Introduction to Algebraic Geometry and Commutative Algebra
Professor Dillip. P. Patil.
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 54
Example to support the term "Spectrum"

So welcome back to this second half of today's lecture. We have seen some properties of the spectrum and now I want to give some examples how do we and also the why did we use the term spectrum or why did people choose to use the term spectrum. Spectrum was already there in because spectrum in physics was studied long back and it was there.

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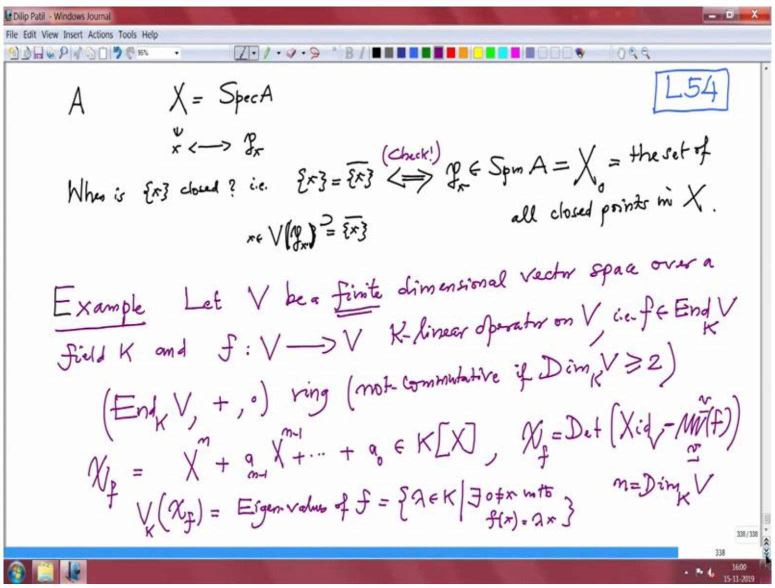

So first of all one more I forgot, so if I have commutative ring A and if I have this x is a spectrum. So if you take any x in x that corresponds to px the prime ideal, so when is it close point close point means so when is single term x close that is when is x equal to x closer, answer is very nice this is if and only if px is a maximal ideal. That is very easy because suppose it is close, then what is vpx we saw in earlier part this vpx is precisely the closer of this single term x. That is very clear because this is close and this contains x and this containment is clear because this is a close set and this is smallest close set.

But on the other hand this one contains x, therefore so equality here, when will the equality happen, sorry when will the equality will happen when there is no more point, there is no more point then x that means there is px has to be maximal ideal because if px is not maximal px will be contained in some maximal ideal and this vpx will have at least two points. So therefore, this is clear, I would still insist that you check here that it is correct. Closer of, so we have to check this closer of x is precisely vpx that we have noted in the earlier discussion also, so when it is a close point the pi has to be maximal ideal, otherwise this will have more points. Alright, so therefore if you remember I denoted this maximal spectrum by x0. This is precisely the set of all the set of all close points in topological space x.

So now, with example that I was what I was saying, example let us start with finite dimensional vector space over a field K. So V be a finite dimensional vector space over a field K and f be a linear operator on V K linear operator on V we will also write that is f belongs to endomorphism's of K vector space V.

This is ring this endomorphism is we can add two linear operators by adding point wise, and you can compose them and with that it becomes a ring. Unfortunately, this is almost never commutative, not commutative if dimension of V as a K vector space is at least two. But that does not matter for us and in function analysis or in physics also there is a study of linear operators and for linear operators we have attached finite set, a finite subset of the field K which called the spectrum which is actually called an ideal spectrum. This is finite dimensional if the vector phase is not finite dimensional, eigenvalue values make sense but Cartesian polynomial does not makes sense.

So let me recall the notation for the linear operator f, we have this characteristic polynomial of f. This is a monic polynomial and the dimension the degree of this monic polynomial the dimension of the vector space. So, this is x power n plus an minus 1 x power n minus 1 plus plus plus plus a0, this is a Cartesian polynomial coefficients are in K.

