Algebra Geometry and Commutative Algebra
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Lec 24
Consequences of HBT

Welcome back to this second half of this lecture. In the earlier part we have seen Hilbert Basis Theorem some of its consequences. Now, I want to give some more consequences of Hilbert Basis Theorem. But firstly, I will make, alright so, this is like linear algebra. Linear algebra you would have, you were studying linear operators on a finite dimensional vector spaces. This will be more general. Now, I want to do this for a more general setup, where modules are over Noetherian ring or at least modules are finitely generated or arbitrary commutative rings.

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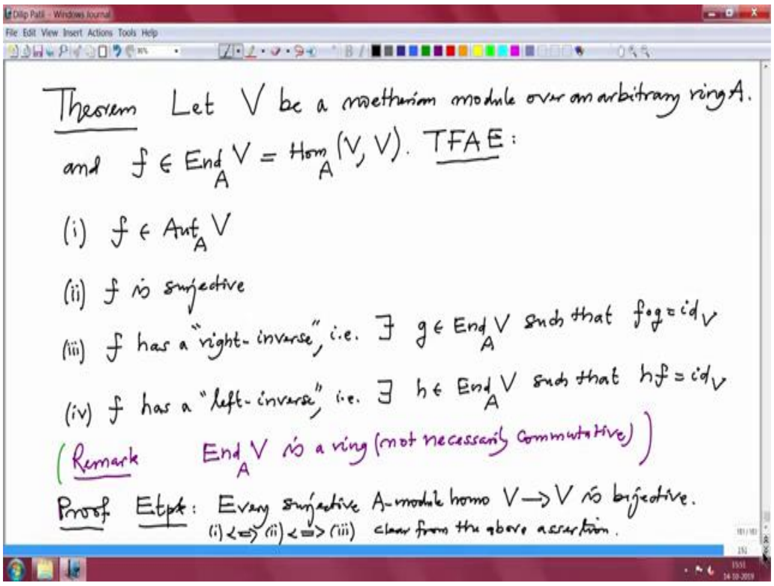

So, I want to prove a theorem like this for example, this was very easy to prove in case we were studying vector spaces cases. So, this is a theorem. So, here let V be a Noetherian module or an arbitrary ring A and again I want to remind you whenever we say ring, it is always a commutative for us and f is an endomorphism of V End AV again these are called endomorphism, they are (homomor) A-module homomorphism from A to, V to V. So, this is also same as Hom A V, V, homomorphism of A-module from V to V.

Then the following are equivalent. So, proof, no, not proof, the statements. What are the statement? 1 – f is actually an Automorphism, Aut AV. 2 – f is surjective. Remember you would have seen in a linear algebra course that if you have a finite dimensional vector space, then a linear operator on that V is surjective, if and only bijective, if and only if injective. So, this is a analogous of those statement or an arbitrary ring, but the module should be Noetherian and in the corollary I will deduce you do not have to assume really and A-module is Noetherian we just have to assume that it is finitely generated module.

So, we will do that in a corollary. So third statement, f has a right-inverse, that is there exist g another endomorphism of V such that f compose g is identity on V. I will explain these why am I using this language and forth f has a left inverse that is there exist another endomorphism h such that hf equal to identity on V and when I say hf the DH compose f. Now, before I go to the proof about this language, so we are working in this. So, this is I would write this as a remark.

See this end AV is a ring, not commutative, not necessarily commutative with addition of obvious addition, point wise and the multiplication is composition. So this is ring and this ring is very important to study in representation theory for instance, and now because this is not commutative, left-inverse and right-inverse the elements may not be invertible and somebody may be left invertible but may not be right invertible and all these things will crop up, this is precisely non commutative algebra, but only for this ring is more interesting, arbitrary non-communicative algebra is not interesting for us at least.

So, this is right inverse in that sense, this is, there is a right, on the right side there is a g so that is inverse, this is a left-inverse in that sense. Okay, so that was one and now let us prove first of all, I will show you that, so proof, proof I will write here it is enough, enough to prove that, enough to prove the following statement that under the given assumption, that means our assumption is V is a Noetherian module over a arbitrary commutative that every surjective homomorphism is bijective.

Every surjective A-module homomorphism from V to V is bijective. Let me show you why is it enough? So, this if I prove this statement every surjective homomorphism is bijective then I would establish the equivalence of 1, if and only if 2, if and only if 3 so, let us see how? So, first of all 1 implies 2 is trivial because one says it is automorphism therefore bijective and therefore surjective So, 1 implies 2 is and this statement which I stated above every surjective homomorphism is bijective that we are assuming that we approved then that implies 1 also and how 3 implies 1 or 3 implies 2 or 3 equivalence of.

