Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 32
Consequences of the Correspondence of Ideals (Continued)

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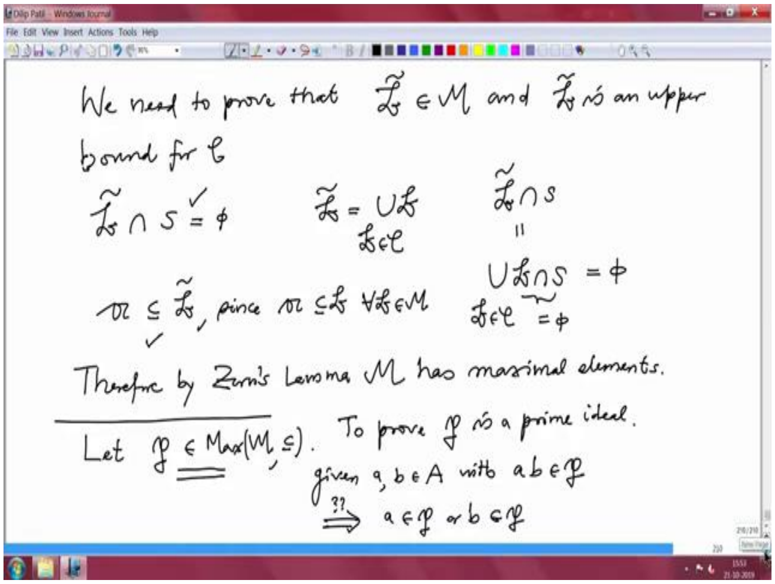

Okay, the next part we have to prove that the maximal elements in m are prime ideals so, let p be a maximal element in this m with respect to the inclusion to prove p is a prime ideal, we need to prove that given a, b in the ring A with a times b is in p, we want to prove a is in p or b is in p, this is what we to prove.

So, I will assume that both elements are not in p and look for a contradiction and contradiction to what we should guess for that p is a maximal element in it m. So, let me go to the next page.

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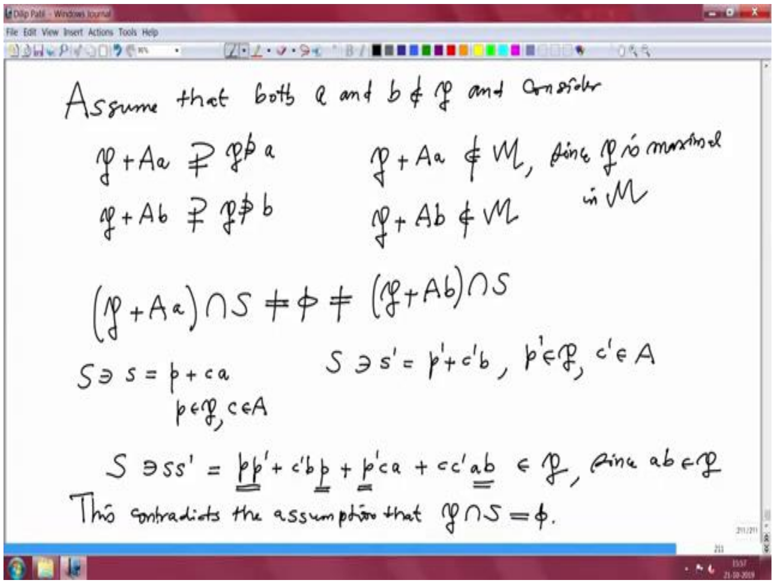

So, let assume that both a and b they do not belong to p and consider p ideal generated by p and this a, that is this and p and ideal generated by b. So, both these ideas strictly contain p that is because a is not here, b is not here, but a is here, a is not here, b is here, b is not here. So, this is a strict containment. Therefore, p plus these ideal cannot be in m because if it is in m that will contradict the maximality of p. Similarly, ideal generated by p and b cannot be in the set m because p is maximal, since p is maximal in m.

What does that mean? That means, it already contains p. So, therefore, the only thing which could go wrong is p intersection p plus a, this intersection S is non empty. Similarly, p plus Ab intersection S is also non empty, but that means, what I choose an element s here which is in S and these ideal, so that means s of the form p prime plus ca and I choose similarly an element s prime, not p prime, p where p belongs to p and c belong to the ring A. And here I choose s prime in s, this is an s, s prime is in s, which is also have the form p prime plus c prime b where p prime is in p and c prime is in A.

