Introduction to Algebraic Geometry and Commutative Algebra
Professor. Dr. Dilip P. Patil
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 31
Consequences of the Correspondence of Ideals

Welcome to this lectures on Algebraic Geometry and Commutative Algebra. In the last lecture, we have studied the ideal structure in the localization of a ring. So, let me recall briefly what we studied.

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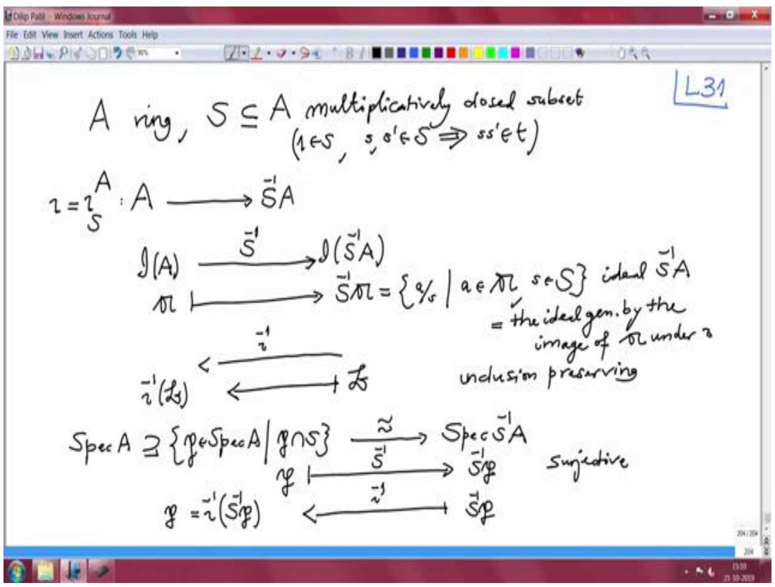

So, A is our ring commutative always. And S is a multiplicatively closed subset of A, that means 1 is there, 1 belong to S and whenever two elements are in S, their product is also in S. And then we have using this multiplicatively closed set, we have constructed in new ring S inverse A, which is called ring of fractions of A, where elements of S have now become invertible elements and also we have constructed a ring homomorphism from A to S inverse A and this we have denoted by iota suffix s upper A, but I am going to aggregate this for iota, whenever S is fixed and A is fixed.

So, this is ring homomorphism and then we were comparing the sets ideals of A and ideals of S inverse A. And what we did was we define a map here, that map is also let us call it S inverse only, for each ideal A we have defined this S inverse A, S inverse A, this is by definition all those A fractions a by s such that a varies in the ideal A and s varies in the subset S. And we have checked that, this is indeed an ideal in S inverse A, in fact it is the ideal, this is the ideal generated by the image of A under iota.

And we have noted that if A is A do not intersect with S, then this is a proper ideal and also we have, we have a natural map in the other direction, namely the iota inverse, this is any ideal b in S inverse A, we can always pull back to ideal in A, that is simply iota inverse of b, and this is ideal in A. And we have also noted that, this correspondence become better, when you study prime ideals.

So, what we did was, we have the spectrum, prime spectrum of S inverse A, this and we have a prime spectrum of A, and among them we have taken all those prime ideals in A, which do not intersect with S. Then there is a bijection here, and the bijection is this coming from this map S inverse and iota inverse.

This bijection is also this earlier maps are, all these maps are inclusion preserving, all the maps are inclusion preserving, that means whenever A is contained in A prime, then S inverse A will be contained in S inverse of A prime.

So, this is what we did last time, this bijection, you can any P here, any P here goes to S inverse P. And we approved this map is surjective and therefore, we can define a map on the other direction, that S inverse P goes to iota inverse of S inverse P, and this map is this is equal to p. That mean, this S inverse and iota inverse, these maps are inverses of each other. So, this is what we did, now I want to deduce some consequences of this.

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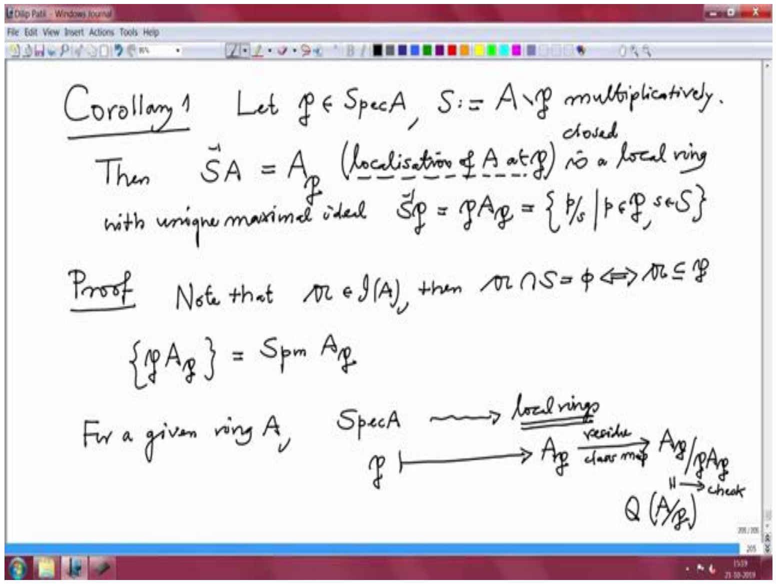

