**Introduction to Algebraic Geometry and Commutative Algebra**
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**Lecture 35**

Welcome to this lectures on Algebraic Geometry and Commutative Algebra. At present last couple of lectures we were studying localization of rings and modules and today I will state more general form of the universal property of the S inverse for modules which is and I will show you how useful it is to study homomorphism modules.

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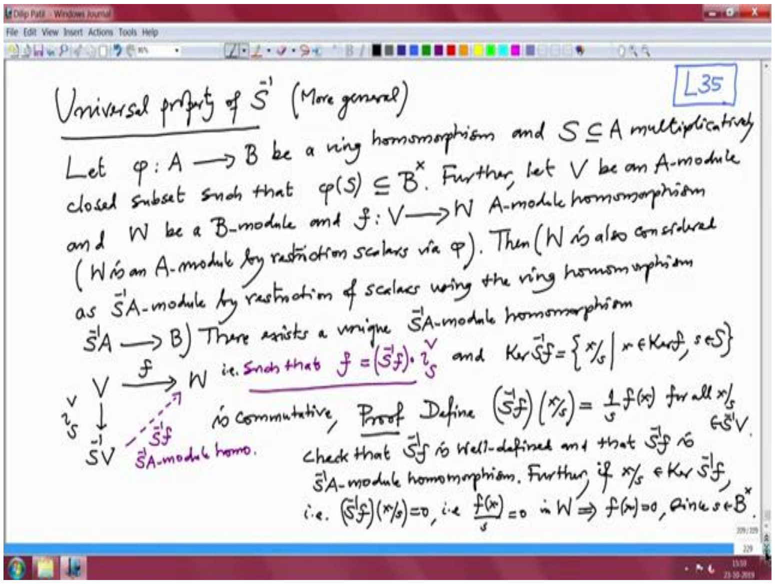

So, let us start with as usual. So, this is the Universal property of S inverse. I would just say this is more general than the previous one. So, let me state precisely what it is. So, let phi from A to B be a ring homomorphism. In the previous version of universal property there was only one ring but here now there is other ring B also and S is a multiplicatively closed subset in A such that the image of S under phi are all units in B.

This is the unit group of B. That means every element small s in S image of that phi S should be image in B. So, further let V be an A-module and W be a B-module and f is a homomorphism of A-module homomorphism. Here we have considered W as A-module also. So, W is an A-module by restriction of scalars via phi. So, its a A-module homomorphism.

Then first of all we can also consider then because we know the there is a so let me write a statement first then W is also considered as S inverse A-module by restriction of scalars using the ring homomorphism from S inverse A to B. This exist by the universal property of the S inverse for the rings. So, this is not so serious statement. So, I just put this in a bracket. Then there exist a unique S inverse A-module homomorphism. So, V is here W is here.

This is a given A-module homomorphism and then we have S inverse V here and this map is our iota V suffix S. Then we are saying that let me use a different color. There exist a unique map here which I will denote S inverse f and this should be S inverse A-module homomorphism, S inverse A module homomorphism. So, that this diagram is commutative such that so the diagram is commutative to means f if go by f that is same thing is going from here to here such that f equal to S inverse f compose iota V S. This is what we need to prove.

But this is very simple, this is so that let me write here such that this diagram is commutative that is this that is such that these 2 maps are equal f equal to this and I want to describe this Kernel and Kernel of S inverse f is precisely all those x over S such that x belong to the Kernel of f and s is in S. Proof is very simple. I will just write the definition of S inverse f and leave the checking to check for you. This is very simple.

So, proof we need to define. So, define S inverse f, I want to define this map. So, it should be map from S inverse V to W. So, it is defined on the fractions like this. This we define this to be equal to 1 over s fx for all x over s in S inverse V. Now, we have to check number of things. I will just see that check that S inverse f is well defined that it does not depend on the representative of A equivalence class and that S inverse f is S inverse A-module homomorphism.

And to checking the Kernel so further if x over s in the Kernel of S inverse f. That means what? That is S inverse f evaluate at x over s is 0 but this means so that is by definition was fx over s is 0 in W. But W is a B-module and where this s in B is in multiple so that will imply fx is 0 since s is unit in B, image of s so same as s. So, that proves that the Kernel of S f is precisely this as you saw. So, that proves this statement. This is more general than this. I will show you why it is more general and we will give application of this to modules of homomorphism.