Now, when I multiply s and s prime, this is in s, because s is multiplicatively closed, on the other hand I will multiply it out then I will get pp prime plus c prime b times p plus p prime ca plus cc prime ab, but it is clear that this is also in ideal p because pp prime is in ideal p, p is in ideal p, p prime is also in ideal p and ab is also an ideal p by assumption. So here since ab is in p, but then this is in s, so this contradicts, the assumption that p intersection s is empty. So therefore, this complete the proof that the maximal elements in this m are prime ideals.

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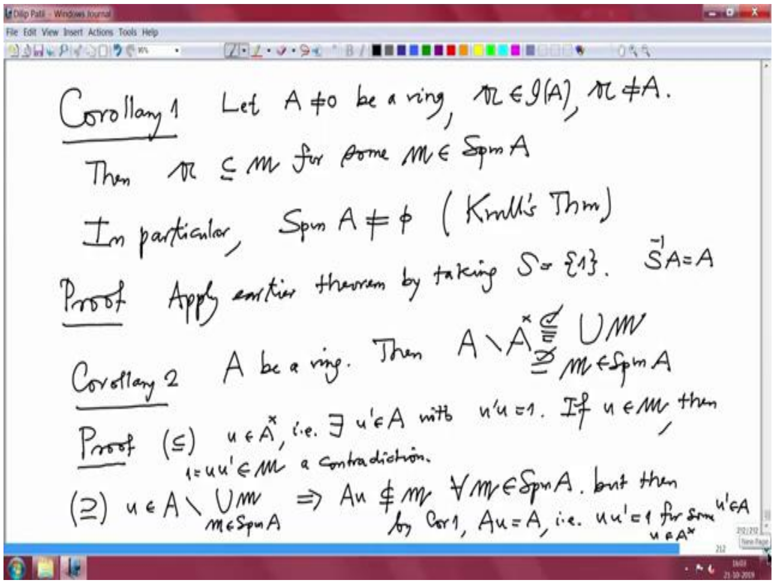

Now let us continue, so, I still want to write some more applications of these theorem now, so, let me deduce some consequences from this theorem. So, let us write it as a Corollary 1. So, let us start with a nonzero ring, let A not zero be a ring and A is a ideal which is not a unit ideal then, a is contained in m for some m belonging to the maximal ideals.

In particular, Spm A is non empty, this is again we have proved, this is Krull's theorem. Proof, earlier we have proved it as a application of Zorn's lemma, of course here in the theorem also we have used Zorn's lemma, but we proved that theorem has a consequence of the localization.

So, similarly, this corollary will be consequence of the earlier theorem. So, apply earlier theorem by taking S equal to singleton 1, then because A is not equal to the unit ideal, this A and S they do not intersect and therefore, I can apply a theorem and then you can say that there exists a maximal ideal actually.

So, next one, next consequence, corollary 2, remember if you take S equal to 1 then what is S inverse A? S inverse A is just given ring A. So therefore, by the next part in the earlier theorem those prime ideals are actually maximal ideals. So, corollary 2, let A be a ring, then this describe the units, complement of the units, A minus A cross this is precisely the union of m where m varies in the maximal ideals of A. Proof, well, if you have this containment first, this containment. So, let us take u in unit that means, there exist u prime in A with u prime u equal to 1.

So, if so, I have taken element which is not here because I omitted here a unit started with u. So, I should prove that it does not belong to any of the maximal ideal. So, if u belong to some maximal ideal, then uu prime will also belong to m, but 1 that is 1. So, contradiction a maximum ideals cannot, maximal ideals are proper ideals, so, that proves this inclusion.

So, this inclusion if I take element in maximal ideal in the union, then similarly I will take a compliment u belonging to A minus union in the m in maximal ideals, I should show you this is a unit but this means, this means u cannot be in any maximal ideal. So, that means ideal au, ideal generated by u cannot be in any maximal ideal for every m. But by earlier corollary, but then by earlier corollary, by corollary 1 au must be the unit ideal. So, because if it were a proper ideal then it is containing some maximum ideal. So, that is uu prime equal to 1 for some u prime in u so, that means u is a unit in A.

So, we have check the compliments, compliment of this right hand side, each content in the compliment of this which is units. So therefore, we have proved this inclusion so, that is already proved. So that proves corollary 2.