So, the first one, that is I will call it corollary 1. So, let us seeks a prime ideal, let p be a prime ideal in the ring and let us take S equal to the complement of p, then obviously this is multiplicatively closed, that is a definition of prime ideal, in fact the other way is also true. If complement of an ideal is multiplicatively closed, that mean, that ideal should be a prime ideal. This is almost a definition.

Then, the S inverse of the ring A, this we denote it by A suffix p and also it is called localization at P of A at p is what we called it. And then this ring is called its localization at p, this ring is a local ring, is a local ring with unique maximal ideal S inverse of P, and this is also denoted by pAp, which is by definition all those elements p upon s, fraction p upon s. So, where p varies in the ideal P, and s varies in the multiplicative set S.

So, let us proof this, so note that, if I have an ideal, so note that if I have an ideal A, in A, then in this case because S is a complement of P, A intersection S is empty, if and only if A is contained in the compliment, A is contained in P. This is very clear because A is the complement of P, therefore pAp is the only maximal ideal. So, this is Spm of A localized at p, so that true the certain that it is local and pAp is the only maximal ideal. So, that is the statement.

So, now what we have done, so this corollary has a good implications. So, that means whenever I have a ring A, whenever for a given ring A, we have defined a map, from spec A, to a collection of local rings, any p goes to pAp Ap, A localized at p, this is a local ring.

And whenever we have a local ring, we have seen in earlier lectures, there is a unique maximal ideal and therefore there is unique residue class field, that is precisely Ap localized p, modular the maximal ideal pAp and we have this natural residue class map. And one can check this is the in fact the quotient field of, the integral domain A mod P, this is the quotient field, this equality we should check.

So, for each prime ideal, we want to understand a local ring and that we will give you the local information, when we go back to study of topology, Zariski topology. So, this we will take up after we have finished this section on the localization of rings and modules.

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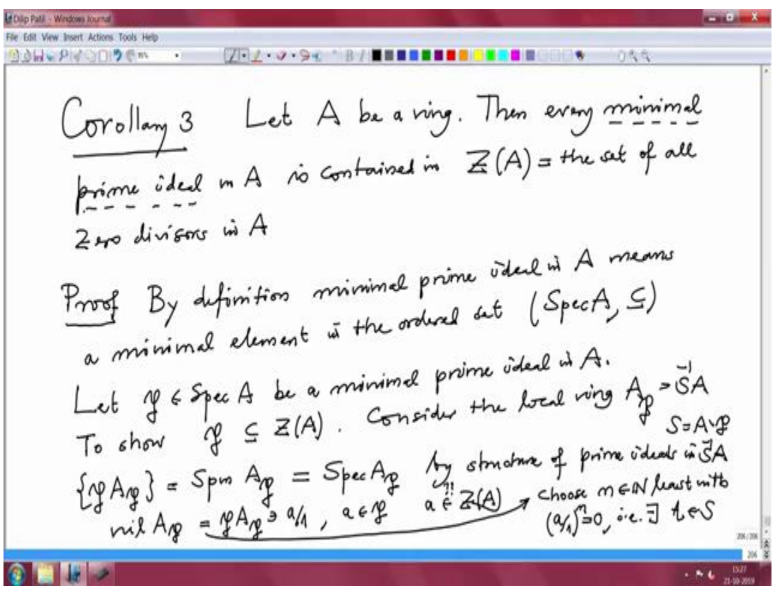