By definition, it is actually kai f is actually determinant of this operator x idv minus f. So you write down the matrix of f, so I am sloping the notation here should I have written matrix of f. Matrix of f with respect to some basis and this determinant will depend on the choice of that basis and you can see it is the monic polynomial of degree n, where n is the dimension of V.

Now what do the Cayley Hamilton theorem says? Cayley Hamilton theorem says that the zeros of the Cartesian polynomial in K are precisely these are the zeros this are precisely eigenvalues of the operator f. That mean this is precisely the set of all lambda, lambda in K such that there exist a non-zero x with f of x equal to lambda x, so the those are the eigenvalues.

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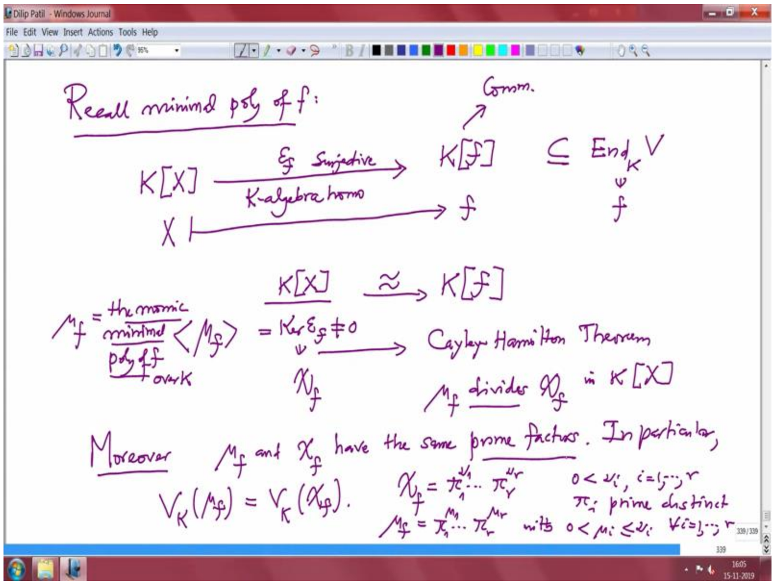

Now, so but on the other hand we have recall the minimal polynomial. What is the minimal polynomial of f? For that look at the evaluation map from the polynomial ring in one variable over K, This is evaluation map into the endomorphism ring of this, where this is K algebra homomorphism, so I only have to give values on K, so this x going to f, this is an algebra and this was an element there, so to define a map from K algebra morphism from the polynomial ring to this algebra I just have to give values on f.

This is K algebra homomorphism, and what is the image of this? Image of this is a sub algebra of this endomorphism algebra generated by f, so it is Kf. But now, this not commutative but this is commutative because f commutes itself this is commutative and this map is surjective image is this.

So in particular this Kf by isomorphism theorem this is isomorphic to Kx modular the cornel of this evaluation map. Note that, cornel of evaluation map is non-zero this is non zero because this cornel will always contain the characteristic polynomial kai f this is precisely the Cayley Hamilton theorem. So this cornel is a ideal non zero ideal in this polynomial algebra over a field, therefore it is generated by single it is PID therefore cornel is non zero so it is generated by a monic polynomial uniquely determine.

And that minimal that is called the minimal polynomial of the operator f. So, this is generated by mu f, where this mu f is called minimal demonic minimal polynomial of f over K. That is the definition of minimal polynomial, but this kai f belong there means what? That means, mu f divides kai f in Kx but this is not this steel crude relation what is the right relation is, they have moreover this is a non-trivial not very difficult but easy and interesting many application of this theorem what you say mu f and kai f have the same prime factors multiplicities may be different.

But if have a prime factor of mu then it is a prime factor of kai and if you have a prime factor of kai, then it is prime factor of mu. This is in particular they have the same zeros, so Vk mu f equal to Vk Ki f, and does mean by the same prime factor? That means, so I will write I will spell out this, that means if kai f equal to pi 1 power r1 or let me call it mu because I want to pi 1 power mu 1 pi r power mu r 0 less bigger than mu i, i is from 1 to r and pi 1 to pi i is our prime polynomials distinct. Then mu f has to be of the form pi power mu 1 pi r power mu r with 0 strictly less than mu i less equal to mu i or all i.

