**Introduction to Algebraic Geometry**
**and Commutative Algebra**
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**Lecture 27**

Welcome to this course on Algebraic Geometry and Commutative Algebra, in the last couple of lectures we have been studying modules, modules which n conditions and so on. Now, I will come to a topic which is more useful for geometry, it is called Rings and Modules of Fractions or Localization.

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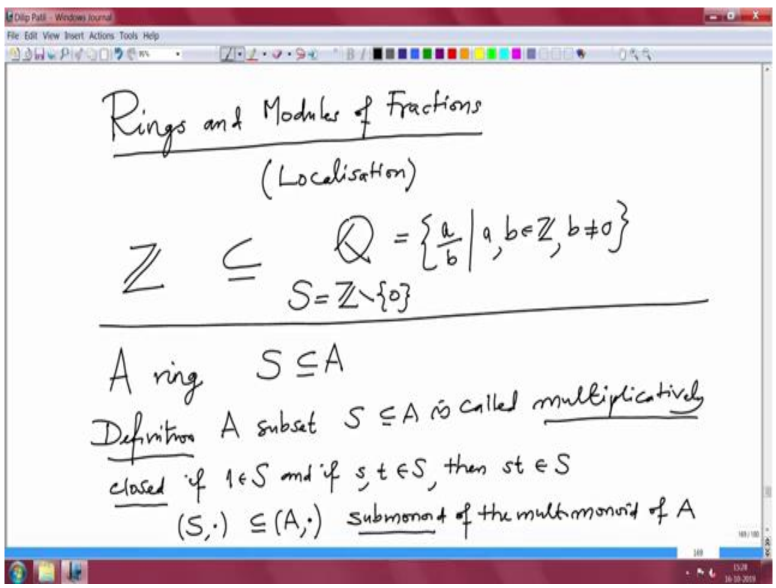


So, let me write the title, this is rings and modules of fractions, or also it is called as localization. This is one of the most important technical tool from commutative algebra which is use in algebraic geometry and it corresponds to the attention of a particular open set and also particular point in an open set.

So, it gives a local information about a point in a topological space. That is one of the reason why it is called localization and why is it called fraction is it is a process which is modelled on the following examples. Likes suppose, we have this integers, ring of integers and from there we have constructed rational numbers and how was it done? These are fractions so, these are a by b, a, b are integers and b non-zero.

These are fraction but then, the way I have written they may not be in a lowest fractions, lowest form. But this is very useful when you want to concentrate it locally. This will be clear from the examples. So, todays notation we be it is more general than this so, our ring may not be even integral domain so, whatever standard things works here it may not work in general. So, we have to be very careful in the definitions.

Alright so, let us start. So, what am I going to do? So, A is a rings always commutative remember and S is a set, S is a subset of A, which I will call it, S is called so definition, subset S in the ring A is called multiplicatively close, close. If one belongs to S and if two elements s and t belongs to S, then the product also belongs to S, that means S is closed under multiplication of the ring A and 1 is also there.

So, in other words, another word is that S with the dot operation, dot is a multiplication in A, this is a submonoid of A dot, submonoid of the multiplicative monoid of A. So, that is a submonoid means the identity should be there. So, 1 is there and it is a closer under product.

Now I am going to construct a new ring, where these elements of S will become units right, but also one has to be careful that ring also might become zero. So, we will see precisely what happens when you do this operation. So, what are we going to do? So, I am constructing a new ring. Alright so, on the set, so, the further construction of the ring. And in this example, in the above example, this here we have taken the set S equal to all non-zero integers, Z minus 0. So, everybody got inverted.

So, that became a field so, this is typically will happen when this is an integral domain when A is a integral domain and if you take S equal to all non-zero element, then the new ring actually will be a field, in fact it will be the smallest field that is also called quotient field of the integral domain. Okay so the next one, so, what is the construction, is as follows.

