Introduction to Algebraic Geometry and Commutative Algebra
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Prime Ideals and Maximal Ideals
Lecture 05

Let us recall that in the last lecture we have studied ideals in a commutative ring and operations on the ideals and then we have used ideals to construct new ring from the given ring and that is called residue class ring of the ring by the ideal we were under consideration. Now today I will study, define and study some special ideals in a ring.

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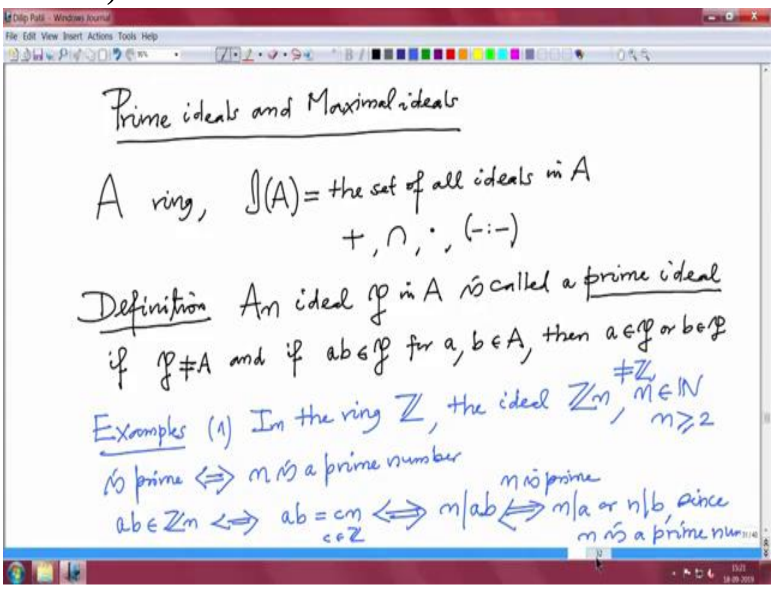

So these are called prime ideals and maximal ideals. The two types of this special ideals. As usual, let us denote A to be ring and let me remind you in the last lecture I have introduced the notation for I of A.

This is the set of all ideals in A and on this we have these operations plus, intersection, product and one more, that is colon. These operations we have defined and then some other properties I will write in some of the exercises.

But today, so let us define first. An ideal p in A is called a prime ideal if, so let me just remind you, this prime ideal will be defined...will be a generalization of prime numbers. See we have studied prime numbers right from the school. They are in the ring of integers. These ideals, prime ideals will correspond analog of the prime numbers.

So the definition one can guess, so an ideal p is called a prime ideal if, first of all p is not a unit ideal and if a times b belongs to p, a product belongs to p for a, b in A then one of them should belong to p, a belong to p or b belong to p. So such an ideal is called prime ideal. Let us see some examples. Immediately after the definition we should see some examples.

So some examples, in the ring Z, you know any ideal is a principal ideal generated by a natural number. The ideal Z n where n is a natural number is prime if and only if n is a prime number. So how is this, let me just, so one has to get used to it. First of all when you say, I should say a proper ideal, ideal Z n which is not Z that means n has to be at least 2. So what do I have to check?

The product a times b belongs to Z n, so I am proving first if it is a prime number then it is a prime ideal. So suppose the product belongs to this. What does that mean in terms of divisibility condition? That means this a b is a multiple of Z, multiple of n, a b equal to c times n. But this is, where c belongs to Z.

But this is if and only if n divides a b. But this is, this implies n divides a or n divides b since p n is, n is a prime number, and obviously the prime numbers are... So we have proved if it is a prime ideal then it is generated by prime number.

Conversely if you have a prime number then we have these properties. Therefore this is if and only. This is if and only if n is prime. So in Z we know all the prime ideals. They are precisely generated by the prime numbers.