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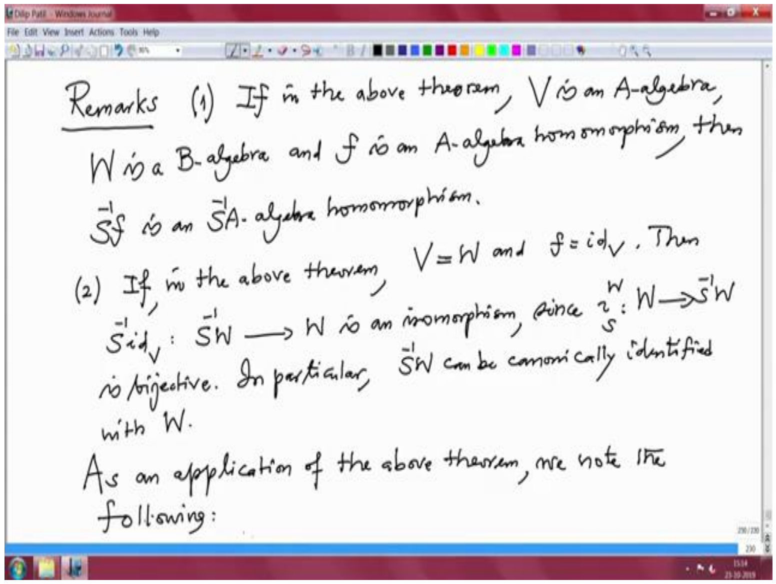


So, first I want to write some remarks. So, remarks number 1. If in the above theorem, V is an A-algebra and W is a B-algebra and f is an A-algebra homomorphism then this map S inverse f is an S inverse A algebra homomorphism. That means it respects a multiplication but that is very clear from the definition of S inverse f because f has that property that was first remark.

Second one if in the above theorem V equal to W and f equal to identity map on V, then S inverse of id V this is a map from S inverse W to W is an isomorphism since iota W S from W to S inverse W is bijective. In particular, S inverse W can be canonically identified with W. So, this decision is very easy to check it for the immediately from the definition of the map S inverse f we defined in the theorem. Now, I want to give application of this theorem. So, as an application of the above theorem we note the following. So, let me repeat some of the assumptions.

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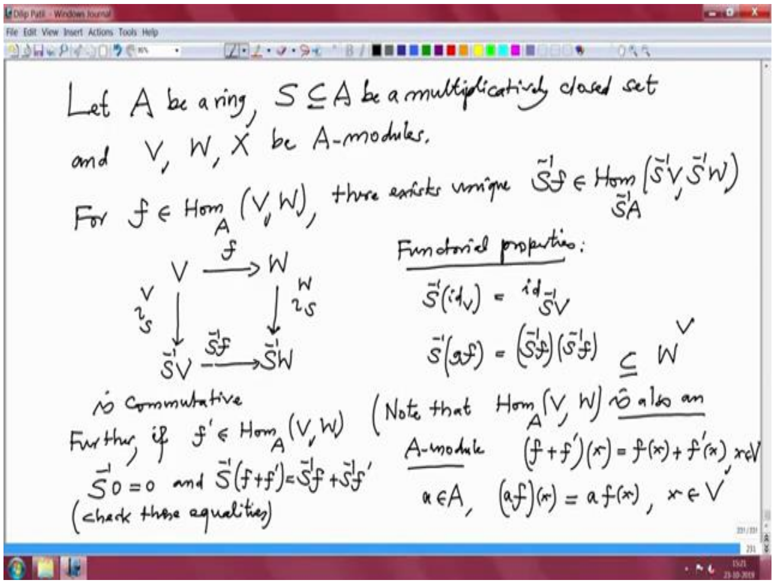


What are the assumptions? Let A be a ring, S contained in A be a multiplicatively closed set and V and W, V, W, X be A-modules. The 3 A-modules and for a homomorphism f from A-module homomorphism from V to W we know there exists unique S inverse f which is a homomorphism as S inverse A-module from S inverse V to S inverse W. So, that means we have these diagram is commutative. This is what we have proved in the earlier lecture V to W this given f, there is a iota map here S inverse V, this is iota V suffix s.

Similarly this is iota W S and thus is that map S inverse of f. So, this diagram is commutative and we also saw that they satisfy the functorial properties that is namely functorial properties once again I will repeat here, functorial properties that is S inverse of id V equal to id S inverse of V and if you have two homomorphism's g and f, so S inverse of g compose f is same as S inverse g compose S inverse f. This we saw last time but it also follow from the earlier theorem.

