Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 48
Irreducible subsets of Zariski Topology Finite type K algebra

Come back to this second half of today's lecture. As I said that, now we will discuss some Topological Properties of the Zariski topology. So, let me just recall or mention that, what was our situation?

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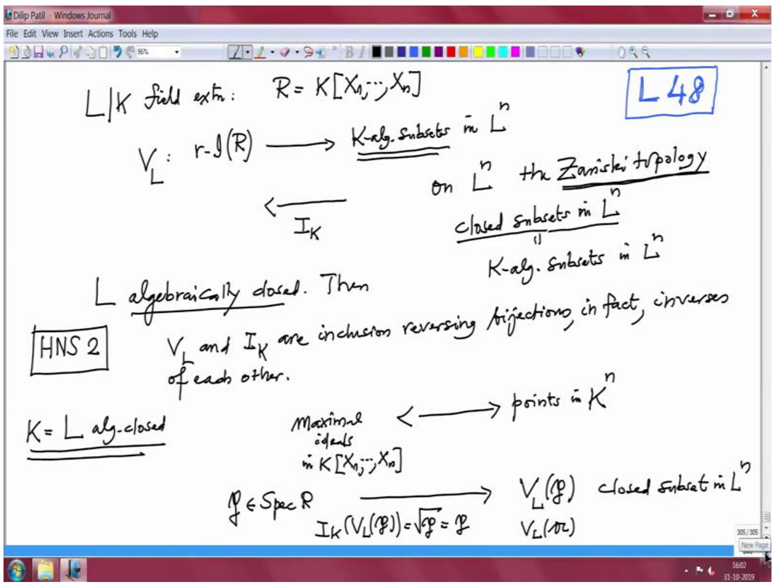

Our situation was L over K field extension and then we have defined these maps, VL, these maps are defined from r, root ideals or radical ideals of A. I do not want to call A, we have not call A earlier, so and R is the ring, which is a polynomial ring, in n variables over a field K. Then we have this map, from the root ideals of R, radical ideals of R to K algebraic sets, K algebraic subset in L power n and also we have defined the map in other direction. This is, this map is denoted by IK and this we have defined it for arbitrary set.

So, anyway our results are better, better in the sense that, this gives a topology on L power n. So, we consider L power n, Zariski topology on L power n, the Zariski topology, is defined by saying that, closed sets, closed subsets in L power n are precisely K algebraic sets, K algebraic subsets in L power n.

So, that is how we have defined a closed sets and we have checked that, they satisfy the required properties of the closed sets in a topological space. That means empty set is a closed set, whole space is a closed set, arbitrary intersection of closed is closed and finite union of closed is closed. So, this properties are satisfied by this algebraic subsets and therefore they form a closed sets in a topology on L power n and that topology, we are calling it Zariski topology. This was defined by Zariski.

So, we have, so really one should think that, you have defined a map from VL, from the radical ideals of their polynomial ring, to the closed sets in, closed subset in L power n and the map, the converse map IK, converse means the inverse we have given, the other way map and these two maps are inverses of each other, only when K is algebraically closed.

So, K algebraically closed. Then VL, and IK, are inclusion reversing, bijections. In fact, inverses of each other. This is very important, this was precisely HNS 2, HNS 2. This follows from HNS 2. So, these allowed us to study topology on the Zariski topology, on L power n, in terms of the radical ideals in the ring, polynomial ring and therefore it gives interplay between the ideals on one side and on the other side, this K algebraic sets or closed subsets in the Zariski topology, on L power n.

Also in particular when L equal to K equal to L algebraically closed, this is a special situation. In that case, we are even better then the points in this L power, the points in L power n, they corresponds to the, the maximal ideals in the polynomial ring, K equal to L.

So, I should strictly written this K power n, this is K power n. So, points give the maximal ideals. So, this is, this is precisely what classically may be more than 200 years or 100 years back, the algebraic geometry was studied in this way. But this language did not exist in a precise way those days, therefore they were lot of confusion.