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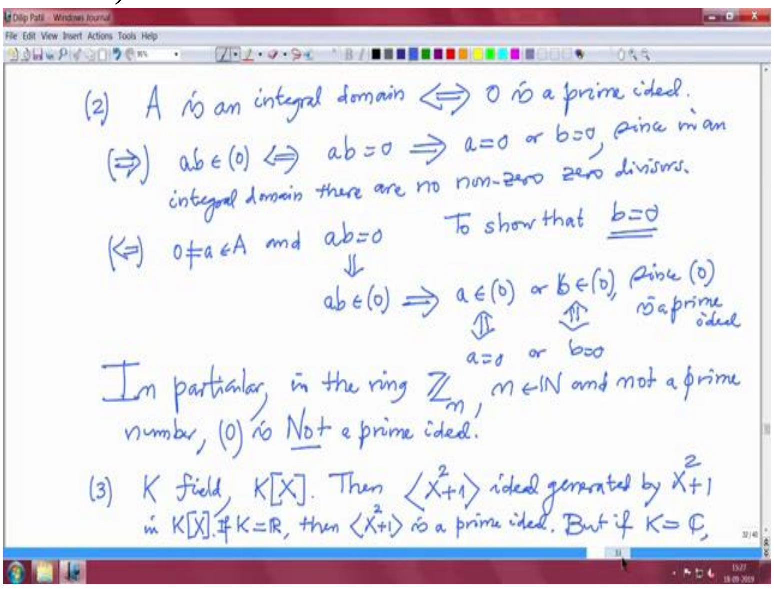

Alright so some more examples, 2, suppose you take any integral domain, so A, ring A is an integral domain if and only if the ideal 0 is a prime ideal. So let us check this again. So I am checking if it is an integral domain then 0 is a prime ideal.

So I am proving this way. So I want to prove 0 is a prime ideal. So let us take a product a b belongs to ideal 0. But that means a b is 0 because 0 is the only element in the 0 ideal. But you are in the integral domain so there is no non-zero 0 divisor.

That means, so this should imply either a is 0 or b is 0, since in an integral domain there are no non-zero 0 divisors. So that proves this, conversely this way. That means we are assuming 0 is a prime ideal and we want to prove A is integral domain. So how do you prove some ring is an integral domain? You have to check that there is no non-zero 0 divisor.

So suppose a is non-zero element and suppose a times b is 0 then what do you want to show? Then we have to show, to show that b must be 0. Because if b were non-zero then this a will be a 0 divisor. So, but a b 0 means, this means a times b belongs to the ideal 0. But then 0 is a prime ideal.

So product of elements belongs to the ideal 0. 0 is the prime ideal. So that should imply either a belongs to 0 ideal or b belong to 0 ideal. Since ideal 0 is a prime ideal. So, but this means a is 0 or b is 0. So that in integral domain, 0 is a prime ideal.

So third, so therefore in particular in the ring Z mod n where n is a natural number and not prime, not a prime number. 0 is not a prime ideal because we know, because if n is not a prime ideal then Z mod n is not integral domain. Therefore 0 cannot be prime ideal.

So third example, now let us take a field. K is a field and let us take a polynomial ring over K in one variant. Then just look at the ideal generated by X square plus 1. This is an ideal, ideal generated by X square plus 1 in K X. Now if K equal to R this ideal is, if K equal to R then this ideal generated by X square plus 1 is a prime ideal. That is very easy to check. But, but if K equal to C then it is not prime ideal.

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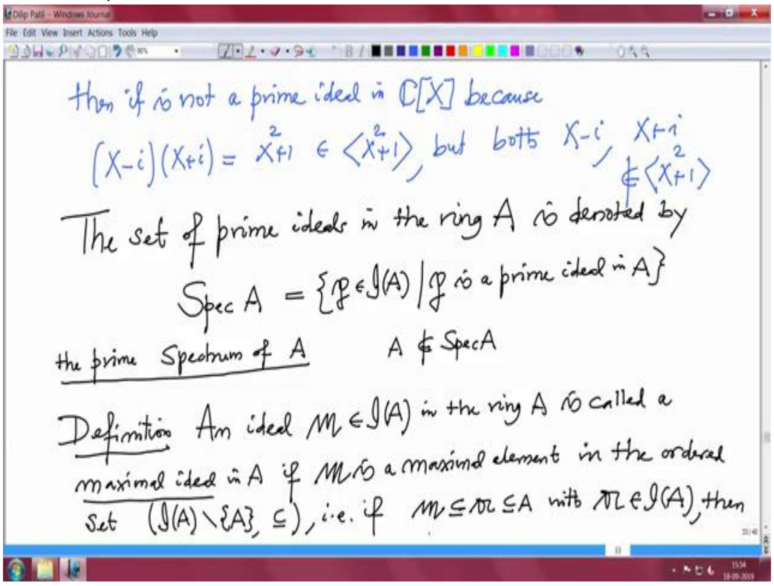