Further now these are the functorial properties but it is it satisfies some more properties. Further if f prime is another homomorphism from V to W first of all note that these homomorphism, this is so I will write here. So, note that Hom A V W is also an A-module and what are the operations here? Additionally of course if I have f and f prime then this is defined by evaluated at any x evaluate individually and add then f W. So, this is defined for every x in V and what is the scalar multiplication?

If I have a in A then af evaluated at x is you take fx this is in W and use a scalar multiplication in A to define this. So, this is also defined for every x in V. So, with this operation this becomes an A-module that is now it is very easy to check that this is an A-module with this addition and this scalar multiplication.

This also can be seen directly because Hom A W is a subset of W power V. See W power V is the set of all maps from V to W and that also as a structure of A-module because you take any tuples which are intakes by the set V and terms in the module W and then you can add component wise and you can scalar multiply component wise.

This is also same as this operation. So, in fact this one is the sub module of this. Anyway so we have this is also module and now the question is what happens to the sum of these homomorphism and when you apply S inverse. So, I am writing that first of all. 0 map the constant map 0 is also homomorphism. So, S inverse of 0 is 0.

That is one and if I have another homomorphism f prime and if I take S inverse of f plus f prime that is equal to S inverse f plus S inverse f prime. This is also very easy to check. You want to check these two maps are equal just evaluate on any x over s and then you know this definition x over s. That means you have to evaluate f plus f prime at the numerator and denominator is the same and similarly here and then you can just check. So, I will just say here check these equalities.

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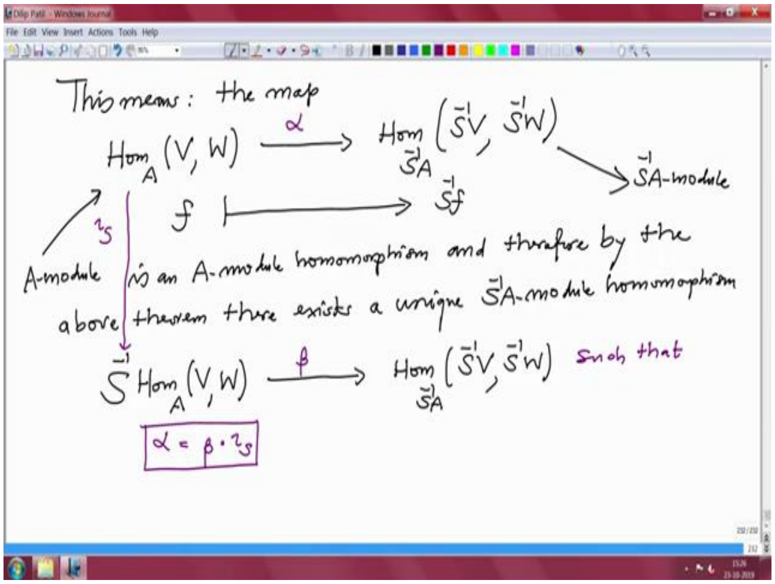


So, what are these mean? I want to write this the meaning of these. This is just a simple equalities but this has bigger meaning or at least important meaning. So, this means the map we have defined a map from where to where, from Hom A V, W to Hom S inverse A, S inverse W to S inverse V to S inverse W map is any homomorphism map going to S inverse of f. But this is a module, this is also a module.

This is an A-module, this is a S inverse A-module. So, this one is an A-module and this one is S inverse A module. So, I want to use this general universal property that is why I stated in that general form. So, and hence the map is an A-module homomorphism because we just now checked that if the 0 goes to 0, the addition goes to the addition that means it is an abelian homomorphism and scalar multiplication will come out. So, this is A linear map is obvious.

So, this is an A-module homomorphism and therefore by the above theorem there exists a unique S inverse A-module homomorphism from S inverse of this module, S inverse Hom A, V, W to Hom S inverse A, S inverse V, S inverse W. The unique one such that from here to here we have an iota map. See I will use the different color. So, here to here we have an iota map, iota and I now I will write only suffix s I have to write here this module. That to S inverse and then this but this is same as this so this diagram should be commutative.

So, you follow it by this and then otherwise you can go by that. That is commutative. So, there exists a unique homomorphism. If you give names, so the if this is alpha and this is beta such that iota compose with beta is same as alpha. Alpha equal to iota s compose with beta. So, this map, there are interesting cases where this map is actually this map is bijective but it is not always bijective.

But there are some cases where this map is bijective and some of these I will write as problem in the assignment sheets. So, some more we already noted that S inverse preserve the exact sequences so that is it is map exact sequences to exact sequences. So, I will also note one little bit more general than the last time.