So, I will tell you one of the confusion, when I have enough vocabulary with me, anyway. So, the point to note is that when L is algebraically closed, then there is a nice correspondence from geometry to algebra and algebra to geometry and this corresponds we are going to study more and more and this study precisely is algebraic geometry. Because lot of geometric problems we are going to prove by using algebra and some algebraic problems, we are going to prove by using geometry and this is precisely study, this study is precisely algebraic geometry. But obviously there are some difficulties in this, I will note down the difficulties after I finish.

So, now you might ask in this situation for example what will and also actually we have also analyzed points in Ln what happens to them, what do they corresponds show and we know they correspond to some ideals and then in the last lecture we have given some corollaries, which describe them, when are they maximal ideals, and so on and the closure and so on.

So, I am going to do it that more now, I will not assume K equal to L, but I will certainly assume L is algebraically closed. That is because I want to use this Nullstellensatz, without Nullstellensatz we cannot go and come back alright. So, I want to describe now, when what happens with the prim ideal, what do they correspond here to, what subset do the corresponds to?

So, if we take a prime ideal p here. Suppose I take p, a prime ideal in the ring, polynomial ring, R. Then what do they corresponds to? Obviously this map is VL, VL of p. So, what kind of a, this is of course, this is a closed set. This is a closed subset in L power n that is by definition. But what kind of a closed set it is and can you recover?

Of course you can recover back this because of the Nullstellensatz, if you assume, your L is algebraically closed. Then you can recover back this because we know we approved in HNS 2, that IK of VL p, this is the radical ideal of p. But that is p, because p is a prime ideal. So, we can recover back.

But I want to explain more what kind of a subset is this, is a closed subset is fine. But does it have some more property. Because this p is a prime ideal. For example, if I take arbitrary ideal, what is a differences between this closed set and if you take the arbitrary ideal and take VL of that, this is also closed set. So, what is a difference? That is what I going to explain in the next, also one more thing to note, that I will write in the next page.

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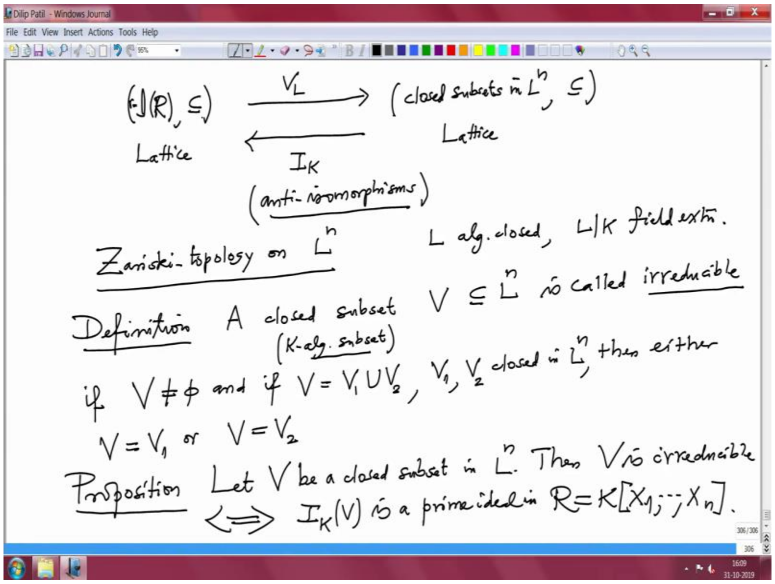

One more thing to observe, that if I take ideals, ideals of the ring R that with inclusion, this is a lattice that means it is an ordered set and any two elements have the supremum and any 2 elements have the infimum such as, such a ordered set is called a lattice, that we have seen earlier and also if you take the closed sets, closed subsets, in L power n with the natural inclusion, that is also lattice.

So, that is, it is ordered set, ordered set is clear. Because ordered set is in the, with the natural inclusion and what is the supremum and what is infimum of two of them. In fact, the supremum should be the union and infimum should be the intersection and this makes sense in a any, any topological space, if I take the closed subset that form a lattice and the map, we have given, this map.