So look here three, there is a g says that a four g is identity, identity is surjective, but once identity is surjective then this f is also surjective. It is clear from this. So, right-inverse implies surjectivity. So these 3 will definitely imply 2 and 2 implies 1 already we know and 1 implies 2 implies 3 is also clear. So, all these equivalence is clear from the above statement above assertion, which we will prove, alright.

And now, how to connect 4 with one but we see 4 says there is h like this, but now this will prove that this mean that h is surjective but once h is surjective we have already proved equivalence of 123 then h will be bijective and the inverse of h will be f so, so I will not say, I will not write much but for 4. So, Let us not write much so it is enough to prove that every surjective module of homomorphism is bijective.

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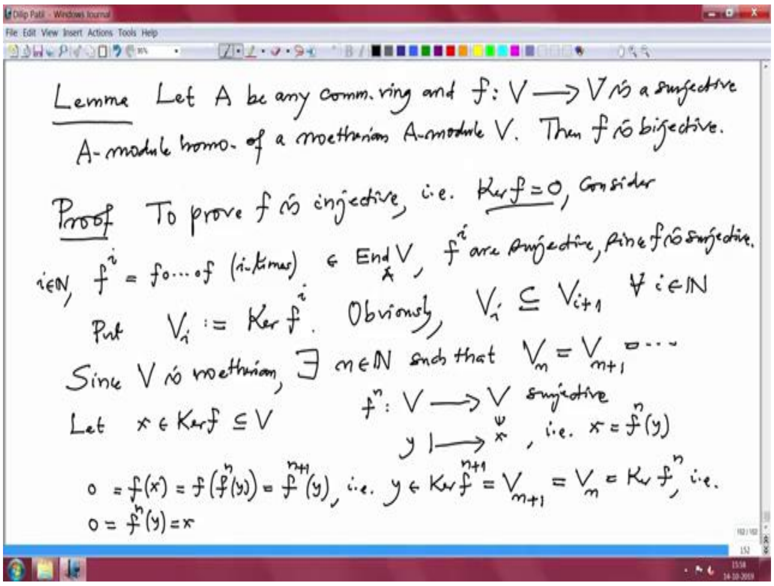

So, that let us write it as a lemma. So, lemma let A be any commutative ring, any commutative ring and f from V to V is a surjective A-module homomorphism of Noetherian A-module B, then f is bijective. This is very simple almost it follows from the definition. So, proof: We want to prove, we have given it a surjective and we want to prove it is bijective that means we want to prove it is injective.

So, to prove f is injective, that is kernel of f is 0, injectivity is equivalent to proving the kernel of f is 0, we consider the powers of f, f power i, f power i means what? For i in natural numbers. f power I means f compose f compose f i times, these are also endomorphisms and if f is surjective, f power i is also surjective, since, f is surjective that is given to us. And now we consider their kernels.

So we will put Vi equal to kernel of a fi, f power I, then it is obvious, obviously Vi is containing Vi plus 1 for all i big or equal to 1, for all i in N this is clear because if somebody is here that means f power is zero then f power i plus one which is composed f with f power i that is also zero. So, this is clear, so we have an ascending chain of sub modules of V. So, since V is Noetherian, there exists a natural number n in M such that Vn equal to Vn plus 1 and then all the way are equal.

So, now we will prove that kernel is zero. So, let x belong to the kernel of f. I want to prove f is zero. So, this is contained obviously in V, but we have given that f is surjective therefore, fn's are subjective and therefore, I have given this x, so fn is from V to V again this is surjective and the x given here, so there must be y here which goes to this. So, that is we have written x equal to f power n of y, but then fx is zero, we know kernel x is in the kernel. So, for x is zero on the other end this x is f power n y.

So, this f of f power n y, which is f power n plus 1y which is zero. So, that is y belongs to kernel of f power n plus 1 which is Vn plus 1, but this is Vn which is a kernel of f power n. So, that means y belongs to kernel of f for n that is when I apply fn to y it becomes 0. 0 equal to f power n y, but what is f power n y that is x. So done. So we have proved that kernel f is zero that means it is injective. So, we have proved the lemma and remember we have only use the fact that V is Noetherian and that means that every ascending chain of some modules of V should become stationary after some time. This is a definition of Noetherian model.