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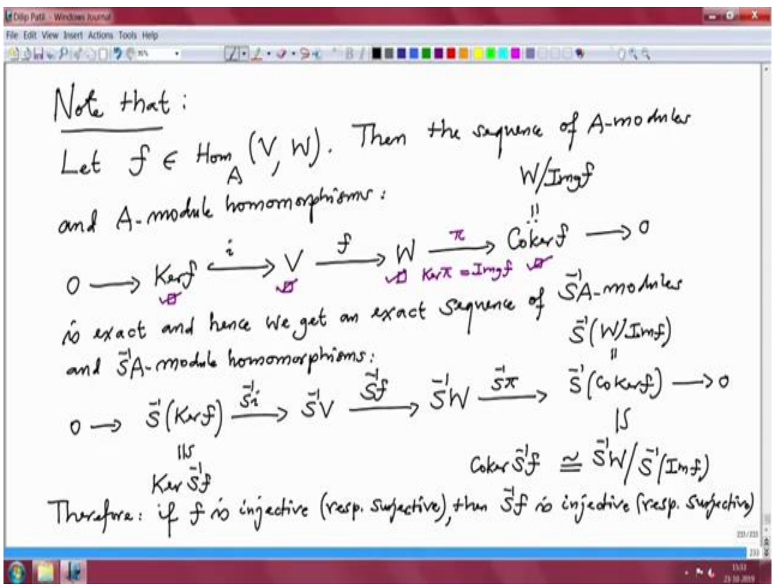


So, note that this statement which will so let f be an A-module homomorphism from V to W. Then you can write down the exact sequence from this homomorphism. Then the sequence of A-modules, last time also we wrote one sequence but that was not so useful. The one which I will write is much more useful. The sequence of A-modules and A-module homomorphism's. So, what is the sequence?

0 from this f we have a Kernel of f and Kernel of f is a sub of V. So, from here to V there is a natural inclusion map and from V to W this is given f. See yesterday I removed W and wrote image of f but now I would not keep that W and where do I go further to what is called Co kernel f to 0 and what is Co kernel f by definition? W module of image of f. This is the quotation module or residue class module of w by image of f.

Now, I want to say that the sequence is exact and hence we get an exact sequence of S inverse A modules and S inverse A module homomorphism's. So, before I write that sequence I want to explain here why this sequence exact? So, we have to check exactness at these places. Exactness here is clear. So, I will write we have to check to here.

But this is clear because here this map is injective. This map is injective that is because this is a natural inclusion. Here we have to check. So, what do we have to check here? We have check that the Kernel here equal to the image here. But this map is natural inclusion map. So, the image is Kernel f under Kernel f as a also Kernel f. So, this is also clear. Exactness here we have to check that this is the usual residue map pi. The kernel of pi we know.

This is the residue class map from w to w module or image of f. So, obviously Kernel of pi is image of f. But this precisely means exactness here and exactness here means this map equals 0 is here. This map should be surjective but the residue class map is always surjective. So, this is also clear. So, therefore this sequence is exact. Now, what is the sequence we will get? We will get a sequence when I apply s inverse to this. So, 0 S inverse of Kernel of f to S inverse of V. This map is S inverse of this inclusion map.

Then here the map S inverse f. This goes into S inverse V to S inverse W and then the S inverse of pi map which is S inverse of Co kernel of f to 0. But this S inverse of Co kernel of f is S inverse of W module of image of f and we saw in the last lecture we saw that the residue class map and operation S inverse, they commute. So, this is isomorphic too. S inverse W mod S inverse of image of f, this is this.

Similarly, we saw that the Kernel of this is also from this exactness. We will know the image of this is precisely the kernel of this. So, this is isomorphic too. Kernel of S inverse f and similarly this is also, this is same as this is also isomorphic too because the sequence is exact the S inverse of Co kernel of S inverse of f.

So, in short we have checked that the Kernel and Co kernel they commute with the operation S inverse. We have checked the (())(32:13) S inverse and this residue class operation they commute with each other but Kernel and Co kernel also commute this operation. Therefore conclusion also one more conclusion I write, therefore if f is injective, respectively surjective, then S inverse f is injective respectively surjective.

This means S inverse operation preserve the injectivity of maps and so preserve the surjectivity of maps. So, now I will make some observations about the chain conditions on the module. What happens to the chain conditions on the S inverse A modules? How we in aerate them from the given module? So, this I will do after the break. Thank you.