So, if pi i occurs here pi i also occurs here. But the multiplicity may be smaller the smaller because it is divide. So now what am I doing? What do I want to explain you?

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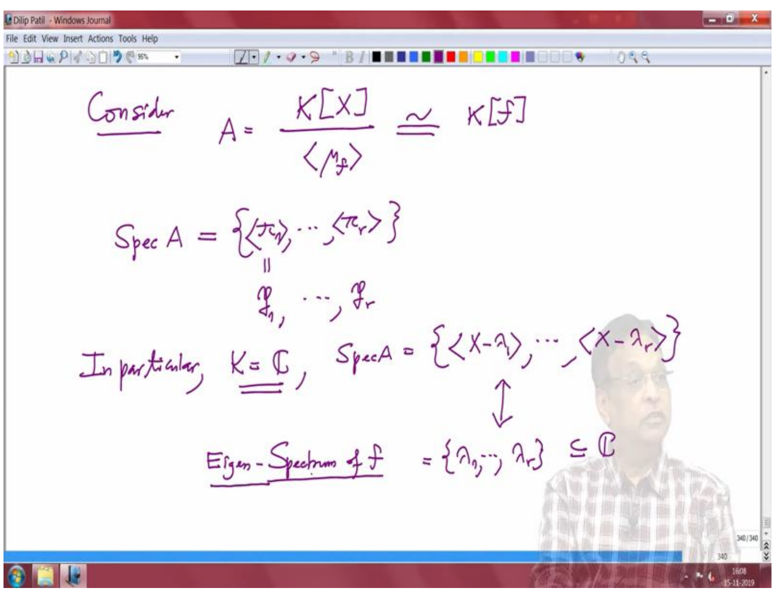

So what is our ring now? Our ring is, consider K X modular ideal generated by mu f, and this is we know they are the K algebra this is isomorphic to K f and if I call this as A, what is a spec of

A? Spec of A is precisely this pi 1 to pi r, I mean the ideal generated by pi 1 and 2 pi r, because they are prime polynomial these are the prime ideals and that is a spectrum.

And in particular when pi 1 is linear for example, that will give you Eigen values. So this is so in particular if you consider the field equal to C algebraically close, then this pi are linear polynomial because their unit is (())(15:25) polynomial, prime polynomial in CX they are linear. Then, Spec of A is precisely this on one our notation is X1 minus lambda 1 X naught X1 X, X minus lambda r these are r ones and this we are identifying this set we are identifying with lambda 1 to lambda r, which is a subset of C, the finite subset of C.

This is called spectrum. This is called spectrum of f spectrum. This is called Eigen spectrum to be very precise. This is, this term was use in physics and therefore, that matches with our spectrum. And that was a reason why people have started using this terminology. Alright, so some more examples.

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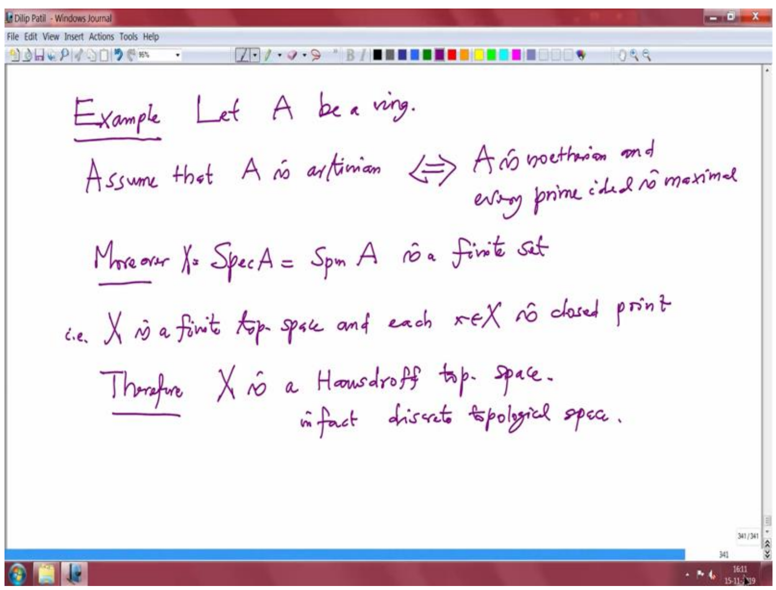