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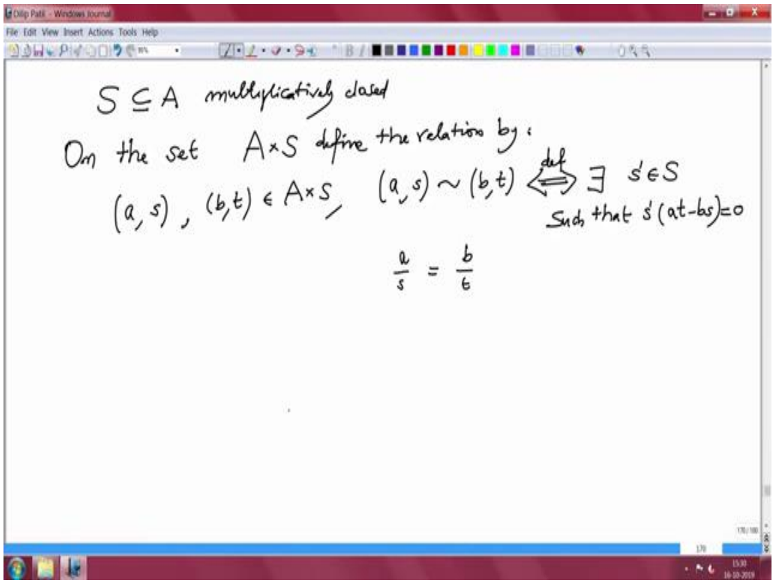


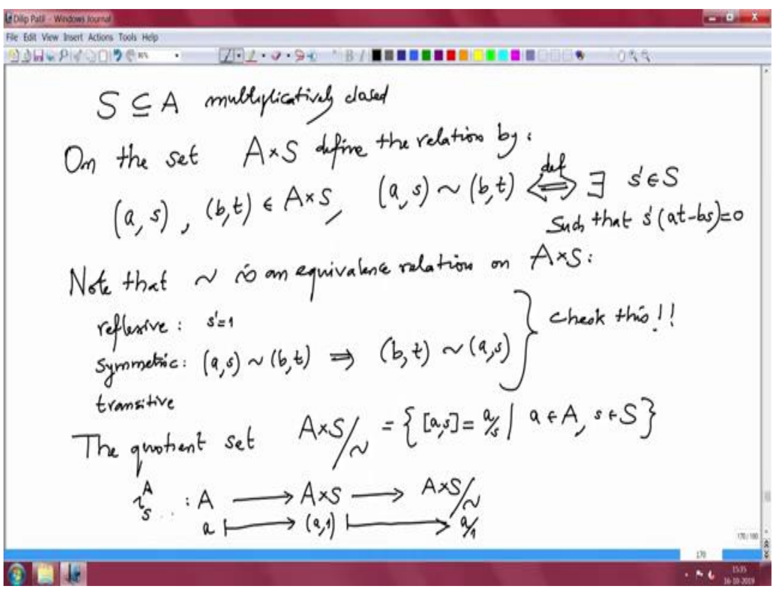


Alright we fix this notation throughout the lectures, S is a multiplicative close set in A, sometimes I will just write multiplicative, multiplicatively close then on the set A cross S define a relation, define the relation by, see if I have two elements here, a coma s and b coma t in A cross S we will say that they are related to the tilde b comma t. This is a definition, if and only if there exist S, there exist S prime in S such that a t minus b s multiplied by S prime is 0.

Now, remember that I have added this extra condition S prime in S. So, that this became 0, actually think of this as a fraction a by s, this is a notation I am writing, in the rational number case we wrote like this and when they were equal when you cross multiply there were equality. So, that is this, but I have to add this condition because your ring may not be integral domain, so, it might have zero divisor.