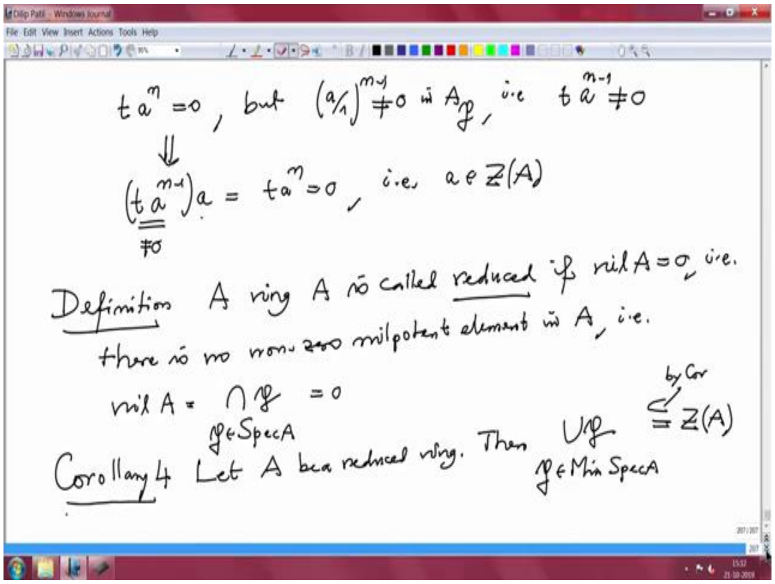

So, next corollary, corollary 3, so let A be a ring and as you will see this course will send a structure to study the spectrum of a ring, and various aspects of the spectrum of the ring properties, geometry, extra. And therefore, many times we will make a statements about the spectrum. So, then every minimal prime ideal in A is contained in ZA, where ZA is by definition the set of all zero divisors in A.

Proof, okay, before I start the proof formally, I would like to make a comment about this minimal prime ideal. I want to make a comment about minimal prime ideal, that is by definition, by definition, minimal prime ideal means minimal elements, minimal prime ideal in A means a minimal element in the ordered set spec A and with the natural inclusion, this is ordered set and minimal elements means that is minimal with respect to the inclusion.

Whether do they exist or not, that is a question and we will prove at the end of this lecture, that minimal elements in the order set spec A with inclusion exist, but we do not know right now they are finitely many, or how many, and when we will prove later that when the ring is Noetherian, the set of minimal prime ideals in a ring is a finite set. This is very useful and these minimal prime ideals will corresponds to some irreducible components in a Zariski topology of the spectrum of the ring A.

So, right now our definition of minimal prime ideal is, it is the definition that it is precisely the minimal element, a minimal element in the spectrum. And we want to prove that this P, so let P be a minimal P in Spec A be a minimal prime ideal in A. Then to show p is contained in the set of zero divisors of A. And as we are writing it as a corollary to the localization ideal structure of the localization. That means, we have to use, we need to use somewhere localization.

So, look at, so consider the local ring A localized at p, remember this is S inverse of A where S is the complement of p and this is a local ring and the maximal ideal is pAp, pAp precisely is the only maximal ideal Spm of Ap, this is noted in the earlier corollary. And it is minimal and then we know that, there is a one to one correspondence between the prime ideals, which do not intersect with the multiplicative set S and the original ring.

So, therefore, because this P is minimal, there cannot be anybody who is contained in, properly contained in, in the ideal P, therefore all prime ideals will have to intersect with S, therefore they had showed that, this maximal spectrum is also equal to the spectrum, that is the only prime ideal. This is by structure of prime ideals in S inverse A, but then because only one prime ideal and that is this.

So, therefore what is a nil radical of the local ring? Nil radical of A localized at p will precisely be that pAp, that means every element in this ideal is nilpotent. So, take any element a by 1, for example, this that is we started with an element. So, we so, where a is in p, a is arbitrary element in p and I want to show that, this is a belong to 0 divisors of A, this is what we want to show and we consider for that we consider a by 1, a by 1 is in p by Ap by definition and therefore it is nilpotent.

So, choose because of this, because of this equality, choose n in N least with a by 1 power n is 0, that is this is 0 means there exist an element t in S with, there exist an element in t and S.

Such that t times a by, a power n is 0, but we know that, but we know, because we are chosen n least with that a by 1 power n minus 1 is not 0 in A localized p, that means t times a power n minus 1, this is nonzero and this equation will then tell you t times a power n minus 1 times a, which is t a power n which is 0. So, that means, this element a is killed by this element, which is nonzero, so that is a is a 0 divisor. So, that proves the statement.

Now, the next one, before I define the next one, I want to, before I continue, I want to define definition. A ring A is called reduced if the nil radical of A is 0, that is there is no nonzero nilpotent element in A. So, by the earlier characterization that also means that we know, we have proved that nil radical of A is the intersection of all those p. So, that p in spec A, so this is 0, that mean the ring is reduced. If you are ring is reduced, then I want to prove.

So, next corollary, that is corollary 3, let A be a reduced ring. Then the union of minimal primes, where p belongs to Min spec A, this is precisely the zero divisor of the ring A, remember that we proved in earlier corollary that every element, every minimal element is an element in minimal, every minimal prime ideal is contained in the zero divisors, that is what we proved but here.