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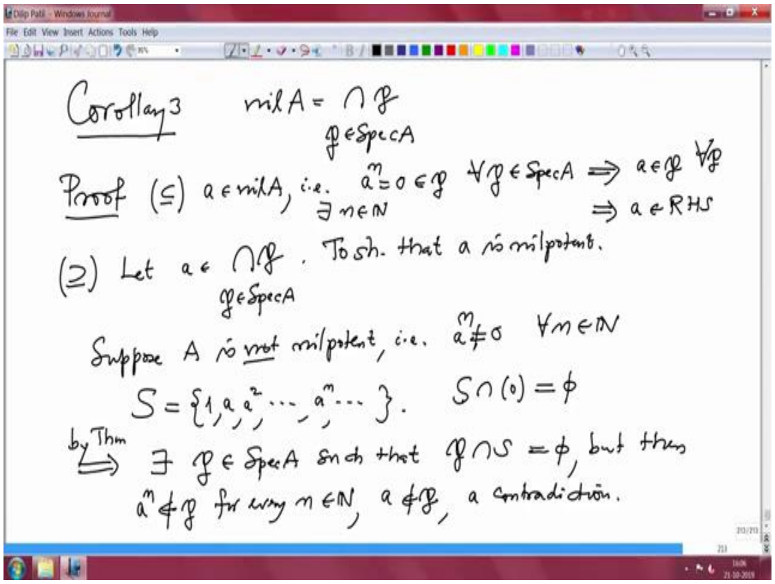

Now, next corollary, corollary 3, this I want to prove again that nil radical of A equal to intersection of p, p belonging to Spec of A. So proof, so first of all this inclusion. So, if you take an element a in the nil A, that is a is nilpotent, a power n is 0, for some n in n there exist n in n, such that a power n is 0, but 0 belongs to p for every p, but p is a prime ideal, therefore by induction, you prove that a belong to p and therefore, a belong to RHS, for every p therefore a belong to RHS.

Now, the other inclusion, I have to prove that an element A which is in every prime ideal, I want to prove that it is nilpotent. So, start with let A belonging to the intersection p, p belonging to Spec A to show that a is nilpotent, suppose not, suppose A is not nilpotent that is a power n is never 0, then we should get a contradiction.

Now, you consider the set S which is a multiplicative set generated by a, that means you take all powers of a and none of the power is 0. So, that means S do not intersect with the ideal 0. So, by the theorem, so by theorem, there exists a prime ideal p such that p intersection S is empty, but then no power of a, a power n will not belong to p for every n, in particular a will not belong to p, but on the other hand we started with a belong to the intersection p. So therefore, this is a contradiction. So, this proves the corollary.

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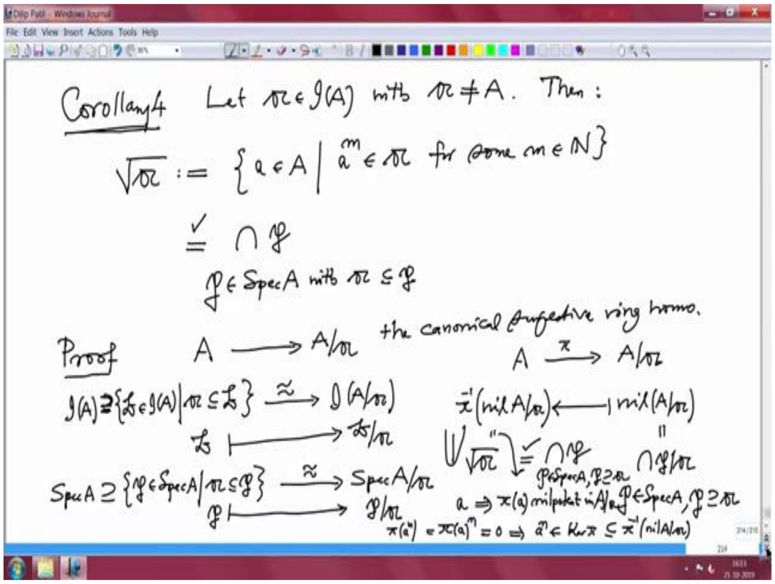

Now, one more so, this is very very important for as far as the geometry is concerned. So, this is corollary, next corollary, so which is this corollary 3, corollary 4 it is, let a be an ideal in the ring A, with a is a proper ideal, A is not equal to the whole ring, then radical of the ideal a is precisely the.