So, to take care of that, I have added this. Now, first of all, note that this tilde is an equivalence relation, relation on the set A cross S. So, for this, what do we have check? We have to check it is reflexive, we have to check it is symmetric and also we have to check it is transitive. So, for these you need to choose S prime, in that case you choose S prime equal to 1 and 1 is in S so, that is okay, the details you fill it up. So, in this case what do you have to check? If a coma s is related to b comma t then you want to check that b comma t is related to a coma s.

But then the same, same and we are in a commutative case so we do not need so, S prime the whatever this should imply b comma t related to a comma s this is very easy, the same S prime will work because it is commutative so, how you write these relation, whether you write this term first or that term first it is the same.

Alright so, that is an transitivity in the product of elements will work. So, I would simply say check this formally, commutative is very-very important because this concept does not exist in non-commutative rings and that is one of the drawbacks of this subject that such a construction does not exist there.

Alright so now, whenever we have the equivalence relation on a set we consider the quotient set. So, the quotient set that is A cross S mod this relation, this is the set of equivalence classes. So, the equivalence classes I will denote so, this is a set of equivalence classes. So, these are sometimes it is denoted by square bracket a comma s.

But this I will denote by A over S like a fraction where the numerator a is arbitrary element in the ring A and the denominator in this subset S and what is the quotient map? The quotient map so, that is A cross S to A cross S modulo this relation and then we have an natural map here A to this, this is a natural map any a here goes to a comma 1 and this goes to its equivalence class which is we have denoted by a by 1. So, therefore, this is the map from the ring to this.

Now, I am going to put a ring structure on this quotient set by using the ring structure of A and then study this new ring and that new ring has nice properties compared to the ring, original ring A. For example, when A is an integral domain this new ring and if S everybody other than 0, then this new ring will become a field that will be called a quotient field.

In fact it is the smallest quotient field and then there is a universal property for this homomorphism and this composition, I will denote this by iota S and A and when the things are fix, I will just write it Iota, when there is no confusion, but this notation should consist what it S and what is A and so on.

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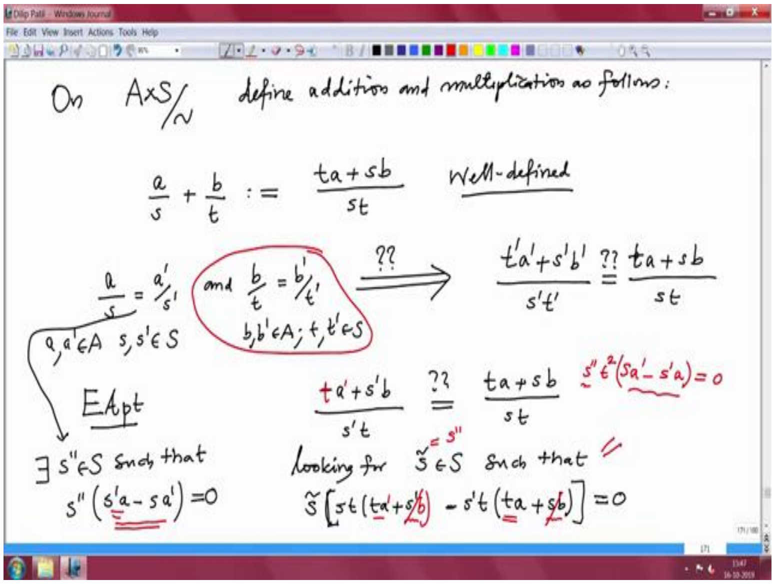


Alright so we have to put the ring structure on this, this is also denoted by, so, on this quotient set we want to define, on this define addition and multiplication as follows. So, take any two elements, a by s plus b by t and define this to be, so, how will you define usually the fractions? So, that s t and then t a plus s b, so it makes sense. Now what is very important is one has to check that this is well defined.

So I will check for one, that means what? This does not depend on the equivalence classes, this does not depend on the representative elements of the equivalence class. So, for example, it could be a by s equal to a prime by s prime. When you write like this denominators are in s and numerators are arbitrary elements in a, a, a prime are arbitrary element in a, then we should also check that.

