Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 59
Lying Over Theorem

Welcome to this course on Algebraic Geometry and Commutative Algebra. In the last couple of lectures we have been studying integral extensions, and yesterday we were trying to prove the theorems of Cohen Seidenberg. Let me recall briefly what we are trying to do.

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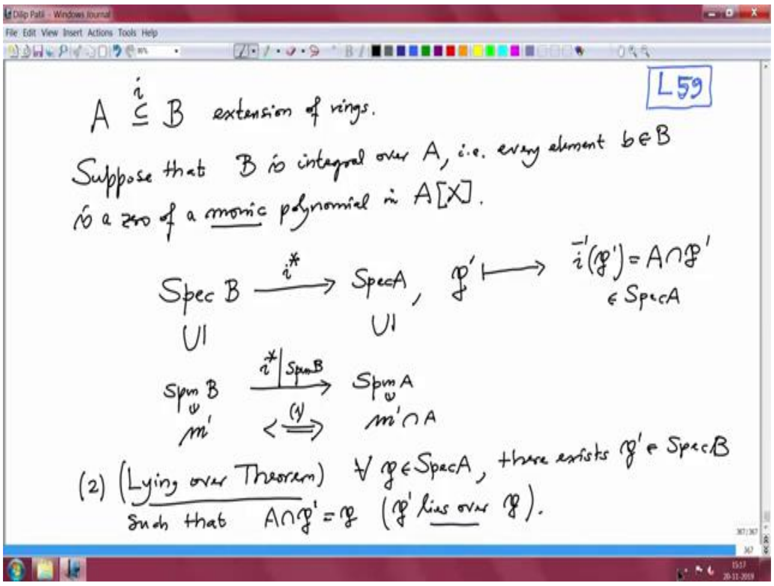

So, we have a ring extension A contained in B, this is the extension of rings. And we are assuming that B is integral over A that means that is every element b in B is a 0 have a monic polynomial in A X, coefficients are in A and the polynomial is monic and B is the 0 of that polynomial then we say B is integral over A and capital B is integral over A means every element of capital B is integral over A.

And in this case, we are trying to study so, this extension will induce a map on the spectrum level, spec B to spec A this map is so this is, if you call this inclusion as i, this is i star, this map is any primordial P prime in B that is map 2, just pull back that i inverse of P prime. This we have denoted A intersection P prime and we have checked that this is a prime ideal deal in A.

And Lying over theorem says that this map is Surjective that is given any prime ideal deal in A whether it is coming from some prime ideal deal in B that is what we wanted to prove. And yesterday what we proved was, if you take a look at the maximal ideals Spm A, this is the set of maximal ideals in A and then Spm B which is contained here, what we check that is if some m prime then m prime intersection A, this is indeed maximal ideal and if this is maximally if and only if this is maximal we have proved. So, in any case, they these i star will induce a map here, this is a restriction of i star to the maximal spectrum of B.

This we proved yesterday, this was precisely so this is belongs here, if and only if this is what we prove, this was what I called it statement one. And that followed from the fact that if you have extension of integral domains and if it is integral, then A is the field if only if B is the field from that this followed easily.

And now, today I want to prove that the second statement so this is what we want to prove, this is also called Lying over theorem that says that for every P in the spectrum of A, there exists P prime in the spectrum of B such that P prime lies over P that means, if I contract it to A, I get P. Then in this case we say that P prime lies over P, this is equivalent to saying so, I will write down the next.

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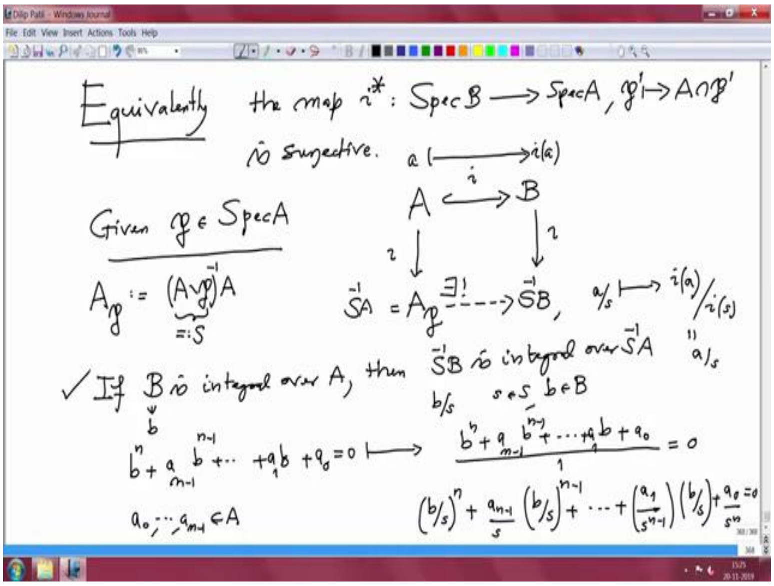