So next example, so example, so suppose let A be a ring commutative always. Let us assume it is artinian. Assume that A is artinian and we approved long back we approved a result we say that ring is commutative ring is artinian if and only if A is Noetherian and every prime ideal is maximal.

Not only that, moreover we have also proved, moreover this every prime ideal is maximal means spec of A equal to Spm of A and it is a finite set. That is what we have proved earlier, that is has finitely many maximal ideals and every prime ideal is maximal, therefore spectrum is also finite and that mean this is a finite set of point, so this X is a finite topological space that us X is a finite topological space.

And we have also known that each X in X is a close point because it is maximal ideal. We have noted maximal ideal are closed point. So that means single terms are close, but then therefore X is a Housdroff space topological space because finite therefore, finite each point is closed therefore each point is also open. And therefore it is actually discrete space. In fact, discrete topological space. So this is how the algebraic geometry will always go on commutative algebra to geometry, geometry to algebra. So some more observation I want to write it down, maybe for future ease.

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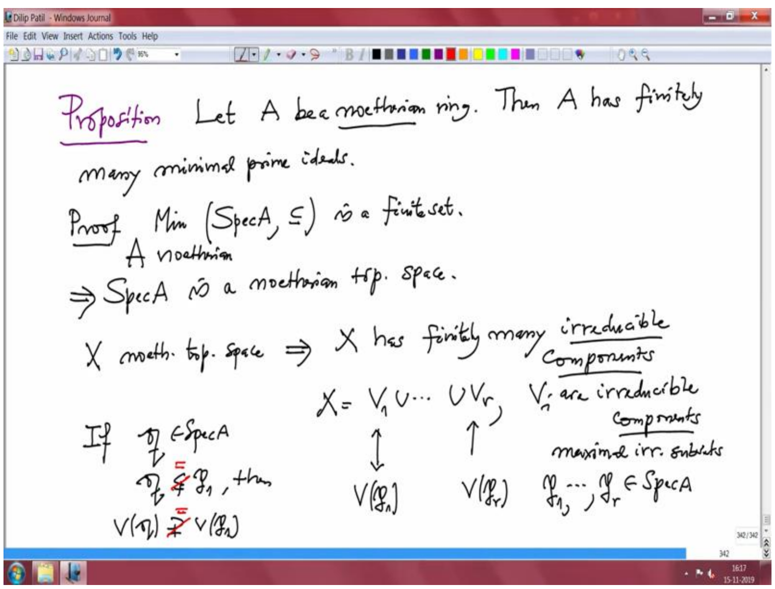

So this should be well let me call it a proposition. So let A be a Noetherian ring, then A has finitely many minimal prime ideals. Proof, note that when I say finitely minimal prime ideals that means we are considering this order sets spec A with inclusion. And the minimal elements in this ordered sets we keep calling minimal prime ideals.

So min of this set. And we want to show that this is a finite set. So recall that we approved in a first half that if A is Noetherian, then the spec A is a Noetherian topological space. So A Noetherian implies spec A is a Noetherian topological space, that means open sets form satisfies a sending chain condition or it has the every non empty subset of open sets has an minimal element, a maximal element.

Where the other way for the close sets. So and we have also proved when recall also we have proved when X is a Noetherian topological space, then it has finitely many irreducible components. Implies X has finitely many irreducible components. Now, what are irreducible we have just said component means that the maximal irreducible closed subsets. So, that means X is a union of the irreducible components, so that is let me call it V1 union union union Vr, where Vi are irreducible components.