So, VL, so even the radicals that is also the lattice. Because you can take can appropriately define a supremum and infimum there and this map VL and IK, these maps are actually lattice. It is does not keep the same order. But it is anti, anti isomorphism of lattices. So, these VL and IK, they are anti, anti isomorphisms of these two lattices, anti means, they change the order.

So, we are studying this, now you have noted in Hilbert basis theorem that Hilbert basis theorem, that this is the polynomial ring, R is a Noetherian ring, that means equivalently the set of ideals satisfy, ascending chain conditions on the ideal. So, what will happen here, that if I take the corresponding images that will satisfy the descending chain condition. So, maximal become minimal and so on, I will make this more precise in the statements now. So, that is what we are heading to.

So, we are studying now, Zariski topology on L power n and I will assume L is algebraically closed and L over K field extension. Actually not all the statement down L algebraically closed is needed, it is needed only when I use Hilbert's Nullstellensatz, then only it is needed. If there is no use of Hilbert's Nullstellensatz, if it is only use is this maps and their properties, that it is inclusion reversing and so on. Then algebraically closed is also not really necessary.

So, first is a definition, this definition is standard in a topological space. So, a closed set, a closed set, a closed subset V in L power n, of course when I now say closed, open etc that will be in the Zariski topology. So, a closed subset V in L power n, means this V is a K algebraic set that means what? That mean this V is of the form V of VL of some ideal. So, that is audit is.

So, when I want to stress more on the topology, I do not try this K algebraic set. When I want to stress more on algebra I will write this. Because then it will be no other ideal. So, this is a closed set in V in L power n, is called irreducible, if first of all it cannot be empty set and it cannot be broken into union of two closed sets and if V equal to V1 union V2, where V1, V2 are closed in L power n. Then either V1 should be equal to, V should be equal to V1 or V should be equal to V2. So, therefore it is called irreducible.

So, now the next preposition will describe, how do you take some subset in Ln, is some closed subset is irreducible or not. So, I will write, so proof, proposition, let V be a closed subset, in L power n. Then V is irreducible, if and only if, the IK of V is a prime ideal in R and R we always denote it by the polynomial ring over the base field K. So, this is very nice, so irreducible closed subsets corresponds to the prime ideals. So, we are improving the dictionary, which is given to us by this VL and IK, maps.

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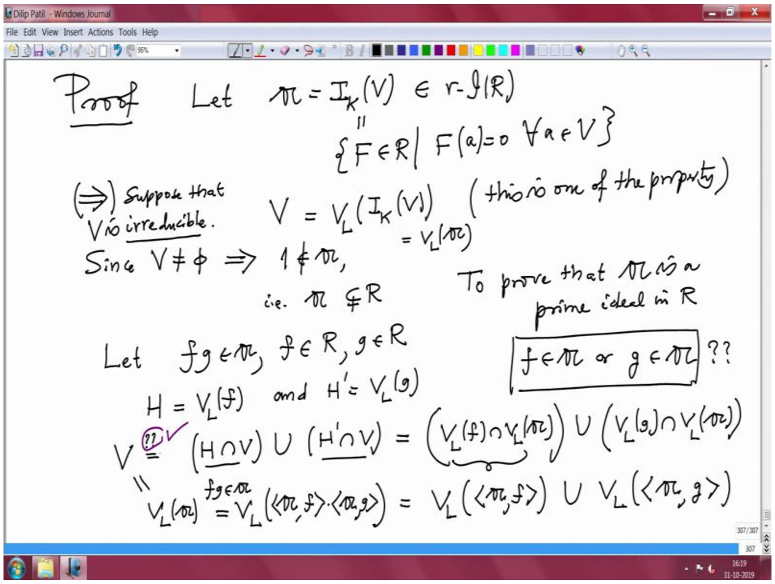

So, proof, proof is very simple, proof. So, let me put, let a is IK of V, this is an ideal in, this is a root ideal in R, radical ideal in R, we have noted that earlier, in fact this is precisely what? This is intersection of, intersection of all I a, where. So, we do not need that, so why unnecessary break our head. So, we do not need that right now. So, it is a radical ideal by definition, because this is precisely all those polynomials, what were their definition? All those polynomials capital F in R, capital F in R, such that F should be 0 on every point of V.