So, radical of the ideal A, what was the definition? Let us recall, this all those a in A, such that some power of a belongs to the ideal A for some m in n, but as you see it is difficult to check what is this ideal. So, this is same thing as, this is the assertion, intersection of all those p where p varies in the spectrum of A with a is containing p, all the prime ideals which contain p if you take their intersection, then you get the radical of it.

So proof, so proof I will recall the following observation which you have already done earlier. So, given the ideal A, we have this residue class map, this is a canonical surjective ring homomorphism, and what do we know about this the ideal structures? So, we know that the ideals of A by a and some ideals among the ideals in A so, that is all those ideals b in ideal of A which contain a where a is contained in b, this set, this is not all set of ideals, this is contained in that, only I am taking this subset and we have bijective correspondence between them, these two sets, namely these b goes to b by a and inverse is obviously the contraction map.

So, this is a bijection not only this moreover, if I consider the prime ideals, so Spec of A which contains all those prime ideal p, p in Spec A, such that a is contained in p, this subset is precisely the spectrum of A by a, this is a bijection under this bijection prime ideals will go to prime ideals of the residue class ring, this is a bijection and these bijections are also inclusion preserving.

So, we know this structure. So, if we look at these residue class map, ring homomorphism A by a, now in this, let us look at the nil radical of a, nil radical of not of a but A by a, this is by definition, it is the intersection of all prime ideals in this residue class ring, but they are of the form p by a, where p is a prime ideal in A and p should be containing a, but when I try to pull back this, so when I, you call this map is pi, then if I take pi inverse of the nil radical of A by a, this is a contraction, what is this contraction?

Then each one of them I have to contract and intersect, so and what do I get? First of all note that this is precisely the radical of a on one hand, because if somebody goes to the nil radical, so this is I am explaining this, this is because if a belongs here, a belongs to this that means pi of a goes so, that means, pi of a is nilpotent in A by a, but that is by pi a power n is 0, but pi a power n because pi is a ring homomorphism, this is pi of a power n that is 0. Therefore, a power n belongs to the kernel, kernel of pi and this kernel of pi is contained in pi inverse of nil radical of A by a.

So therefore, a belongs to the radical of A, a belong to the radical of A, on the other hand, when I intersect each one of them the contraction commutes with the intersection and when I contract each one of them, then I get p. So therefore, on the other hand this is intersection of p where p should be a prime ideal and p should continue. So, we have proved this equality, so this equality is proved, that was precisely the assertion.

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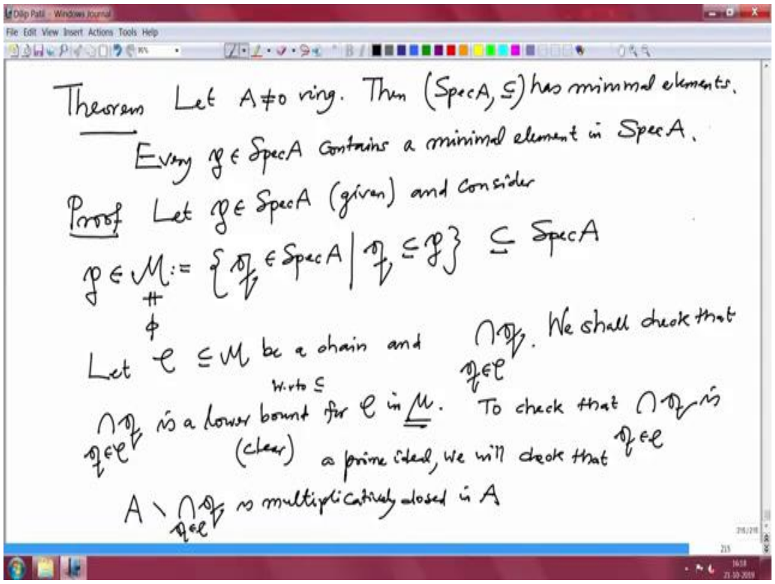

So, that proves this corollary, now I want to prove the existence of minimal elements, this is a very short theorem, so let us prove it and then we will end the lecture here. So, theorem so, let A be nonzero ring, A equal to nonzero ring, then the ordered set Spec A with inclusion has minimal elements.

So, in fact we will prove that every prime ideal p contains a minimal element in spec A. Proof, consider so, let us start with the prime ideal, so let p be given and we know such a p exist by earlier discussion and consider m which is all those prime ideals q, such that q is contained in p. This is a subset of prime ideals and it definitely contains p. Therefore, this is non empty.

