Introduction Algebraic Geometry and Commutative Algebra
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Lecture 56
Properties and examples of Integral Extension

In the last lecture we have saw, we have seen a definitions and some basic properties of the integral extensions. Today first I will, we will see some examples and then we will continue the connections with the prime spectrums of a extensions of rings. Let us see some examples.

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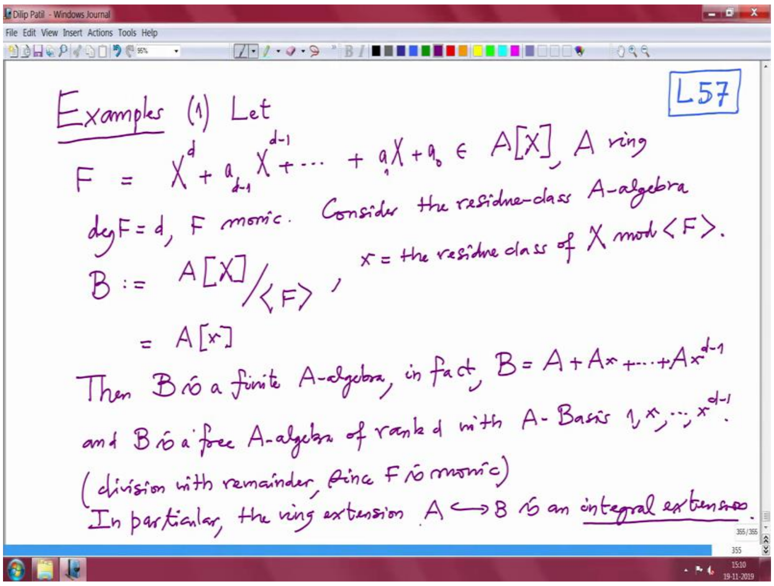

So, 1 let FX be a monic polynomial of degree d with coefficients in a ring A. This is a monic polynomial with coefficients in a ring A and as usual A is our base ring commutative. I will not write. I will just say A is a ring. So, this is the monic polynomial of degree d. A degree F is d and F is Monic and then considers residue class plus A-algebra.

Polynomial ring B, let us call it B which is polynomial ring modulo the ideal generated by F and I will denote X to be the residue class of X modulo the ideal generated by F that is the notation. So, this is also equal to A X. So, it is obvious that this is finite type algebra but we will prove more. So, then B is a finite A-algebra, in fact B is generated as a module by 1X etc etc up to A power d minus 1 and in fact B is free A-algebra of rank d with A- Basis 1, X, etc up to X power d minus 1.

All these things one can check easily by using division with remainders. So, I will just write here division with remainder. That is possible for arbitrary base ring A provided the polynomial was monic. So, this we can use it since F is monic. So, in particular the ring extension from A to there is inclusion up to B, this is via polynomial algebra, so this one is an integral extension. When we say ring extension is integration extension that means the every element of B is integral over A. That is it satisfies the polynomial over A, monic polynomial over A.

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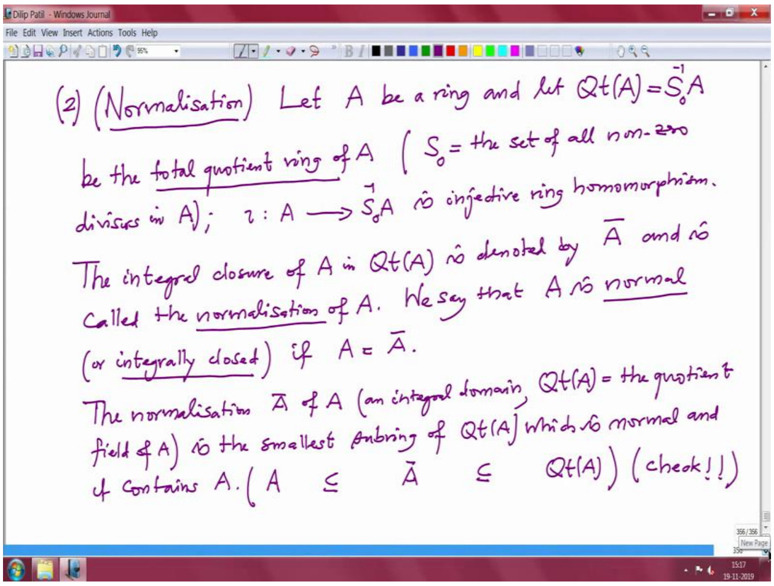