Then it is not a prime ideal in C X. This is very simple because look at the product, X minus i times X plus i, this is equal to X square plus 1 which belong to the ideal generated by X square plus 1 but none X minus i or X plus i they belong to, so none of them belong to, so both of them, but both does not belong. Simply because if they belong, then the degree this is 1, degree this is 2, so this is a multiple of this will mean the degree is bigger equal to 2 which is not true.

Alright so more generally, this I will do it later also, so some more prime ideal, let me see if I have some more, so anyway so I will give some more examples once we have enough vocabulary, now the second definition. So before I go on I will write the set of prime ideals in the ring A is denoted by this symbol, Spec A, this is called the spectrum of A, prime spectrum of A, prime spectrum of A.

Let me just make a comment here that this is the same word spectrum which usually we come across in physics. So there is a nice connection between the spectrum, the word from physics and this spectrum. I will try to digress it when we have enough vocabulary in this course. So this is the set of all p in the ideal of A such that p is a prime ideal in A.

So obviously this ideal, unit ideal A that does not belong to the spectrum, and ideal 0 may or may not belong. We have seen both the cases. In some cases 0 belongs. And in some cases 0 does not belong to the spectrum, alright, and this will be one of our main object of study in this course. One has to study the prime spectrum and gain knowledge about the given ring. So that is how the, it is.

So the first very important result we will need that this is indeed a non-empty set. If it is empty set again, we will have nothing to study and then our goal of studying algebra with geometry and geometry with algebra will not take place, alright. Another very important definition is, we will soon prove that this is a non-empty set. So definition, now this is the definition of maximal ideal.

An ideal m, I of A in the ring A is called a maximal ideal. In A if there should not be bigger ideal in the ring than this m. But of course, the unit ideal is the biggest ideal that way. So you should avoid unit ideal. So if m is a maximal element in the ordered set I A, remove A from that unit ideal and this is an ordered set with respect to the inclusion order. So this is my ordered set and this m should be maximal element there.

So what is a maximal element in an ordered set? That means there is no element bigger than that it is in that set. So that is, if m is contained in some ideal A, contained in A with ideal A, a is ideal in A then either this Gothic a should be capital A, that is it is unit ideal or it is equal to m.

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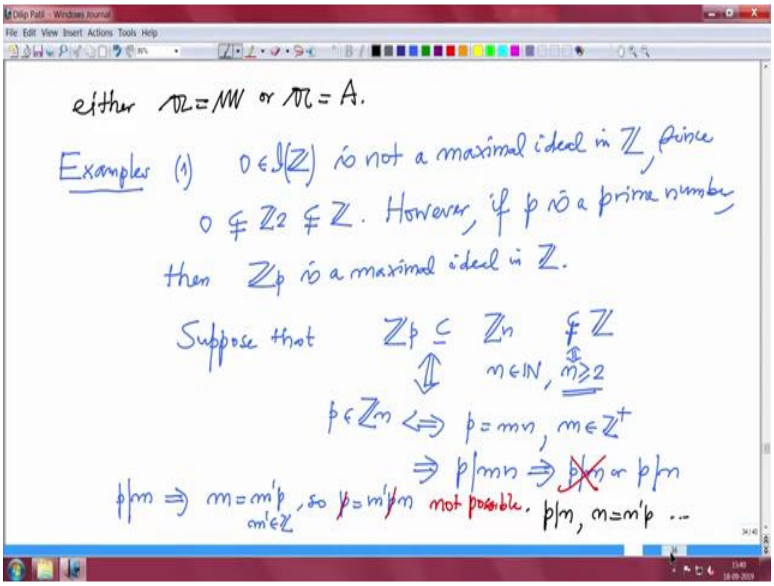

So then either a equal to m or a equal to the unit ideal A then you call it a maximal ideal. Now so some examples as usual, that will be always our strategy. So example 1, 0 ideal, so I will not write any bracket, so the ideal 0, so one way it is clear from our writing what we are considering, this is, so where am I taking, what is my ring? So I should specify my ring also, in the ring of integers.