So, this is some polynomial vanished at every point of V, then the power also if power vanishes in every point of V, then that polynomial will also vanishes. Because we are in field L is a field and therefore it is a, it is a radical ideal. So, now what do we know? So, we know that, if I take VL of IK V, I say this is nothing but V. Because see, this is very easy to check in fact, this was one of the property I stated, that was I think property number 6 or some property. This V compose I, VL compose IK, this is identity on, on closed sets.

So, anyway, so I will simply here, a note here, this was, this is one of the property and we which did need L algebraically closed. So, this is very easy to prove. So, what do you have to prove? Any element here is here and what is a definition of here that means that element all these polynomials should vanish there. But that is precisely a definition. So, this is clear.

So, because V is non empty, since V is non empty, so that means what? So, that means 1 cannot belong to this ideal a, 1, cannot belong to this. 1 belongs here, see this is VL of a, this is equal to VL of a. So, if 1 belongs here, then that means this will be empty set, but this is not empty set given. So, it is a proper ideal. So, that is a is a proper ideal in R and I want to prove that, this a is actually a prime ideal.

So, to prove, to prove that a is a prime ideal in R. Remember we are proving which implication, we are proving this implication, this implication, we have proved. That means we are assuming V is irreducible and we want to prove a is a prime ideal. So, our assumption is V is irreducible, suppose that V is irreducible and I want to prove it is a prime ideal, the ideal of V, this is a prime ideal I want to prove.

So, that means what? If a product belongs to a, then one of them should belong to a and a should be a proper ideal, which already we have noted, a is a proper ideal and now, suppose let suppose the product, f time g belongs to a. Where f is in R and g is in also in R. So, I now denote the elements in the ideal by these letter f and g, because they are polynomials.

And consider and we want to prove that, so what is we want to prove that? Either f belongs to a, or g belongs to a. This is what we want to prove, this is what our goal is. Now and consider now, hyper surfaces H equal to H equal to VL of f, and H prime equal to VL of g. These are two hyper surfaces, these are closed sets by definition. Because this is algebraic set, K algebraic set, defined by the polynomial f, this is algebraic set defined by the polynomial g.

And then what happens? Then we are considering look at V, given V that is same as H, H intersection V, union H prime intersection V. Let us check this. How do you check this? So, look at start with this, this union. So, I am proving this. So, this one is what let us write down. So, this is V, this is V of a, VL of a, this is VL of a and this is VL of f.

So, this is VL, f intersection VL a and then union VL, g intersection VL a. But what were the properties of the, this closed sets? This what is, this is again equal to VL of somebody and that is VL whom? This is VL of ideal generated by a and f and this is union. This is ideal VL of ideal generated by a and g. This was the property we have noted, this is very easy to check and what is the union then? For union you have to take the either intersection or the product, it is same.

So, this is VL of a f, ideal generated by a f, multiplied by ideal generated by a and g. That is union correspond to the product or intersection. But intersection and product also we will give the same VL. But what is this ideal, when I take the, this is generated by of course a square and then a times f and a time g. So, anyway it is contained in a and this is f g.

But so this is equal to VL of a, that is because this equality because f time g, this belongs to a and here we have use a fact that, weather I take VL of a square, that is same thing as VL of a or VL of a Q also it is does not depend on the ideal. But it depends only on the radicals. So, therefore these are equal, but this is same as this. So, you see here, we wanted to prove this, we started with this and came back and prove this.

So, we have proved therefore this is proved, this is proved and this is, so this is proved, now but we are looking for this. But now see this VL, we approved this equality and this is a closed set, this is also closed set and V is union of the two closed sets. But V is given to be irreducible, that means, V cannot be. So, therefore V has to be one of them.