So, second one, this is very important concept of Normalisation and this is more in geometric in nature but to see that we will have to work it more with the language if in this course I think we will not have enough time to go on to this geometry part, but this is done in my existing course on commutative algebra on which is already on NPTEL 2019 January.

Let A be a ring, base ring and let Qt A this is S not inverse A be the total quotient ring of A, that means so this is not you take maximum possible multiplicatively close set so that this ring does not become 0. Take S not be equal to the set of all non-zero divisors in the ring A and you invert them the localisation.

So, therefore we have this iota map from A to S not inverse A this is an injective ring homomorphism. This we have done this while we were studying localisation and so on. So, the integral closure, in the last lecture we saw the integral closure of a base ring in a bigger ring is the set of all integral elements so the base ring that is called the integral closure of A in I will keep using the notation Qt A is denoted by A bar and is called the normalisation of A.

We say that A is normal or sometimes also people use the world integrally closed and when you do not write where that means it is understood that it is in the total quotient ring of A, integral closed if A equal to A bar. Then you called it integrally closed, One more comment I wanted to make, if A is a domain then one call it a normal domain that is not the comment I want to make.

I want to make more serious comment. The normalization, that is this A bar, the normalisation A bar of A, and I assume here A is an integral domain, if A is an integral domain this Qt of A is the total quotient field, in fact the quotient field of A is the smallest subring of Qt A which is normal and it contains A.

So, look here A is here, Qt A is here, this is the quotient field of this ring A and A bar is the integral closure of A in Qt A, obviously A is containing A bar and this is by definition is containing Qt A. There is no other subring here which is normal and contains A. That is what this statement means and I want you to check this, this is not so difficult to, one should check this. So, that means take a sub ring, which contains A, then if it is normal and contains A then it should it contains this A bar. That is what one needs to prove.

I should give you some examples of normal rings or normal domains. Normal study of normal domains is very, very important as far as the algebraic number theory goes and algebraic geometry goes these rings play very, very important role in both the subjects. So, I should give some more examples.

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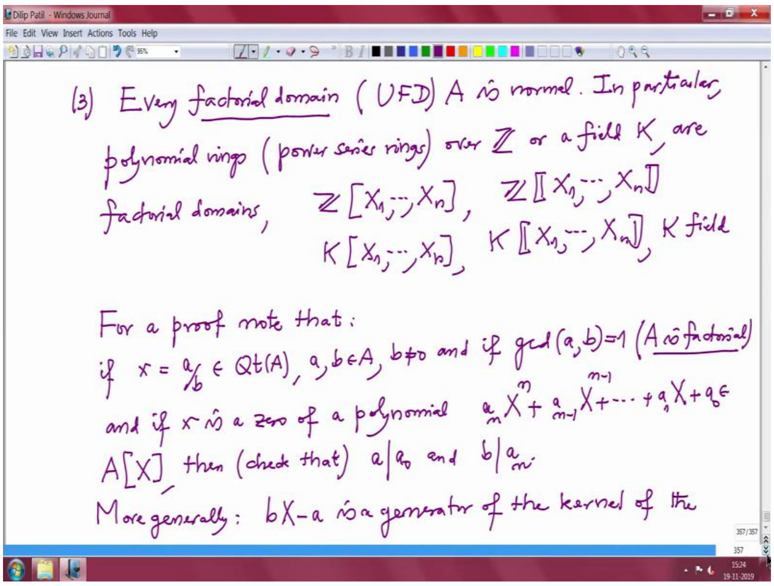

So third, the statement is every, this is a class of examples which are normal domains. Every factorial domain, let me remind you, when I say factorial domain that means UFD A is normal. When I say normal that means it is in the quotient field of that A and a factorial means unique factorization domain.