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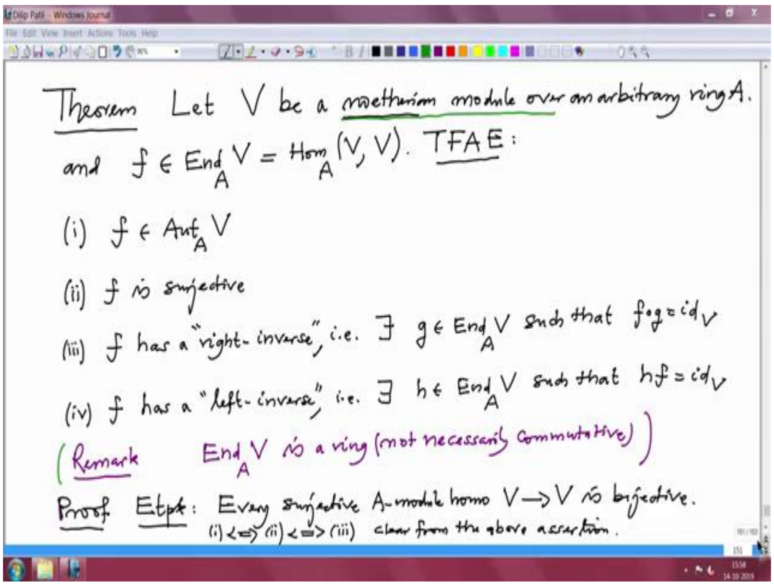

Now, I want to ride the corollary, where I do not want to assume in the above, so I will show you, in the theorem see here, I have said Noetherian, Noetherian module or arbitrary commutative. Now, I want to drop that and just want to say that finitely generated module over a commutative ring. So, the thing I want to prove is this, the same theorem will be proved, if I proved the corresponding statement by dropping the assumption Noetherian and just assuming finitely generated, so this does not work so easily.

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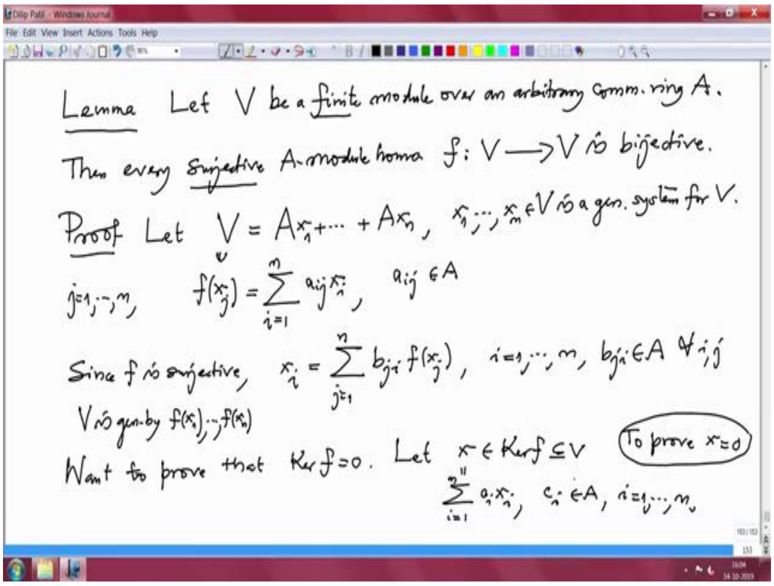

So, the replacement lemma is the following. So, this lemma is more general lemma, but obviously I will deduce to the earlier lemma. So, what now? We have, let V be a finite module over an arbitrary commutative ring A, then every surjective A-module homomorphism f from V to V is bijective. So, proof: We do not have Noetherianness because we know only finite module if the ring were Noetherian, then we have studied finite modules in Noetherian ring and they are Noetherian therefore we can apply the earlier lemma.

So, we have to from this situation someone we have to come down to Noetherian ring and that is where I will use Hilbert Basis Theorem. So let, so we have given a finite module that means it is A-module it is finitely generated. So, let suppose V is generated by n elements X1 to Xn over a. So, our standard notation Ax1 plus Axn where X1 to Xn is a generating system for V, is a generating system for V.

And we have given this f so that means, I have given f of xj, f of xj for any j from 1 to n this is again an element here. So, that means, these guys I can write in (com) A linear combinations of the X1 to Xn again. So, this I can write it as again and the sum summation, on summation is running over i from 1 to n aij xi or some aij in a, I can always write like that because phi is finitely generated and what is the surjectivity means now?

It is given to surjective. Surjectivity means that since f is surjective every element xi, xi is here. So, it is image of somebody. So, therefore, I can write x i equal to summation bji f of xj, j is 1 to n because it is surjective V is generated by f of X1 f of Xn image of generating set is a generating set. So, therefore Xi days I can write it for every i, i is from 1 to n. And what we want to prove?