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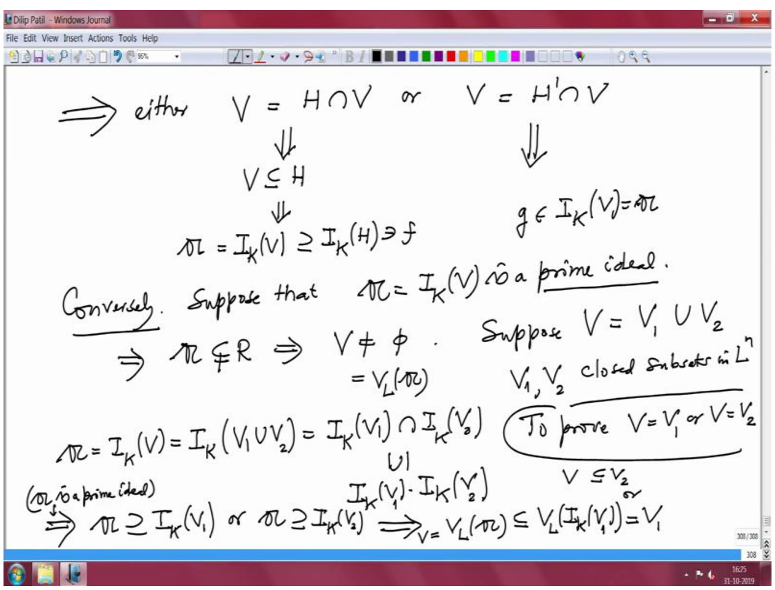

So, that implies, either V equal to H intersection V or V equal to H prime intersection V, one of them. So, if it is equal to this then, so if it is equal to this then this means V is contained in H and then if V is contained in H, then you look at the ideals. So, this will imply, you take the ideals on both sides, that will reverse the inclusion. So, IK of V will contained IK of H, IK of V is the ideal a we have called it and f belongs to the ideal of H. Because it was, it was H was, H was defined by f.

So, f vanishes on H. Therefore f polynomial f will belong to this. So, this equality means a belongs to f. Similarly this equality will mean g belong to IK V, which is a. So, we approved assuming irreducibility, we approved that the ideal is prime. So, that is, this IK V is also called the ideal of V.

So, now conversely, conversely. Suppose that a, which is IK V, is a prime ideal. Then I want to prove irreducible. So, first of all, we are assuming V is non-empty. So, that is we have to show V is irreducible, first this is a prime ideal. Therefore a cannot be the whole ring. Therefore V is non empty. Because V is precisely VL of a. So, it is a non-empty. Because a is non-empty, this was, if it was empty, then the ideal will be the whole ring.

So, here also we have not used Nullstellensatz let us remember that. Now, suppose V is a union of two, V1 union V2 and V1 union V2 are closed subsets in L power n. Then I want to prove V is either. So, to prove V equal to V1 or V equal to V2. This is what we want to prove, this is the (())(29:59). So, suppose we are assuming this, so let us take ideals on both sides.

So, a equal to IK V, which is and IK of the union, IK V1 union V2 is precisely IK V1 intersection IK V2, this is the property of IK, this is very easy to check again just check an element here and take it easier. So, and this intersection will contain the product IK V times IK V1 times IK V2. This is also easy, product is contained in each one of them. Therefore each contained in the intersection.

But now, a is a prime ideal, a is given to be prime ideal and this contained the product of the ideals. Then I say a has to contain one of them. So, that implies a contains IK V1 or a contains IK V2, this is because a is a prim ideal this implication, that is very clear because if it does not contained a both of them, then you take an element here, which is not here and take an element which is not here and consider their product, product is here, which is in a. But product is not in.

So, that shows this (())(31:48), but what does this implication means? Now, we apply V again, if I apply V, VL of a which is V and which is now this will get reverse. So, this will be contained in VL of IK a, IK naught IK, IK V1, but what is this? This is V1, I say I viewed that earlier in earlier proof also V compose IK is good, V compose IK is easier to do it. But IK compose V is not easier.