In particular we have lots of examples of unique factorization domains in particular polynomial rings also the power series rings, over the ring of integers or a field K, are factorial domains. In the notation, so Z polynomial in several variables or power series ring over Z or if you have a field then this K X1 to Xn and power series ring over the field in several variables, where K is a field, these are all factorial domains and therefore they are normal. That means if we prove this statement then they are normal.

So, we have lots of examples. Let me just remind you this four integrals, it was proved by Gauze of course the power series case in one variable is easy but power series case in more variables is more difficult than the polynomial case, more variables. So, for the proof of the fact that factorial domains are normal, let me give you a proof.

For a proof note that it is very easy, if X belongs to the quotient field of A that is a by b in Qt A a, b are elements in a, b non-zero and if gcd of a and b is 1.So, that is where you are using the fact that A is factorial. This is just the reminder, we are used the fact that A is factorial. If A is not a factorial, then gcd does not make sense in arbitrary domain. Therefore this factorial is very important, gcd is 1.

And suppose this X is integral over A that means X satisfy an integral equation which is a monic equation. But and if X is a 0 of a polynomial, I am writing the general fact, so that these results will follow from the general fact. If you have a polynomial like this with coefficients in A and suppose this X is a 0 of this polynomial, this is in A X, then it is very easy to check them.

And say check that a must divide the constant term a0 and b must divide the leading coefficient. If you check this then what happened? The X cannot be indignant because if X is integral then it will be a 0 of a monic polynomial and then their denominator b will divide 1, but he would divides one means it will be a unit and that means this X will be in a. So, that is how the proof is completed. This is a more general proof.

Even more general statement than that I will state it, but I leave it for you to check more generally. And this statement, if X is a 0 of this polynomial then the numerator should divide the constant term and denominator should divide an. This fact will use this to prove this, we will again need that A is factorial. This is what we have to remember. And this we have done, I remember we have done in the school days for integers, this was a school result. So, more generally, more general statement than this what I stated here is the following.

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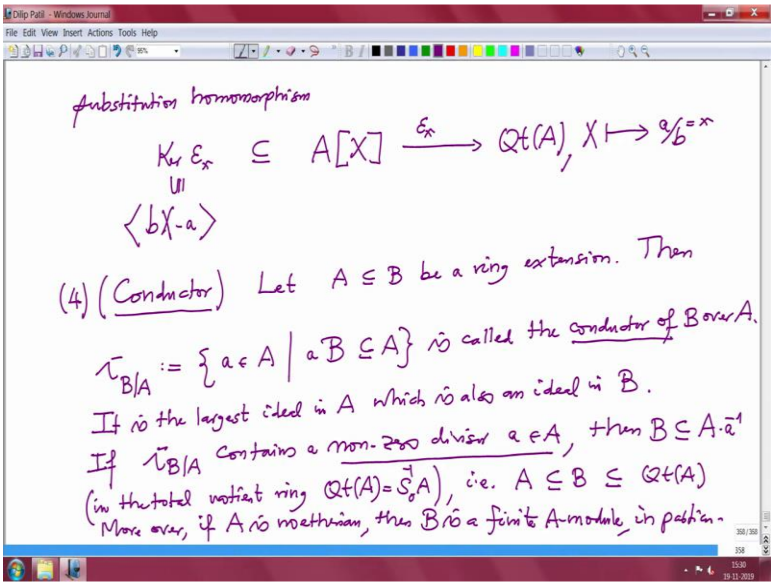