And similarly for b by t so, b by t equal to b prime by t prime, b, b prime are in a, t, t prime are in s, then from here we should check, I am checking, I am trying to check this, this if I use this representation then, we have defined this, if I would have used this representation, then what would we define? T prime, a prime plus s prime, b prime divided by s prime t prime, this also should be equal to t a plus s b divided by s t, this equality we have to check, then only it will mean it is well defined.

So, let us check that and this checking will first, I will first check for this and then we will check for these, so it is really enough to check one step at a time. So, I will only take this equality and check the corresponding thing too. So, let me write this so, it is enough to check. So, this will be two steps, enough to check that.

So, I do not have this so, I will use only this. So that is t a plus s prime b prime divided by s prime t equal to t a plus s b divided by s t, this is the only one we have to check, because the next I will use, I will keep the same and use this. Alright so what are the meaning of the, what is given to first of all and what is to be proved, if we write down it will clear.

So, we have given this so, that means what? These two elements are equal means the cross multiplication, difference and multiplied by some suitable element in s, it becomes 0. So, there exist so, these equality here means, there exist s double prime in s, such that s double prime times s prime a minus s a prime, this is 0. This is what we have given and what do you want to prove? We want to prove this is equal.

So, that means, we are looking for, I will write here, looking for some element in s. So, I will write here s tilde in S such that, when I cross multiply difference than multiply by s tilde it is 0, S tilde times this s t times t a plus s prime b minus s prime t is t a plus s b this, this should be 0. Now just stay ready and you to simplify this expression.

So you see here this is s t, t a, this is also ta here and we have given this right, so if you want to, see if you want to get this term where is it? Did I write correctly? Yes, alright. So what can we do? So, from this equation, there is here t a and this is another t a and this s prime, s prime a and this so it is this term and there is something wrong I have written.

There is something wrong I have written, a by s plus b by t there is no b prime that is what I was, see there is no b prime here. So, what is that we wanted to prove? We have given this and this is not there for us, because we are taking only b by t. So, when you add to b by t here that is what will happen, a by s plus b by t is this, this side, that is this side.

Other side is b by t we have added to the side so, that is t prime a prime. So, that was a mistake. So t prime a prime, not t prime sorry. What was that? So, that it this side so, t a prime this should be t, t a prime minus s prime b minus, no plus we are adding them t, t a prime plus s prime b prime that is correct divided by s prime and t, this is correct.

So, this was a prime here. So, and what do you want? You want this to be equal that means, s t times this so s t times t a prime plus s prime b prime minus s prime t times, t a plus this is equal. So, this is what we want to check and so, we want to rewrite this so, I want to rewrite in this way. So, s prime a, yeah that is here. So, this term we are looking for this and where is this term?

That is this and what is the extra item? s prime t s b, there is no b prime here. So that was the confusion so, this a prime by s prime that is b by t so, it is t a prime minus s b, that was the confusion there is no so, so it is like this. So, here it is there is no b prime so, this I will erase, this is b this, you see now this is getting cancelled, this term is getting cancelled with this term, see st s prime b, s prime ts this is getting cancelled.

So, this got cancelled and then now, what do you wanted is precisely multiply by t and then you can take so, this is simplify you are looking for this right. So, this is what? This is, simplification is, a prime, what is the simplification here? S a prime that is this term multiplied by t square and the same t square is coming here also. So, this minus s prime a, this and this multiplied by some more things you wanted to be 0.

But already you see this is 0, so, minus of this is also 0. So, minus of this when I change the sign it is precisely this and if I multiply by s double prime it is 0, because s double prime times this and this is 0. Therefore, further multiplied by t squared is also 0. So, therefore, this is 0. So, that means, what is the s tilde I can take, s tilde I can take to be so, take s tilde to be equal to s double prime.

So, that is calculation like that. So, that showed that this plus is well defined and you see it is use, it uses the addition in the ring so, it is associativity is also clear. So, I will not take associativity, I will also not take the general thing. Now, let us go on to define multiplication.