So, this is easy, this is also we you can see, V1 is obviously contained here and these points of V1 vanishes here and so on. So, this is again and again we will use this, this does not use Hilbert's Nullstellensatz, I want to stress on that. So, this implication will give this and the other implication, so or V is contained in V2, that is what we wanted to prove. So, therefore we finish the proof of this. Now the next one is I will note it. So, under that, prime ideal we will correspond to the irreducible closed subsets.

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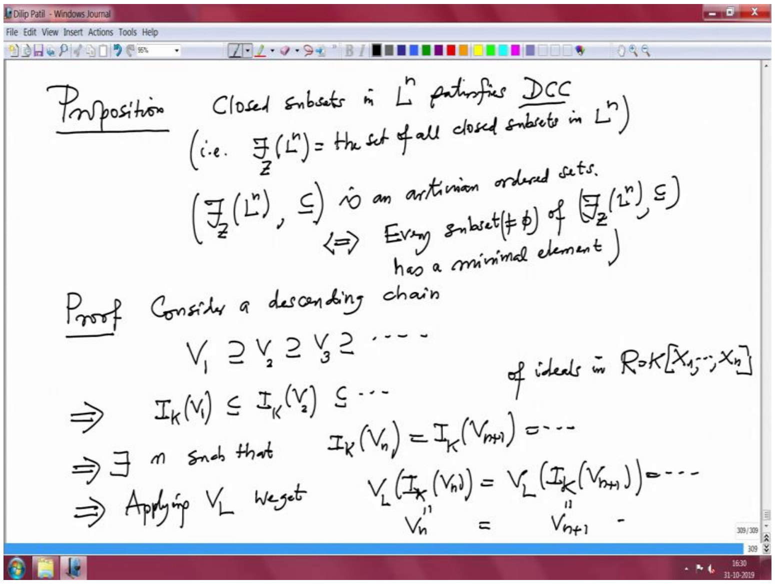

So, the next preposition, a closed subsets in L power n, satisfies a DCC. So, what does that mean? That means in our earlier language, if I take the closed subsets. So, that closed subsets, so let me introduce a notation for that, because we are going to use again and again.

So, the closed subsets I want to denote by f and Z for Zariski topology and L power n, this is the set of all closed subset, subsets in L power n, which is also same as K algebraic sets. That satisfy DCC means what? That means, so this means f Z L power n, with respect to this is an Artinian ordered sets, which is equivalent to saying it has a minimal property. That means every subset non empty subset of f Z, L power n has a minimal element.

So, that means what? That mean when you translate into the closed set language, that means every subset of closed, every subset of closed subsets has a minimal element. So, proof, so consider a descending chain. So, that is V1 contained in V2, contained in V3, and so on and I apply IK to this K.

So, that implies IK V1 contained in IK V2. So, we get an ascending chain of ideals in the ring R, which is a polynomial ring and we approved it is Noetherian by Hilbert basis theorem. So, therefore the chain should be stationary. So, there exist, so that implies there exist n naught such that IK Vn equal to IK Vn plus 1 and so on, all the way they are equal.

So, it became stationary, but then I apply V to this. So, applying V, applying VL we get VL of IK Vn equal to VL of IK Vn plus 1 and so on. But then this is Vn, this is Vn plus 1. Therefore these are equal and so therefore the chain is stationary, that means it satisfy DCC and equivalent that it is minimal that always, that we are approved for arbitrary Artinian ordered set, a DCC is equivalent to the minimal property.

With this I will stop today's lecture and I will continue studying the topological properties of the Zariski topology on L power n and eventually I will point out the difficulty in caring it on further. So, that will allow us to think better option, then the, this K spectrum and so on. So, we will generalize this and in the generalization things will become easier. But on one hand but the definitions are more abstract. So, this will lead to abstract algebraic geometry. So, with this I will stop and we will continue in the next lecture, thank you.