Equivalently, the map i star from Spec B to spec A, P prime going to A intersection P prime is surjective, this is what we want to prove. Before I prove that, I want to discuss how many, not how many, but what is the relation between the elements given P. So, given P in spec of A, I want to know if there a containment relation between the elements in the fibre over i star so that I am making a statement.

So, before I prove that also I want to set up some notation. So, given any P in spec A, we have these A to B this is the inclusion map I, and then I localize this A at P, A localized at P. So, remember a localized at P is by definition, take the complement of P, A minus P and take the inverse of that in S inverse of this, this is my set S.

This is multiplicatively close set because P is a prime ideal and we look at this ring and then B is the module over A, B is an actually algebra over A but in particular module over A. So, I can also localize that B at the multiplicatively close set this S so this S inverse of B. This is S inverse of A and this is the natural Iota map and this is also Iota map.

This S is multiplicatively close in B because it is multiplicatively close in A and the universal property of the localization will tell us that there is a unique ring homomorphism which make the diagram commutative. So, there exists unique in fact, we can tell what is the definition of this.

So, what we do, any A by S where should it go? So, that dictates you, A by 1 if it was S is 1, then A by 1 goes to where? It should come this way. So, A goes to A under inclusion and then that will go to A by 1 only. So, this is going to, if you like iota of a, I inclusion, this is A hear I put to hear and then map it here. So, this is this but because we are identifying these a with the sub ring of b, this is same as a bias so this is the natural map. And this diagram is commutative that because of the way we are defined and also it is a ring homomorphism that is also clear by definition.

So now, the observation I am seeing is, if B is integral over A, then s inverse of B is integral over A integral over S inverse A. This is very simple because you take the we have to show that every element here satisfy a monic equation over S inverse A, but take that element and it is of the form b by s where is in S and b is in B, I am checking that every element here is integral over S inverse A, but now look at that B.

This B is integral over A means there exists monic polynomial so, it will be like B power n plus a power n minus 1, b power n minus 1, etc plus a 1 b plus a 0 is 0 and with the a 0 coefficients a 0 to a n minus 1 they are elements in a. Now, this is an element in the ring B, this is 0 element, but now I want to read this in S inverse B so it will continue to be 0. So, that means take its image in S inverse B it will be b power n plus a n minus 1 b power n minus 1 etc plus plus plus plus b a 1 b plus a 0 divided by 1, this is also 0.

But if this is 0, then I can further multiply up and down by S that will also be 0. Not only that, I want to write the denominator S power n and I have absolve it here. So, this equation is same thing as up and down multiplying and adjusting the thing, this is b by s power n plus a n minus 1. Now, you keep it here, whatever you need to and remaining part you can absolve with that coefficient a, so this is 1.

So, I will write one term at least here, so b power n minus so, b by s power n minus 1. So, I accounted for b power n minus 1, I accounted for this and this is S power n minus 1. So, I have to still so that S I will absolve it here, plus so on so, here it is a 1 by s power n minus 1 b by s plus a 0 by s power n and this will continue to be 0 also because I have not done anything, I have just adjusted s, but from here I multiplied up and down by s power n and then did this. So, this is a monic equation, coefficients are in s inverse A and it is an integral equation for b by s so, we have proved this statement.

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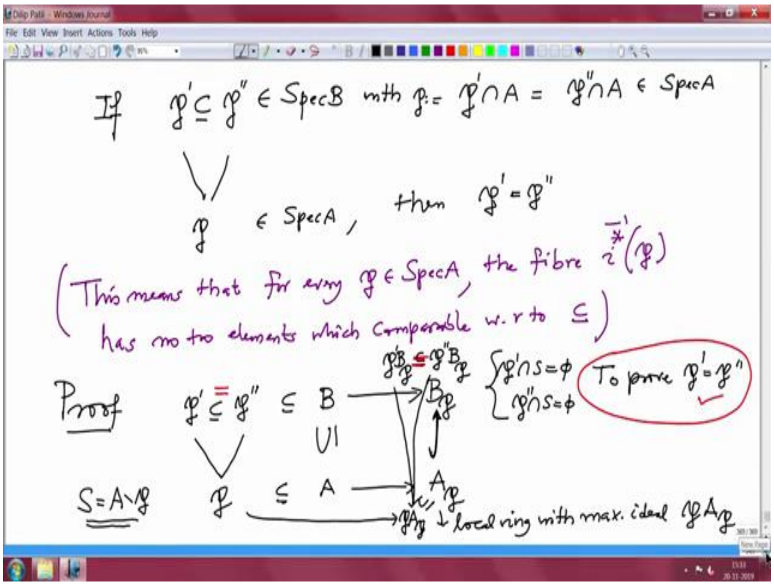