So, therefore, by that this inclusion is clear, this inclusion is clear, this is by earlier corollary, by corollary 3, 2, 3 or 2 I do not know what is it, let me check, that is 3. So, this is 4, this corollary is 4, so this is by corollary 3, so we have to prove for the other inclusion.

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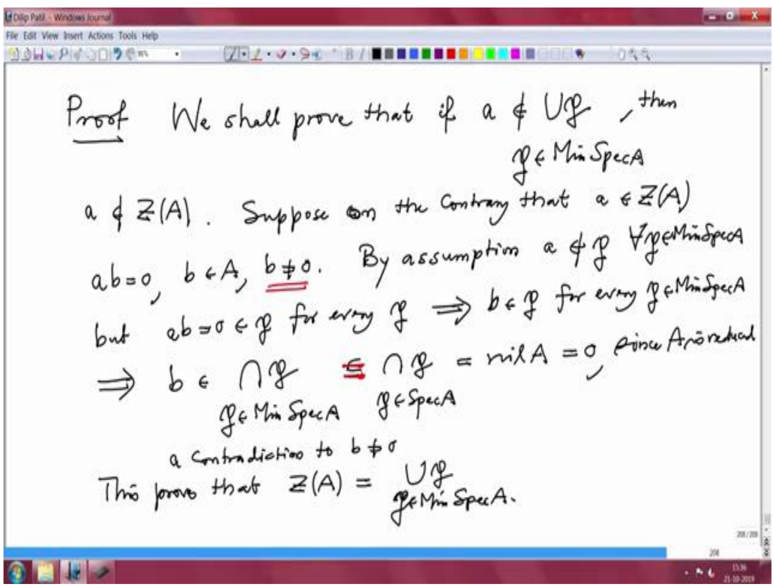

So, proof for the other inclusion. So proof, so I am going to prove that, so we shall prove that, if a is not in the union, where p is a minimal prime in the spectrum, then A cannot be a zero divisor, this what we will prove. So, it is like a indication, contra positive. So, let us take A nod in the minimum, a union of the Min of spec A and suppose, so suppose on the contrary that a belong to zero divisor, that means a times b is 0, for b in A, b nonzero.

Now, we should get a contradiction, so then note that a b, a not in the union means, a not in any one of them, by assumption, a does not belong to p for every p in Min Spec A, but a b which is 0, this belong to p for every p in fact, in particular for the Min, but then if your product belong to prime ideal, then one of them belongs, but a cannot belong, therefore b has to belong.

So, therefore, b belongs to p for every minimal, for every p Min spec A, but then b will belong to the intersection of p, p, belong to Min spec A, but this means this is contained in p, p belong to space A, in fact it is equality here, because of minimal prime ideal, every prime ideal will contain 1 minimal, therefore the intersection will be equal, but this is nothing but, this is nothing but the nil radical of A, but A is reduced, therefore nil radical A is 0. Since A is reduced.

So, that proves b is 0, that is a contradiction to b nonzero. So, a contradiction to b nonzero, so this proves that intersection, Z A 0 divisors of A, must be the union of p, where p where is in Min spec A. So, that proved this corollary 4.

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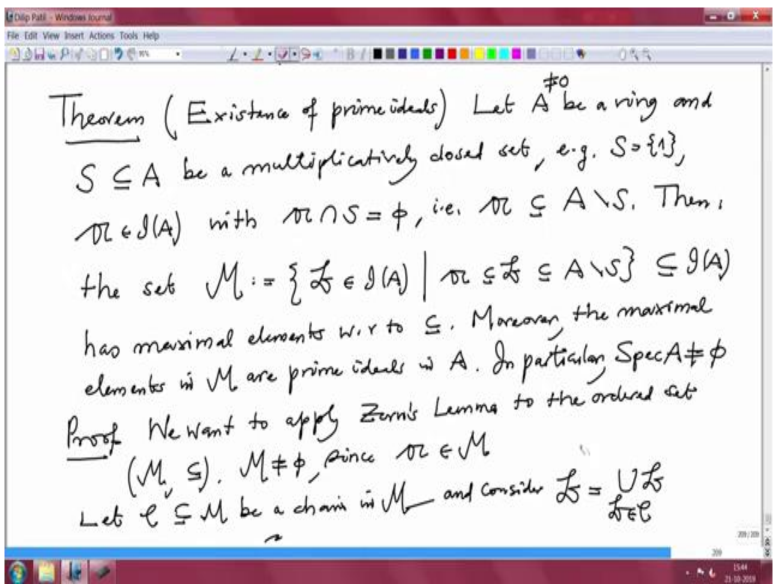

Now, I want to prove a theorem, which is so this theorem, this is very important theorem actually we have proved, we have not proved this but we have approved directly that, there are prime ideals by proving there are maximal ideals. And every maximal ideal is prime ideal, therefore that was precisely the content of the Krull's theorem but this is directly proving that, there is always a prime ideal.