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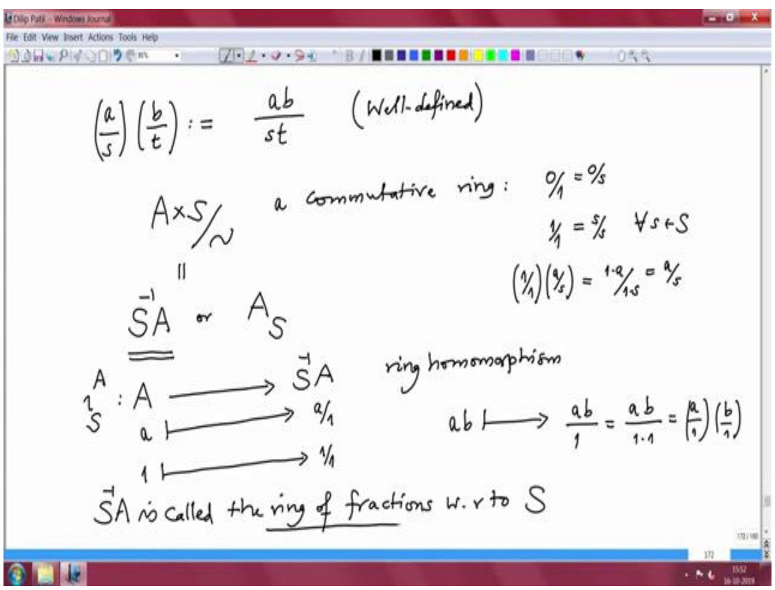


So, multiplication you define by a over s times b over t define this is equal to as usual a b over s t. This is very natural and now we have to still check it is well defined. Whenever you define, whenever you any definition on the quotient set you have to check that it is independent of the representatives set, representative element of the equivalence class. So, that will make this A cross S over this quotient set a commutative ring.

So, and what is the 0 element in this ring? So, additive identity is 0 by 1 because 1 is in S, this is additive identity and multiplicative identity is 1 by 1. So, these are multiplicative identity, but remember this 1 by 1 is also equal to any S by S, for every S in S. And 0 by 1 also can be equal to 0 by S. So, this also can be 0 by S, for any S.

So, this is a new ring and now I am going to show you the properties of this new ring, this is a compulsion notation. So, there are short form for the notation which is also more suggestive. This also is denoted by S inverse of A just because we will check that the elements of S became the unit that is very easy.

So, we will check that or also it is denoted by A suffix that S, but standard notation use in most of the books is this. So, I will stick to this notation alright so, and the map we have defined this commutative ring and the map, that the map we have seen A to S inverse A it is natural map, any A going to the fraction A by 1. This map is denoted by Iota capital above A ends here small s. This is the ring homomorphism.

Let us check that quickly. So, whenever you want to check if it is ring homomorphism we have to check that identity element goes to identity element. What is the identity element? Here is 1 and where does it go? To 1 by 1 and we have just said that this is a multiplicative identity that is also very easy to check we should have checked here.

So, 1 by 1 times any element a by s in this ring that is by definition, it is one times a divided by one time s, which is a by s. Similarly, I mean the other thing we do not have to check because it is commutative. Similarly, you could check that this 0 by 1 is an additive identity. So, this is a ring homomorphism. So, we are passed on from this ring, also the multiplication.

So, where do a b go? Ab goes under this to a b by 1. But that is same thing is a b by 1 times 1 which is same as a by 1 times b by 1. So, that means Iota of a b goes to so it is a ring homomorphism. So, we are going to study this new ring and its properties. So, the most important property is, okay this ring is called this S inverse A is called the ring of fractions with respect to the multiplicative set S.

Alright so what is the most important property? The Universal property, okay before I go on I should give some examples.