Now, let us come back to the proving that what I want to prove the following statement. So, if P prime and P double prime they are two elements in the spectrum of B with suppose they lie over the same P that means P prime intersection A and P prime intersection A, they are two prime ideal deals in spec A because are contraction of the two prime ideal ideals.

And suppose they are equal that means, both these prime ideals lies over the same prime ideal P, let us call this as P. P is by definition this, if they lie over the same P usually such a diagram is drawn, then they are equal, P prime equal to P double prime. So, this simply means so I will write in the bracket what does this mean?

So, this means that for every P in Spec the fibre that is i star inverse of P there are no containment relation fibre has no two elements which are comparable with respect to (())(16:09) that is what it means, proof is very easy as you will see. So, proof of the statement, so I want to prove that if they lie over the same so now, it is better to draw a diagram I think, so B is here, A is here and there are two prime ideals, P prime and P double prime.

They are prime ideals here and they are lying over the same one P, and I want to prove that they are equal, and containment I forgot. So, if these are contained so here also you are contained then what we want to prove? To prove P prime equal to P double prime, this is what we want to prove, this is contained here.

Now, let us see so what do I do? I want to localize this. So, let us take a s equal to A minus p and localize this. So, from here you pass on to the localized ring A localized at P, from here you pass on to the ring B localized at P. Remember this may not be local because P you have only localized this B at S of A minus P and this is not an prime ideal in B, this was it here.

So, this ring is local ring with maximal ideal P which is the image of P. So, here the ideal P this will go to the P A P. This is this is the unique maximal ideal of this local ring, this is what we have done it in the localization. Now, here we have this map, now what happened here now, when you pass on these prime ideals with this localization first of all they will be a proper ideals here because they do not intersect with this S because they are lying over P means that P prime intersection S is empty, and P double prime intersection is also empty.

So, this means, both these will be a prime ideals here, they will be proper prime ideals here so, and the containment will remain containment. So, when I take the images of these in these B localized at P, it will be usually we know it is denoted by P prime B P and P double prime B P and this is contained in this.

And now these are two prime ideals, because they were two prime ideal ideals. These are prime ideals in B therefore, the image under S inverse also prime ideal and containment happens and where are they lying? So, they are lying over this maximal ideal, just transfer the data to the localization, but now, this is a local ring, this is a prime ideal, this is a maximal ideal and over this there are two of them which one contained in the other they are lying over, but our early statement say that if you have a maximal ideal, then only maximal ideal can lie over the maximal ideal.

Therefore, both of them cannot lie over the maximal ideals. So, both of them are equal here, so that shows these are equal here, but then when we pull it back they will be equal here and that is what prove this statement. So, this we have proved statements under localization what happened to the prime ideal.

We will go to a prime ideal under localization when it does not intersect with the multiplicatively close set that we are considering and at one to one correspondence between the prime ideals who do not intersect with the multiplicatively close set S and the prime ideal, so, that proves that in the fibre of i star P, there cannot be any containment relation between any two prime ideals in the fibre. This is very useful fact, this I will use it to prove the lying over theorem.