Now, I want to check that this has every chain. So, let us start with the chain because if I want to conclude it has minimal elements I want to apply Zorn’s lemma, but I will apply Zorn’s lemma not to the m with usual inclusion, but I will apply it to the opposite set. So, I will prove that every chain has lower bond.

So, let c contained in M be a chain and consider this intersection, intersection q, q belong to c, we check that, we shall check that q, we shall check that this intersection q q in c is a lower bond for a c in M. So, that means I have to check that first of all this element is in M and it is lower than every element event in c.

So, it is clearly, this is clearly a lower bond that is clear, this is clear because it is intersection, it is contained in every q in c therefore, it is a lower bond with respect to this inclusion. And now to prove it is in M, I have to check it as a prime ideal, but note that to check that, to check that intersection q, q in c is a prime ideal we will prove, we will check, check that the compliment is multiplicatively check A minus intersection q q in c is multiplicatively closed in A, but what is the compliment?

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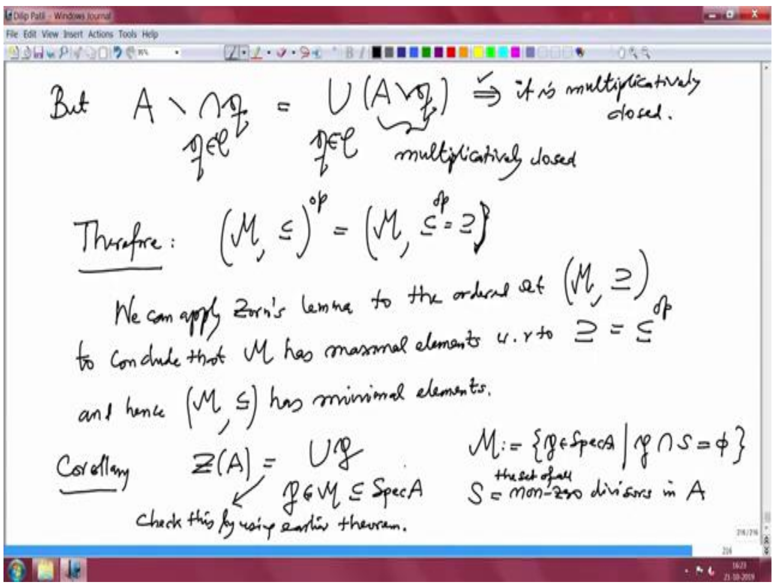

But this compliment, but A minus intersection q q in c this is nothing but and this will become union and A minus q q varies in c and each one of them is multiplicatively closed and it is a chain therefore, the containment relation between them. So, therefore, this is multiplicatively closed that implies it is multiplicatively closes because first of all 1 belong there.

Secondly, if I have two elements here, it will belong to two elements let us say but I will choose one of the q both the q and suppose it is q prime they are comparable. Therefore, these complement will have containment relation among them. Therefore, I will work in a bigger set which is multiplicatively closed therefore, the product will be there so, this implication is very easy to check.

So, we have check so, that means we have checked that c has the lower bond in M. Therefore, if I take now the ordered set this opposite, opposite ordered set that this is nothing but the set is same, but the order is opposite. This means it is this so, inclusions will get reverse that mean this set, any chain in the set that minimal will become maximal.

So therefore, I can apply Zorn’s lemma to set, so therefore, we can apply Zorn’s lemma to the ordered set M. This to conclude that M has maximal elements with respect to this which is opposite of the natural inclusion and hence M, this has minimal elements that was assertion.

Now, as a result, one can easily prove corollary, the 0 divisors of the ring A equal to union p where p belongs to M, which is a subset of, in fact, we can take this M to be the. So, you can take M precisely to be all those prime ideals p, such that M intersection, p intersection S is empty, where this S is, so S is nonzero divisors in A, the set of all nonzero divisors, which multiplicatively closed set.

And now apply earlier result and you will get this equality, right now we do not know this M is a finite set or not, okay right now we do not know whether M is a finite set or not but we will prove that if A is Noetherian, then these M is actually finite set and I will just simply write here check this equality by using, by using earlier theorem.

Earlier theorem says that this set has precisely the property that all the (())(31:33) are in a 0 divisor so, check this. So, with this, I will stop for today and we will continue in the next lecture, localization of modules, which is also very important topic and after that, we will then start geometry again, the geometry of the maximal spectrum of the ring and the prime spectrum of ring. Thank you very much.