This is not a maximal ideal in Z. So what should we produce? We should produce a bigger ideal which is not a unit ideal. So since 0 properly contained in let us say 2 Z, Z 2, Z 2 is a proper ideal. 1 can never be multiple of 2 in Z and this is properly contained in Z. So these are, so this is not maximal in this. So therefore this is not a maximal ideal.

However, if p is a prime number then the ideal generated by p that is Z p is a maximum ideal in Z. So let us prove this of course later on also we will see how do you test some ideal is prime and this will become trivial in that test. So what do I have to check?

So suppose there is a bigger ideal. Suppose that Z p is here and there is an ideal in between. So we know any ideal in Z. It looks, it is generated by one element, one natural number in fact. So this looks like this. So n is a natural number and let us assume, so what do you have to prove, you have to prove either equality here or equality here.

We have to prove either equality here or equality here. So let us assume one of them is not equality. Let us say this. So this is not equal means this is equivalent to checking n is bigger equal to 2 because if n were bigger equal to 2, if it is equality here, 1 should belong here. But 1 is never a multiple of n where n is bigger equal to 2.

So this condition not equal to is equivalent to saying this. This is equivalent. So I have to prove equality here now. So what does this mean? This, this inclusion is equivalent to saying p belongs to n, p belongs to Z times n. That means p is the multiple of n. So this is equivalent to checking p equal to some m times n where m is an integer, n is a natural number bigger equal to 2, p is a prime number and m is integer.

So this equality first of all will force you that this Z has to be a positive integer. Not only that, then p divides m times n. So that will imply p divides m times n and therefore p is a prime number. So p either divides m or p divides n. But, so we will say both of them are, both, either of these give a contradiction. So if p divides m, what does that mean? That means this m is a multiple of p.

So m equal to some m prime times p, m prime is in Z. When you plug it in here then you will get, so we will get p equal to m prime times p times n. But now we can cancel, can cancel this and we will get 1 equal to m prime n. But n is at least 2 and m prime is an integer so that is not possible. So this is not possible. What about this? If p divides n, again do the same.

If p divides n, so if p divides n then p, n is a multiple of p, so this is not possible. So p divides n. p divides n will mean again n equal to n prime times p, but then again you cancel and get a contradiction. So I will just write a dot dot here just few minutes later we will see how to check in a better way. So that was their prime ideal.

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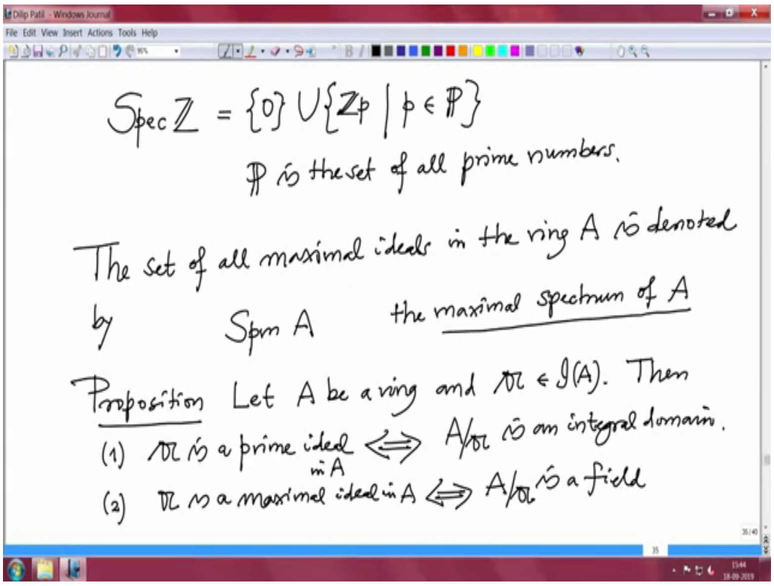

So therefore in this case, so I will write here, so we have described all elements in the spectrum of Z. So this is precisely, obviously 1 element is a 0 ideal. The other one is Z p. So I should write a correct notation, this is union Z p where p is a prime number. So this is, this P is the set of all prime numbers. So like we have introduced the notation for the prime spectrum, also I want to introduce a notation for set of all maximal ideals.