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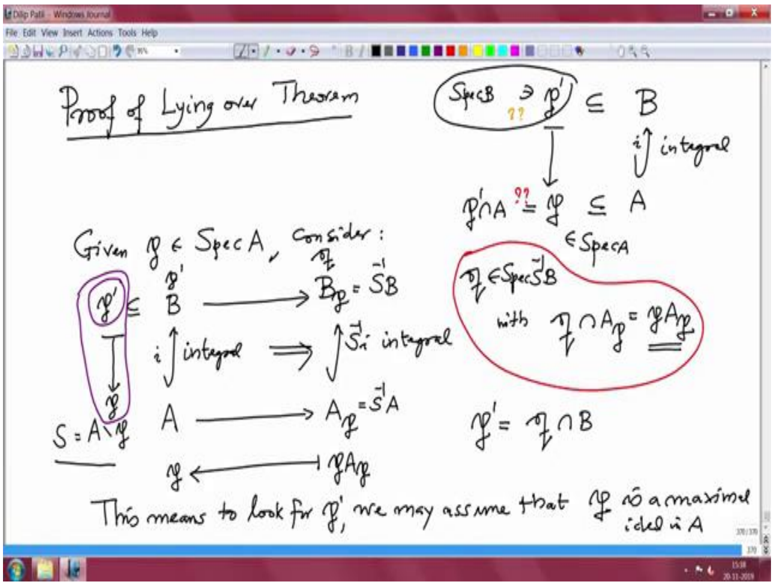

Now, proof of Lying over theorem. So, we have given a situation like this, B contained in A, this is integral, this is our inclusion map integral and P is given to be prime ideal here, P is in this Spec A and we are looking for somebody here P prime, and P prime should be in the prime ideal, this is what we are looking for and also it should lie over this.

That mean this P should be P prime intersection A, this is what we are looking for. So, now what do I do? Again I do the same trick, we have given P. So, given P in spec A, consider the diagram so, A is here, B is here, this is integral then you go to A localized at P and go to B localized at P, where P is A minus P not P S, this is S inverse B and this is S inverse B and this is the S inverse of this inclusion map, this is inclusion map.

And we know this is integral and then we have checked that whenever I have integral homomorphism, then S inverse of that is also integral so this is also integral. And what are we looking for? We are looking for P prime here which lies over P this is what we are looking for.

We are looking for this, which is lying over this, but now I want to push this data to this localization. So, that means what? I am looking for some prime ideal deal here which should not intersect with S and that should lie over this P A P. So, therefore looking for Q, which is a prime ideal in S inverse B with Q intersection this ring A localized at P should be P A P.

Once you succeed in this, this is what we are looking for. Suppose, you succeed in this, then what will you do? Then we will take P prime to be equal to Q, you pull it back to P that is Q intersection B and this will be the required one because that Q was here, you pull it back to P prime and that one is equivalent to saying you pull it back here and then pull it back here.

So, this Q is pulled back to P A P and this one obviously when you pull it back, you will get back your P so, this one P prime and S 2 to go to p so that will finish the problem. And what did we achieve in this reduction? The achievement is this ideal here is not a prime ideal, but it is actually a maximal ideal. So, this proves this means, to solve the problem to look for P prime we may assume that the given P is a maximal ideal that is the achievement. So, now let us start once again with the assumption that P is actually the maximal ideal.

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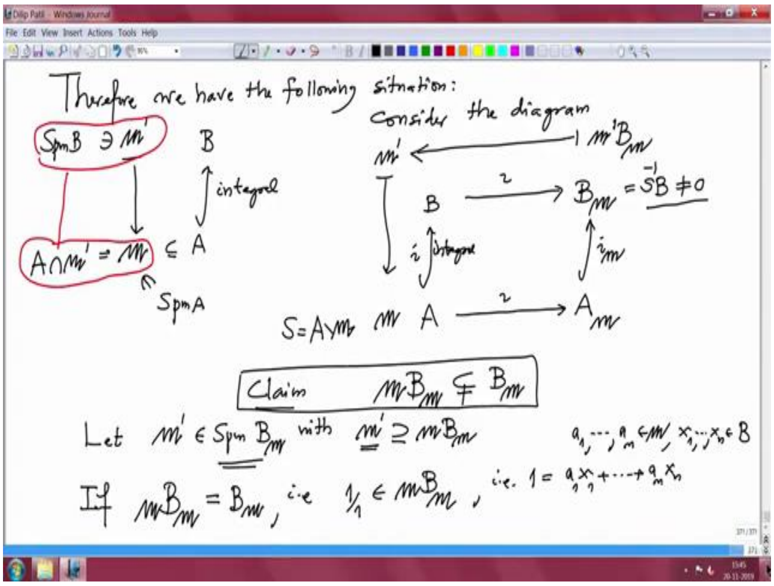

So, therefore we have the following situation, B is integral over A, and m is a maximal ideal in A and we are looking for m prime because we know if at all there is a prime ideal it has to be the maximal. So, in Spm B such that this is lying over this that is this one is A intersection m prime so we are looking for this with this condition. So, what do we do? So again, I am going to localize again at m.