So, this theorem is also one can say it is existence of prime ideals, and we will prove this is an application of the localization. So, let A be a ring and S contained in A be a multiplicative closed set, for example you could take S equal to singleton 1, of course we are assuming here note that we are assuming A is nonzero ring.

So, and a be an ideal in A with which do not intersect with S, A intersection S is empty, but that means that A is contained in the complement of S, with these assumptions the set M of all those ideals b in A, such that A is contained in b, which is also contained in A minus S, this set M, which is a subset of ideals of A has maximal elements with respect to the inclusion.

Moreover, these maximal elements, the maximal elements in M are prime ideals in A. In particular, spec of A is non empty, note that, this also proves the fact that spectrum of A is non empty.

So, this is little bit more general then the Krull's theorem, but this does not assure you that there are maximal ideals, any way but Krull's theorem show that there are maximal ideals and therefore prime ideals. So, let us prove this, this is as one can guess, this is a simple application of Zorn’s Lemma. So, I have to, I have to justify that I can apply Zorn’s lemma.

Proof, we want to apply Zorn's Lemma to the ordered set M with inclusion, first of all note that M is non-empty, since the ideal A belongs to M and all ideals which belong to M, they do not intersect with S by definition of M. And to apply Zorn's Lemma, we need to check that, that is this set is inductively ordered that means we need to check that every chain in this set M has an upper bound. Then we can conclude by Zorn's Lemma that M has maximum element.

So, let C be a chain, C contained in M be a chain in M and consider, and consider b, which is the union of, b which is union of all b, b tilde let us call it b tilde, which is union of all b in b varies in the chain C and this is b tilde. And I will show you that these b tilde is an upper bound for C in M.

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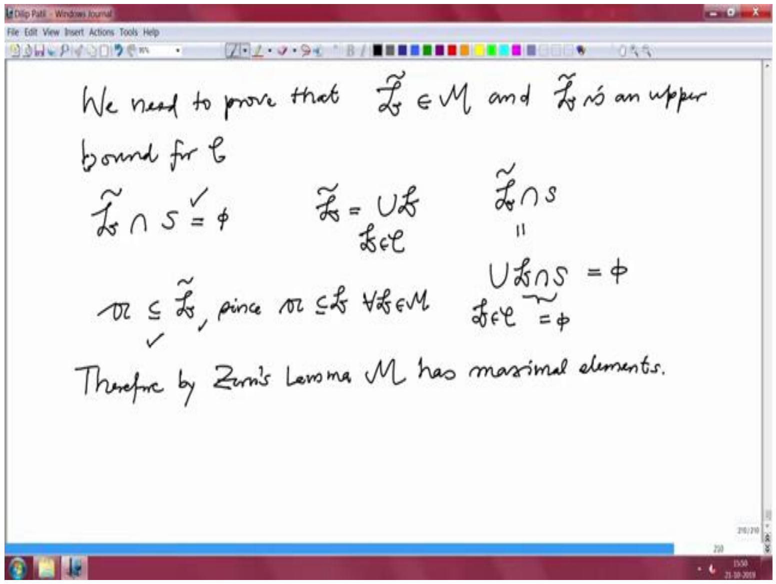

So, we need to check that b belong, b tilde, so we need to prove that b tilde belongs to M and b tilde is an upper bound, clearly this b tilde is in upper bound for C by definition of b tilde, because b tilde contains every ideal in the chain C. Now, to prove that b tilde belongs to the set M, you have to prove two things namely, this b tilde. So, we need to prove that b tilde intersection S is empty, but this follows from the fact that, b tilde is the union of all b's, b in C and each b.

So, when you intersect b tilde with S, this is same as union b intersection S, b varies in C, but each one of this is empty, therefore this is empty set. So, we know, we proved this. Second thing we need to prove that A is contained in b tilde, but again because this is clear, since A is contained in b, for every b, in M, in particular for every b and C and b tilde is a union of all those b, b in C, therefore this is also clear.

So, we proved that a chain, the given chain has an upper bound in M, therefore Zorn's Lemma, it has, M has maximum elements. So with this, we still have to prove that, the next part that these maximal elements in M are prime ideals, this I will prove after the break. So, thank you and we will continue the proof after the break, thank you.