So the set of all maximal ideals in the ring A is denoted by Spm Z, Spm A, is called the maximal spectrum of A. Alright again the same, so let me give some examples. It is very important to see more examples of the maximal ideals. But before I do that I want to check how do you test some ideal, some given ideal whether it is prime or not? So for that there is a nice easy test. So that is, let us call it a proposition.

Proposition, let A be a ring and a ideal in A. Then this test, then this proposition is very handy to check some ideal is prime or not, or maximal or not. So part 1, a is a prime ideal if and only if, in A of course, A, the residue class modulo a, this ring what we constructed yesterday, this ring is an integral domain. 2, a is a maximal ideal in A if and only if A mod a, this also I keep calling it A modulo this a, A modulo Gothic A is a field. So let us prove this.

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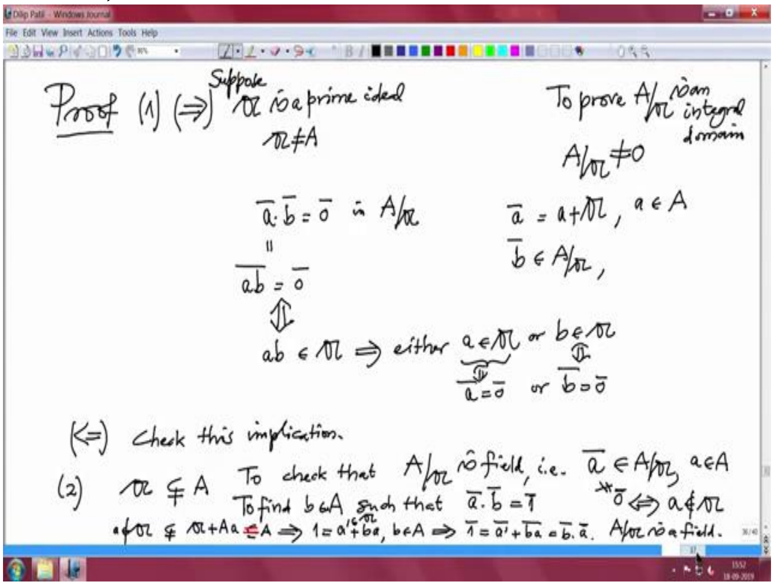

Proof, I am proving 1 and that too this way. So that means we are assuming a is a prime ideal. And then what do we want to prove? So suppose a is a prime ideal then write in a corner somewhere that to prove the residue class ring is an integral domain. That means, that means what do you want to prove?

If product of 2 elements is 0 then one of them is 0, this is what we want to prove. So first of all it is a prime ideal. Therefore a is not A by definition, and therefore this is not a 0 ring, A mod a is not a 0 ring, this is a non-zero ring because it is a proper ideal, and now you remember yesterday we were denoting the elements of this ring by co-sets a plus a.

But I will denote this by a bar. So elements of the residue class ring are denoted by bars of the small letters. And what do we want to prove, if it is, if there are two elements a bar and b bar in A mod a, and a bar b bar is 0, 0 bar in A modulo a then I want to do one of them is 0.

But what does this mean? The way we have defined the multiplication in this residue class ring, this is same thing as a b bar equal to 0 bar. What does that...when are the two elements equal? When the difference belongs to the ideal.

So this is equivalent to checking a b minus 0 that belong to the ideal a. But A is given to be a prime ideal and the product belong there. Therefore either a in A or b in A, but that means, so this means what? This means a bar is 0 bar or b bar is 0 bar. This is equivalent to saying this, same. Two bars are equal means their difference belongs to the ideal. This is (())(32:48). So that proves if a is prime ideal then A modulo a is integral domain.

Conversely if A is integral domain, actually you can reverse this, all these arguments. So I will leave it, I will just say check this check this implication. Now 2, I want to prove a is a field if and only if a is a maximal ideal. So if a is maximal ideal that means first of all a is a proper ideal because it is a maximal element in I A minus singleton A. So this is not a unit ideal.