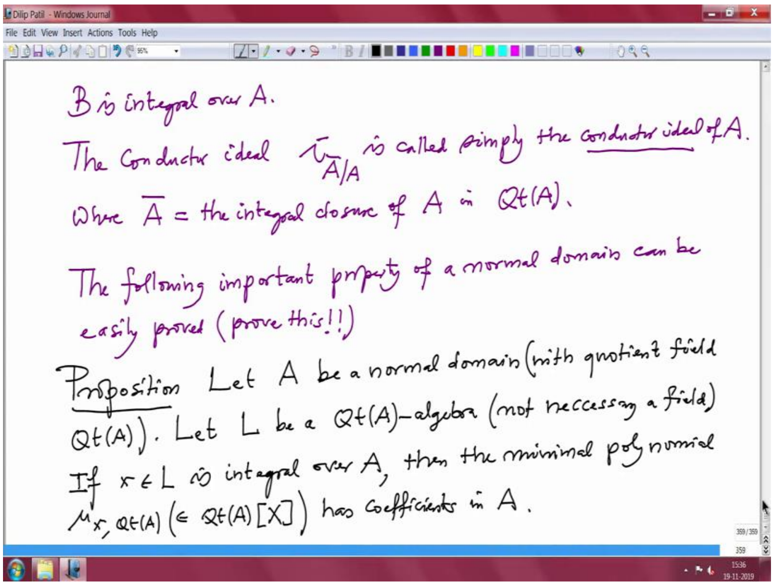

The polynomial bX minus a is a generator of the kernel of the substitution homomorphism from AX to Qt A, quotient field of A, and this is X is mapped on to a by b. This is our small x and this is ex. What is the kernel? What we are saying is the kernel of ex, this is an ideal here and this kernel is generated by the polynomial bX minus a, it is the principle ideal.

So, obviously this generator you see it goes to 0 because X is going to small x which is a by b and you cancel it. It is indeed a element in the kernel, but you have to check that all elements in the kernel are preciously the multiples of these polynomial, AX multiples of this polynomial. And once you check that then, if you remember this kernel will never contain any monic polynomial unless b is the unit.

Because any element, any polynomial in the kernel of ex will be multiple of bX. So, leading coefficient will always be divisible by b. So, it cannot have, cannot contain monic polynomial unless b is a unit. But in that case this X will be in the ring A. So, this is a about the normal factorial, factorial domains are normal. We have many examples.

Now, the next one is the concept of conductors. This is also very useful concept, but again I cannot do more on these things because our course is a nearly coming to end. So, let A contain in B, be a ring extension. That means A is a subring of B and both rings are commutative that we are assuming. Then in this case these ideal is very interesting in variant.

So psi B over A, this is the notation. Psi is for the conductor and this B over A is for the extension. This is by definition, all those elements a in A such that a times B is contained in A. This is an ideal in A is called the conductor of B over A. It is the largest ideal in A which is also an ideal in B. Normally the ideals in a subring they may not be ideals in a bigger ring, but in this case this, particular ideal is also largest ideal in A which is also an ideal in B.

So, moreover see how it will be used, will be clear from the next few of my comments. If this phi conductor ideal contains a non-zero divisor a in A then these B will be contained in A times a inverse. Because by definition if this A is contained in this conductor ideal then A times B is contained in A therefore when I multiply by a inverse on this side, and where are we working when we multiply by a inverse?

Obviously this we are writing in the total quotient ring Qt of A which is we know this is S not inverse of A, where S naught is a set of non-zero devisors in A. Therefore this A is in S not and therefore this makes sense in the total quotient ring.

So, moreover that means you are imbedded this ring in A. So, that is A is here containing B and this B we have imbedded in Qt A. That is where we have used the fact that A is a non-zero devisor and moreover it is finite over A, so it means B, if A is noetherian, then B is a finite A module, in particular B is integral over A.

So, just I want to show you just what did we conclude? We started with an arbitrary ring extinction and we concluded that it is integral but with the assumption that this conductor ideal should contain a non-zero devisor. So, that is very important. So, when we like normalization, when we, the ideal, when I take B equal to the ideal, the conductor ideal, psi A bar over A is called simply the conductor ideal of A. Where this A bar is the integral closure of A in Qt A, total quotient ring of A.