So, consider the diagram so A is here, A localized at m is here. This is a localization map so S is now A minus M, this was B here, this is here, this is integral, this is B localized at M, this is the Iota map this is also iota map and this is the inclusion map i this is i localization of this I suffix m, we have this situation and what are we looking for?

We are looking for now, somebody here some maximal ideal here which lies over this maximum ideal. So, we are looking for that. So, first of all note that this ring B m is S inverse of B and this has to be nonzero ring because if it is 0 this will also be 0, but this is not a 0 ring. So, this is a nonzero ring and now you extend this maximal ideal there.

So, and m extended to this if this is a proper ideal, then I will take any maximal ideal which contain this and then that I will pull it back this and that will be the maximal ideal here, which will lie over this. Actually we only need a prime ideal so, we do not even that is a consequence. So, what we do is, first of all, so we claim that this is a proper ideal.

Assuming the claim what are we going to do? Now, let m prime, this is a proper ideal so, let m prime be a maximal ideal in B localized at m with m prime contains m localized M. So, if this is a proper ideal I can always find a bigger ideal which is a maximal ideal in B M, so that this contains this. And now this m prime has to so here I have chosen my m prime, this m prime when you contract it here.

This is the way I have written it is this will contract to m prime and then this has to contract to m only because this rate is contracting too. So, I only have to prove this claim, if it is not equal, if it is equal then what happens? So, if m times B m equal to B m that means, that is one will belong to the other side one which is actually one by one we should write these belong to m times b m.

That means what? That means one I can write it as so, that is one I will write it as some a 1 x 1 plus, plus, plus, plus plus a n x n where a 1 to a n, they are in a and x 1 to x n are in this m and x 1 to x n they are in the ring B. So, and these are the generators, so these are in fact, they are in m because this ideally generated by elements of m. So, therefore, I can write that in a finite combination of the generating set. So, these are the some elements in m and the combination is 1, but what does this mean what will these contradict?

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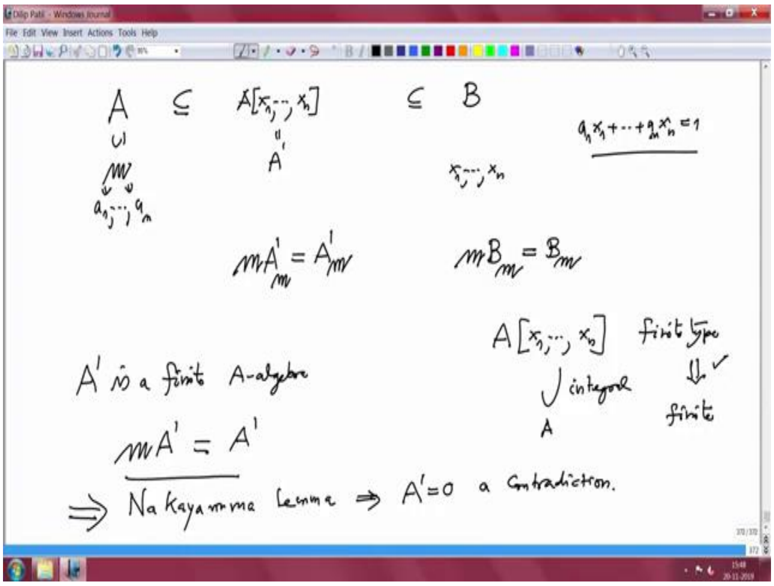

So, come back here we have this situation, a is here, m was given here, B is here and I found elements here a 1 to a n and x 1 to x n we found such that now I take the sub algebra of B generated by A prime. Now, this m which was already m B localized at m this was unit ideal, and that is because this equation but that equation is also valid here.

So, when I localize here that means m extended to A prime m prime m prime m equal to also A prime m, same equation, because that makes sense. So, what is achievement again that means, we may assume and this is integral that we know.

So, we have proved that this one is also integral extension, this one over A is integral and it is finite type. Therefore, finite this we proved yesterday's lecture so this is a finite type finite algebra A prime is a finite algebra. So, that means it is a finitely generated module and then it satisfies this property.

It already satisfies m times A prime equal to A prime because this equation is there, but then Nakayamma says, will imply A prime is 0 actually, a contradicts. So, therefore what we proved is this one, this one cannot be equal and therefore, I can chose a maximal ideal and then earlier proof goes on, so this proves that the map is surjective.

Lying over theorem we have proved and after the break we will see some more the third theorem of Cohen Seidenberg that is also very important that is called a Going up theorem. Thank you very much. We will meet after the break.