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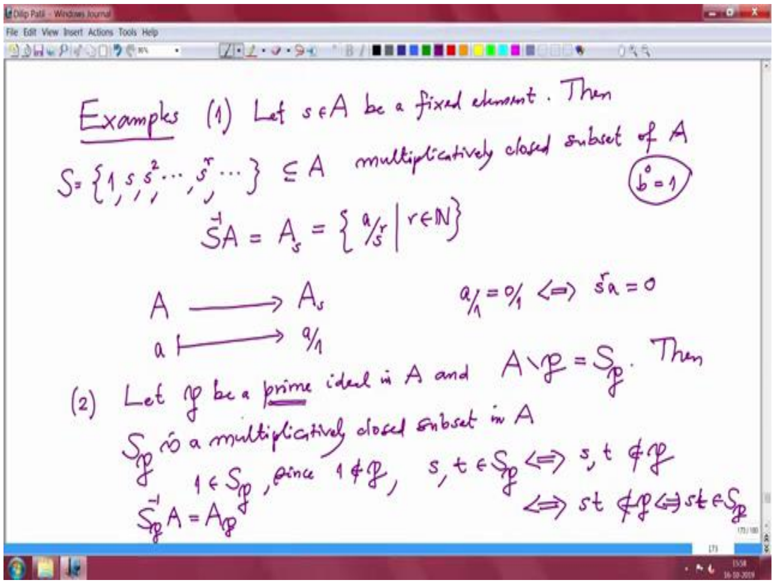


How do you, which multiplicatively sets are interesting? So, 1, also it is very important to know we will discuss that but we know what is the Kernel of the ring homomorphism. Alright so, the first is we can take any fix element a in A so, let a in A or s in A, let me call it s, (s in s) s in A be a fixed element, then 1, s you take all the powers of x, s power r, this is obviously multiplicatively closed set, closes subset of A. So, in this case what happens, the ring, the new ring S inverse A.

So, if you call this as S, this I will denote because this only depends on this elements small s because once you know s you know A square and so on, so, this is also sometimes denoted by A suffix S and this things will give the basic open sets in as risky topology. So, that is what we will do it, but before that we have to check some details.

So these new ring, so, what are the elements here? You see now the denominator allowed is only power of x, power of s so, there are a power s power r, r is varying in natural numbers so, note our usual convention the standard, it is not a convention it is a force that any element power 0 is 1. So b power 0 in any ring is 1.

So that is r equal to 0 it is a by 1. Alright so, what is the map? Let us also see the map that will give you an understanding what goes wrong sometimes, A s this map is any a going to a by 1 and let us see whether a by 1 can become 0 or not, when can it become a by 1 0? So, a by 1 is 0, 0 in the ring means 0 by 1. That is if and only if a and multiplied by some element of s, now s is only this power of s so, there exist some power of s, s power r times a is 0.

So, for example, when s is an nilpotent element, then the drastic thing will happen, then every element will go to 0, when s is an nilpotent, otherwise also, for example, if s was a 0 divisor also they will be troublesome. So, such cases can lead to problem but we have to be careful. So, that was example one.

Second prominent example is what we will keep using is, suppose our prime ideal, let p be a prime ideal in A and then and you take the compliment of p, A minus p, this is S, sometimes also denoted because to remember p, Sp. This is then S suffix p is a multiplicatively closed set, subset in A, this follows immediately from the definition of a prime ideal, let it check that.

So, what do we check? 1 is there, well prime ideal is by definition is a non-unit ideal therefore, 1 is not therefore, 1 is in the compliment. So, 1 is obviously Sp because since 1 is not in p. Also I want to check that if s and t are elements in Sp but that is by definition of a Sp it is that means s and t are not in the prime ideal p but if the both of them are not in p, then the product will also not be in p.

But that means st belongs to the set Sp. So, check that it is multiplicatively closed set and in this case this ring, this ring this is usually denoted by because there is no need of so, it is a suffix the prime ideal p. So, just to show you a particular example.