We want to prove that kernel is zero, want to prove that kernel of f is zero. So, let x belong to the kernel and I will write here to prove X is 0 that is what our aim is. Well x is in the kernel in a it is in V. Therefore, in any case x as the expression of the type summation i is from i to n, CiXi where Ci is our elements in A for all i from 1 to n and here also I should have said bji, their elements in A for all ij. Now, what you see, look at this data, this data involved these aij, BJi and the Ci. So, obvious thing to do is

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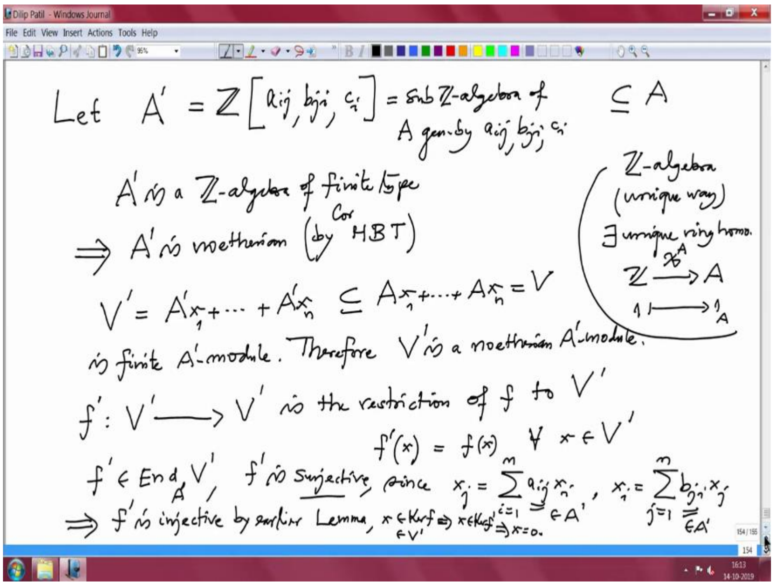

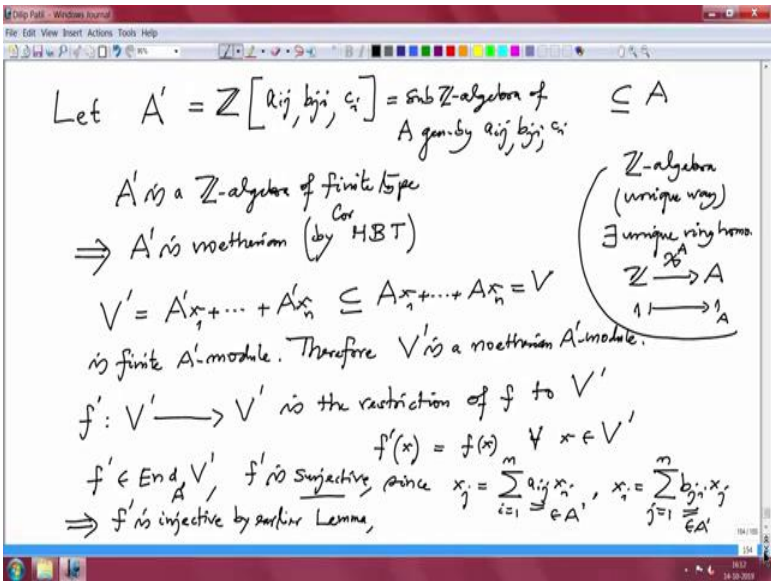

So, let A prime be the sub-algebra of A over Z see every ring is every ring A is a Z-algebra in a unique way, there is no other algebra structure on ring of Z algebra because there is a unique, there exists a unique ring homomorphism from Z to A, these are denoted by kai A namely 1 goes to 1 of A and these dictates everything if you want a ring homomorphism and therefore, as an algebra over Z, every ring is an algebra over Z is in a unique way.

So, take that Z algebra structure, there is only one and generate a sub-algebra over Z so my notation is this by the elements, aij, bji and ci this is a sub-algebra, sub Z-algebra of A generated by aij, bji, and ci for all, so they are anyway finitely many elements. So it is this A prime is a finite Z algebra of finite type. Therefore, A prime is Noetherian by HBT or by corollary to HBT. Corollary some one of the corollary. Finite type algebra over a Noetherian ring are Noetherian. That is the corollary I am using. So it is Noetherian.

