sub algebra first, let us say sub algebra, like we have defined in case of ring we have defined ideals, in case of modules we have defined sub modules, quotient modules, quotient modules I have not defined but I will come to it, we will define it.

So, similarly, we will define now for sub algebras. So, sub algebras. So, an A sub algebra of B what is that is a sub set B prime of B such that so whatever operations we have when you restrict those operations to this subset B prime first of all they should restrict and with those operations it should become an A algebra. So, what does that mean? That means first of all B prime plus this is a sub group of B plus.

So, in this it is the same plus, not only that the dot also, this should be a sub ring, remember we will have identity in the sub ring many books they do not insist that the subrings should have the identity element also, identity with respect to the multiplication. So, it is a subring and also the scalar multiplication of A should restrict to the scalar multiplication on B prime. So, B prime so B scalar multiplication.

So, A cross B to B we have this scalar multiplication, this is A scalar multiplication, scalar multiplication that is the same one should restrict to B prime. So, you see here this is a subset here and that map the image this way should go inside B prime. So, this will induce A scalar multiplication on B prime then you call this B prime is a sub algebra. For example, how do you find examples of sub algebras? Again you do the same stuff, same funda.

So, examples, if you take for example, if you take any element X in B. Now, if I just take this, this is an A sub module, but this may not be closed under multiplication. So, what do you have to do is, you have to take powers of X and generate a sub module, so that means we have to take the sum X power n, n in N. Now, it is closed under multiplication I made it close under multiplication.

So, this is A sub algebra of B and we will say it in generated by x. So, what are the elements here? You see elements are finite sums, finite A linear combinations from 1 X, X square and powers of X but and the elements are not unique remember two finite sums may be equal without the corresponding quotient being equal because this X is not a variable, it is not they are polynomials in X but the expressions are not unique.

So, this is also actually strictly speaking it should be denoted by A square bracket X. So, this is a sub algebra generated by X. So, this just this notation is not enough. So, this is and you know specialty about 1X. So, that is where I want to assume the algebras are commutative if I have finitely many elements X1 to Xn then I should write this notation A X1 Xn. Now, what is this? These are A linear combinations among the monomials in X1 to Xn.

So, this is same thing as summation, now this summation is running over the tuples nu equal to nu 1 to nu n and coefficients in A. So, I will write A and here I will write X1 nu 1 Xn nu n. So, this is clearly an algebra and this is called A sub algebra generated by X1 to Xn and again it is, we can always take not only finite family but arbitrary family and we can consider A sub algebra generated by that arbitrary family.

So, this is when do you call a algebra to be finitely generated, when there are finitely many elements such that this sub algebra generated by X1 to Xn equal to B.

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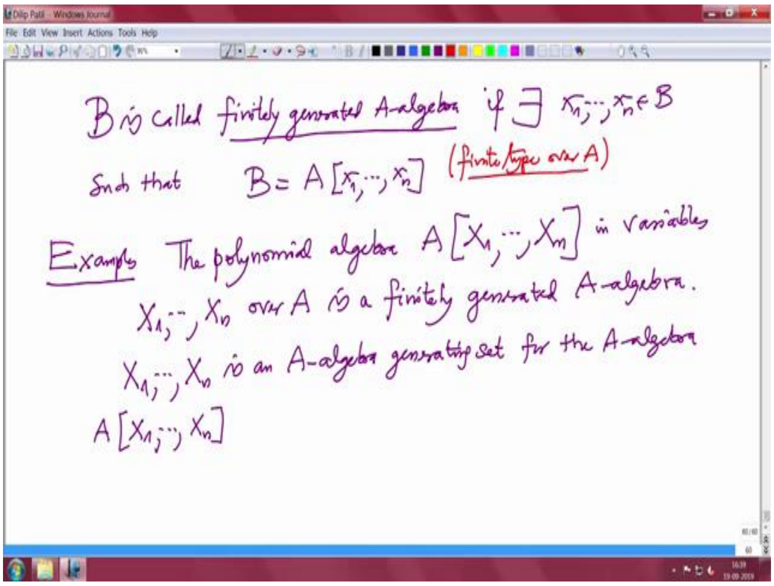

So, B is called finitely generated A algebra if there exist finitely many elements X1 to Xn in B such that B equal to A X1 to Xn. Some examples, though the most prominent example is this if I take the polynomial algebra, the polynomial algebra A capital X1, Xn in variables X1 to Xn over A is finitely generated A algebra. More over this X1 to Xn is an A algebra generating set for the A algebra A X1 to Xn.

Also this finitely generated algebra also it is also called finite type over A that is also used in the definition. So, I think with this I will stop and when we come next time I will give more examples and then eventually we will get back to our study of algebraic sets and the geometry. So, I will keep shunting between algebra and geometry now and then to recall the concepts of algebra and use them in a geometry language. So, that will always be our strategy in our throughout this course. So, we will continue in the next lecture. Thank you.