And now what do we want to check? I want to check that, to check that A mod a is a field. That means what do I have to check? So that is, I have to check that every element a bar in A by a, a is in A has a inverse. That means what? We want to find, to find b in A such that a bar times b bar equal to 1 bar. This is what we want to find. Alright so a bar is, a bar is here and non-zero, I should have written a bar is non-zero, a bar is not 0 bar.

But what does that mean? That means, so this is equivalent to saying the difference is in the ideal, so a minus 0 that is a is in the ideal, not in the ideal. Not 0, so the difference is not in the ideal. So this is not. So obviously what do I consider? Look at a here and take ideal generated by this a and that small a. So that is plus, so this is this. This is clearly an ideal which contain all this and a also.

So this is smallest ideal which contain the Gothic a and small a. So this is, and this a is not here that means this is proper. So this is a bigger ideal. This is because a is not here. But then this is an ideal in A. So because of the maximality of A this has to be equality here. This has to be equality. But that will mean, so that means 1 here is a combination of somebody in A and some multiple of this small a.

So that is 1 equal to some a prime plus b times a for some b in A. But now when you read this equation modulo A then what do you get? Then you will get 1 bar equal to bar of this, a prime bar plus b a bar which is, a prime bar is what? a prime is in the ideal a, I forgot to write here. This one is in the ideal a. So therefore this is 0, so 1 bar equal to, no so this, so this a bar, a prime bar is 0 so 1 bar equal to, this is b bar times a bar.

So I have produced the inverse for a bar. So therefore, so this proves A mod a is a field and conversely similar. So I will not prove the converse, alright. So we have easy way to test some ideal is prime or not. You go modulo and then check that it is prime or not.

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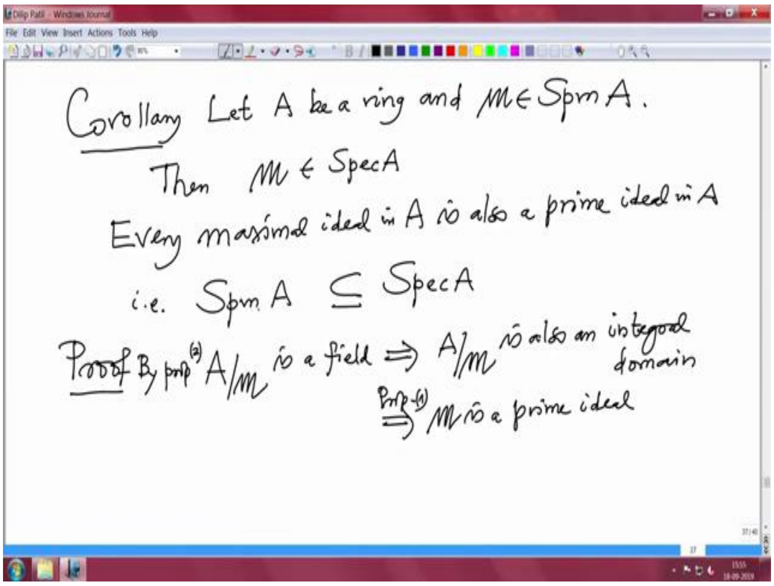

So just one example and then, so example, before I write an example, maybe better to write one corollary. So corollary, let A be a ring and P be a prime ideal, or not P, m, m is a maximal ideal, Spm A. Then m is also a prime ideal. So in other words, every maximal ideal in A is also a prime ideal. Converse we have seen, it is not true because 0 is a prime ideal in the ring of integers but 0 is not a maximal ideal there.

So this is in symbols Spm A is contained in Spec of A. So that is this in symbols. Proof is very simple. Let us finish it off. Proof, m is a maximal ideal. That is given. So, therefore A modulo m is a field by the proposition second part but once it is a field it is also an integer domain, so therefore every field is an integer domain therefore A by m is also an integer domain.

Therefore, again by proposition, so this is by proposition again, p, m is a prime ideal. This is by proposition. This is also by proposition. This is by part 2 and this is by part 1. So we have proved that every maximal ideal is prime.

Now the question what we were thinking that we need this set to be non-empty. So actually we will put this set is, we will prove that this set is non-empty and that will be very important for us because then we will have a nice flow of many ideals which are maximal ideals, which are prime ideals and so on, and I will continue in the later half by many more examples in some special rings, mostly geometric examples. So we will meet after the break, thank you.