So, now how do we test? So, some important and important property of the normal domain I want to write which is very, very important property and use quite often. So, this is what I will state, maybe I will leave the proof to the as an exercise. So, I will just mention the following important property of a normal domain can be easily proved. Prove this.

So, what is the property that I want to write as a proposition? This is also very useful. This proposition is also very useful in the especially algebraic number theory. So, let A be a normal domain, remember normal domain means it is a domain and integrally closed in its quotient field, with quotient field Qt A.

If X is an element in Qt A. Not element in Qt A. So, if so I want to further write something so let L be a Qt A algebra, that means it is a algebra over this, this may not be a field, not necessarily a field. I will indicate here what will be the normal situation where one can apply this. So, if an element X in L is integral over A, then the minimal polynomial mu X, Qt A, this is the monic polynomial in coefficients in Qt A, where X is a 0 is the degree least polynomial.

Then after all this is the polynomial in Qt A, then the minimal polynomial has coefficients in A. Already it has coefficients in A. Now, I will just indicate why this propulsion is very important. Already prove I have left you to check. This is not so difficult, but where will it be used that I will indicate. So, just 2 minutes and then we will make a break.

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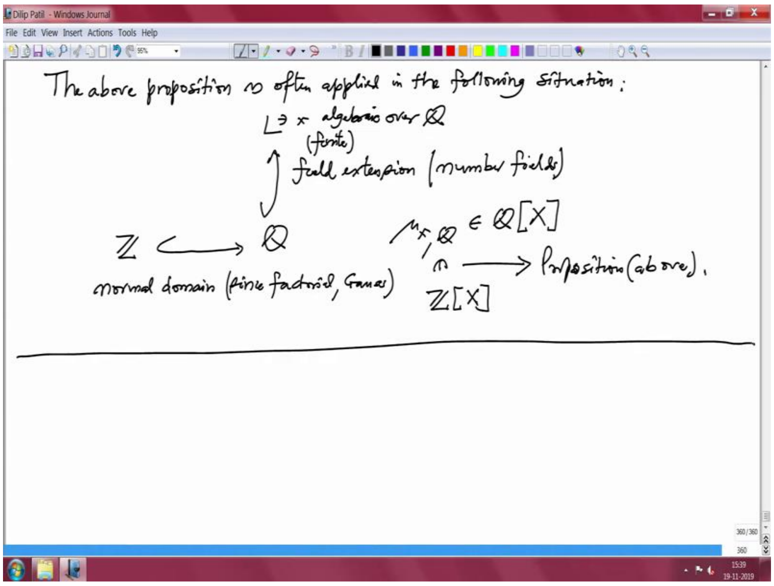

So, here is the ring of the integers. So, the above preposition is often applied in the following situation. So, what did the situation? So, Z is a ring of integers here. We know the quotient field of rational functions and we know this Z is normal. They are normal domain that we know because it is factorial, since factorial, this was theorem of Gauss and then when you take an field extension of, any field extension, even finite field extension such fields such called number fields and we have an element X here.

Suppose this element X is algebraic over, this is a particularly in the finite field extension case, so suppose these elements was algebraic over Q. In particular that will happen for every element when L is the finite extension of Q. Then it will have a minimal polynomial mu XQ. This is a polynomial, monic polynomial with coefficients in Q X.

The above theorem says this polynomial is actually indeed belongs to integral coefficients. This is by above proposition. This is very useful fact and the proof is very simple and proposition is also stated in more general case where L may not be field, L is simply the algebra over Q and algebraic elements makes sense.

So, an algebra over a field, so this is more general situation. So, I will leave it for you to write the proof for the proposition. Probably I will also add in exercises with some hints and with this I will stop this first half of today's lecture and we will continue after a break. Thank you very much.