And now this V I am considering V as A-module over A prime or V prime. V prime is A-module generated by, generated over A prime by the same generating set. So this is A prime X1 plus plus A prime Xn, this is a submodule, sub of Ax 1 plus A Xn which is V. So, I am forgetting this V and concentrating on this V prime now V prime is finite A prime module.

Finite means finitely generated therefore, by earlier results, by earlier lectures we have proved finite modules over Noetherian ring they are Noetherian, therefore, V prime is a Nigerian A prime module and what happened to that f? Now, f still makes sense. f is from V prime to V prime. Where does where do the X Xi go? f of Xi. If I restrict this V, this f. So better to write like this. This f is, is the restriction of f to V prime. I am using the same notation.

So, that means what so, okay let me write it f ring here. So, f prime of any, so f prime of any X equal to f of x for all x in V prime. So, it is a module, it is clear that it is a module homomorphism because original f was a module homomorphism. So, f prime is also a module homomorphism. And f, f is all, this f prime is also surjective, first of all f prime belongs to the end that is clear, f prime is surjective is also clear.

Since, you know this I need to only prove that all the generating system X1 to Xn of V prime also in the image but that is also clear because we have written Xj as summation aij xi i is from 1 to n this and this aij are actually in A prime. Therefore, these makes sense. So, it is surjective. Also this makes sense and also surjectivity make sense because Xi we have also written it as bji xj j is from 1 to n.

So, this means also it is surjective because these are in A prime So, this is surjective this makes sense because it is Xi going to this, so it is indeed a surjective endomorphism of the module. This is over A prime I should say. And also what, what do you know then? Then obviously then by the earlier lemma this f prime has to be injective. So, this implies f prime is injective by earlier Lemma and also what is clear?

This x or the x, I have chosen we wanted we, have fixed an element takes in the kernel and we wanted to prove it is zero. But that x, whatever that X, I just want to show you that. See we wanted to prove this we have fix X which is zero. And this X we have written in a linear combination of which is Ci let me make it clear here. This is Ci and this Ci is also we have added in that A prime.

This Ci is also we have added in a prime therefore, X is also there in V prime, X is in was which was in the kernel which was fixed that is also belongs to V prime and that also belongs actually to the because f prime is restriction of f this is also equal to kernel of, so I should write here this so, that implies x belongs to the kernel of f prime and therefore x is 0 and that ends the proof.

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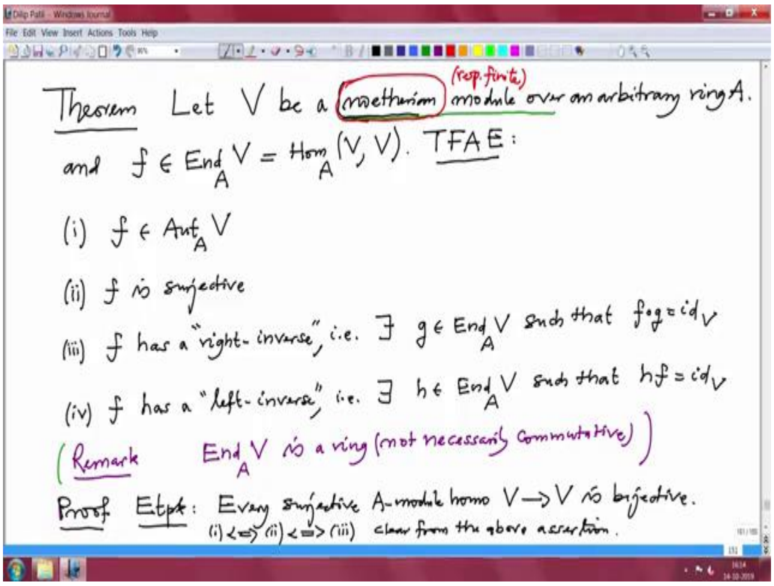

So, we have proved that and now once I approve this lemma which is a replacement for this which was used in this theorem. Now in this theorem, do not use just Noetherian just finite is enough and instead of this statement, you use the second lemma. And therefore, we prove that this theorem is valid for I will correct here.

So, instead of this assumption Noetherian, just or I will say respectively finite then the same, same theorem is true. And the next time we will now interchange the roles for injective and bijective and here we use Noetherian and obviously that time we will have to use Artinian. So, the Artinian and Noetherian are dual to each other. So, like therefore surjectivity and injectivity will be duel to each other and so on.

So, that proved this, this is very important because I will show you the use of these sometime and also I want to write more general statements in commutative algebra obviously connected to the modules over a commutative rings and in particular finite modules over commutative rings. So, with this I will stop this and continue it in the next time. Thank you very much.