That is maximal irreducible subsets, so they are in particular they are close, so they are close and irreducible. Therefore, just now we have noted that these each Vi will corresponds to the prime ideals. So, this V1 will be V of p1 and this Vr will be V of pr, where p1 to pr are prime ideals in the ring A and this irreducible component means maximal irreducible subsets but they if they are maximal, then these p1 to pr has to be the minimal prime because if they are not minimal if anyone of them is not minimal then it will contain further prime ideal properly.

But then this if for example if q is a prime ideal with q contained in p1 not equal to, let us say, then what will happen? Then apply V, so V of q will become strictly bigger than V of p1, both are radical ideals. Therefore, applying V they cannot become equal former (())(25:28) will tell that they are proper. So this cannot be this cannot be maximal irreducible because this is already reducible, because it is a prime ideal therefore this is reducible and it contains properly this cannot happen.

So equality here, therefore equality here therefore p1 is minimal, similarly p1, p2, pr etc. are minimal and therefore that proves that it has finitely many minimal prime ideals. This is very very important proposition. So, with this I will only make one comment which also is very important to understand the topological arguments namely.

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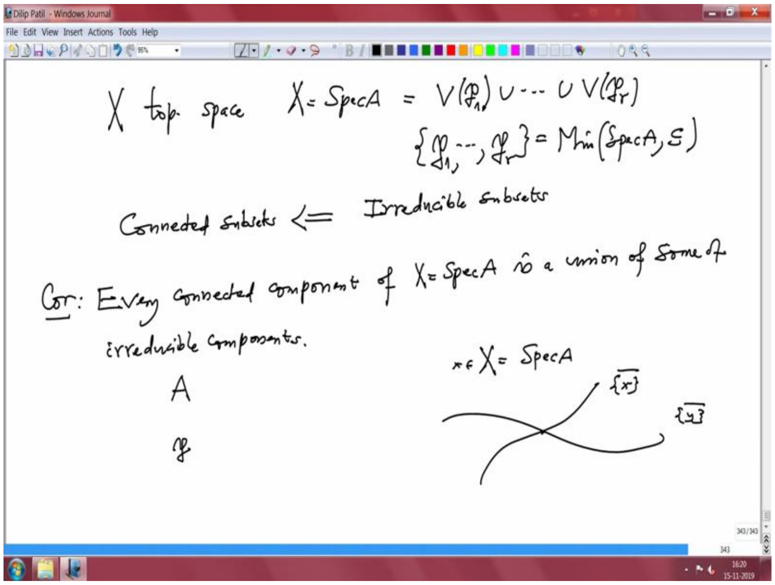

So see if you have a topological space X and suppose it is X is Spec A, then we know already the irreducible component, this X is a union of Vp1 union union union union Vpr, these are p1 to pr are the minimal prime ideals. So this set is precisely the main Spec A with respect to the inclusion, irreducible they are irreducible component.

Now, what is a relation between connected subsets and irreducible subsets? So irreducible is more stronger condition. So therefore, what we have to note is every connected component, so every connected component of X equal to Spec A is a union of some of irreducible components because every irreducible subset is a connected subset. So this is what I wanted to write it as corollary.

So if you draw or try to draw a picture the, so if A is commutative ring and X is Spec A, I will try to show you by picture. So, and p is a prime ideal, so x is a point here, so that is this x closure will be this irreducible subset, closure of x. If you take another one it will be closure of y. If they intersect if they are still connected, so you will have to find components irreducible components which do not intersect.

And that will form a disconnected, so that is how this. So now all this getting more and more intemator with a topology and commutative algebra. So this is a beginning actually, but unfortunately we cannot go on so much in this course. In the next time I am going to start a new topic that also has a geometric content but first I will have to (())(29:28) on the algebraic component what is called integral extensions of the rings and then what happen to the spectrum.

So this is what the topic I will take it and this will be the last topic of this course. With this I will stop and we will continue our lecture in the next time. Thank you very much.