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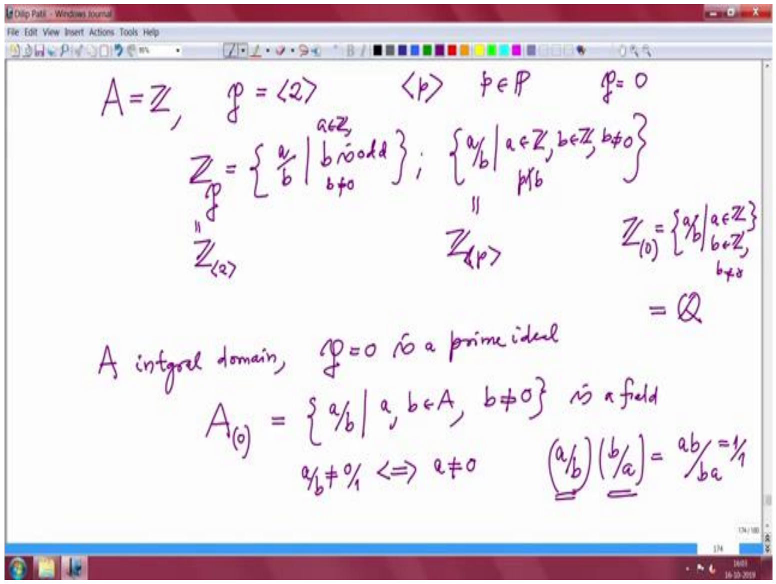


So, let us take A equal to Z and the prime ideal p is the ideal generated by 2 or for that matter ideal generated by any prime number p, these are all prime ideals or 0 or it can be 0, p is equal to 0, okay what are you going to get in all these cases? So, what does that mean? Now, what is the Z localized at p, what are the elements?

They are fractions such that a numerator is arbitrary element and denominator is from the multiplicative set but multiplicative set is ideal, complement of the ideal 2, complement of the ideal 2 means those are the elements which are odd integers, they are precisely the complement of 2. So, this is precisely any integer a divided by b such that b is odd and ofcourse, b is non-zero, that is all.

Such a ring we have considered earlier. You remember that there was a ring that Z and then I kept putting a bracket. The bracket was for ideal generated by 2, this is no different. What will it be here in case of arbitrary prime number? That will be the fractions again a by b, a integer, a is integer, a integer, b integer, b non-zero and what should be the condition that it is outside these prime ideal? That means, p does not divide b.

That will be the localization, this will be Z localize at p. So, I should write like this. And here I should write Z localized at 0, 0 ideal and what will it be? Now, there is no condition because the compliment is all integers, non-zero integers. So, this is a by b where a, b are integers and b non-zero.

So, in this case you get this is Q, it is a field of rational numbers, and this is what will happen in arbitrary if a is integral domain then p equals 0 is a primary ideal because it is integral domain and therefore, I can talk about A suffix p which is what are the elements? They are fractions a by b, a, b are elements in a and b non-zero.

So, this is actually I should have written A 0, p0 but better to so, it is A0 and obviously this is a field. So, let me show you every element as inverse. So, if you take a non-zero element here, if a by b is non-zero, non-zero means it is not 0 by 1. So, that is equivalent to saying, see a has to be non-zero, if a is 0, this will be equality here.

So, a non-zero but then what does the inverse of, already b is non-zero, now what is the inverse of a by b? a by b times b by a, b by a makes sense now, because a is non-zero, it is not in the prime ideal 0. So, this is a b by b a which is 1 by 1. So, which is a unit, it is a multiplicative element, therefore this element has this inverse, remember we are writing like a fraction only.

But equality is little bit different. It is not just the cross multiplication and difference is 0, but I will show you after the break that this requires little attention, but the calculation as far as goes it goes like a usual fractions. And after the break, we will come back and check the most important property of this ring called universal property. Okay, thank you.