Algebra Geometry and Commutative Algebra
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Lec 21
Examples of Artinian and Noetherian Modules

Let us continue these lectures on Algebraic Geometry and the Commutative Algebra and let us recall that we have been, last two lectures we have been studying Noetherian modules and Artinian modules and we have given characterization of Noetherian and Artinian modules and some properties we have even. Now, I want to switch to Noetherian rings and Artinian rings but before that in the last lecture I promise that I will give examples Noetherian module which are not Artinian and Artinian modules which are not Noetherian.

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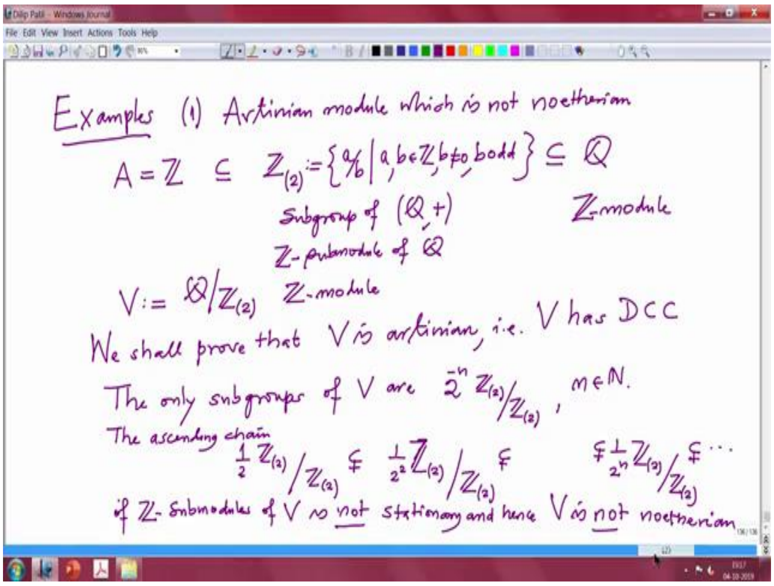

So, let us start with examples. So examples, so one, this I am giving you example of a non Artinian module, Artinian module which is not Noetherian and in the second we have example of a Noetherian module which is not Artinian. Alright so, my base ring, A is the ring, ring of integers. That means we are giving example of an Artinian Abelian group which is not Noetherian, Abelian group. So in this case Abelian goods modules are identical. But anyway, let us keep saying Z modules.

And now I have these big Z module Q. This is the field of rational numbers, this is Z module and in this I am constructing some module which is Artinian. So, what is that submodule I will write that here Z suffix 2 this bracket 2 and what it is? So you will understand why am I using this notation, it will be clear when I start doing localization that time this is the notation for something and I want to keep the same notation.

So, this is by definition, all those fractions a by b such that ab are integers, b nonzero and b is odd. These are obviously the rational numbers. This means they are rational numbers with denominator not divisible by 2. So, it is obvious that this contains all the integers because I can take in the denominator 1, b equal to 1, which is odd. So this is clear, these are also clear and it is also clear that this is a subgroup. This is a subgroup of Q, Q plus, if you want to be little precise, but this means therefore, this is Z-module Z-submodule of Q.

Now, we have a submodule of this Q, therefore, I can consider a quotient module. So, V is Q module of this Z bracket 2 this is also Z module and I want to prove that, so we shall prove, we shall prove that these Artinian let us prove that first, so what do I have to prove? We have to prove that by earlier equivalencies, we have to prove that V satisfies DCC that is V has DCC on some modules, that means any descending chain of submodules of V will become stationary, that is what we want to check.

So for this, for this, we just have to study the subgroups of this this, this, this Z module means subgroup and Z-submodule means subgroups. So, what are the subgroups that I want to list of this module? So, the only subgroups of V are 2 power minus n Z2 on Z2. We saw in last lecture how to describe the submodules of the quotient module, they are precisely the submodules of the upper module which contain this they are precisely those submodules.

So, look at this, the upper one, this means an n is a natural number. So, I will spill out a little bit more about this and then the proof will be or more or less obvious so, what is it, so, if I take 0 some module here, it is just Z2 cross Z2. If I take half this is in rational number, and you take the submodule image of this year, that means this. Take this, and this contain the obviously Z mode, Z this bracket 2 because when you multiply these by 2 we become here?

So, this is 1, then 1 by 2 square, Z2 I have to be clear in the, what I am writing this mod Z2 and increase this, keep increasing the power to natural numbers. So, these are precisely the submodules, grant this claim for a moment then what will happen is, so when you take a ascending chain so first of all, this one is contained here. This one is contained in 1 by 4 and so on. This one is contained in 1 by 2 power n Z2 mod Z, Z2 and so on.

And at every stage this is an ascending chain, this will never get terminated because the, this higher power of half will never belong to the earlier one. So, that is easy to check because this in Z bracket 2 there are no fractions who has a denominator which is even so, therefore, that cannot happen. So, all the chain is proper chain, this is an ascending chain and it's a, it never become stationary.

So, that first of all shows obviously that these module V and all these are submodules of V. So, they are, so the chain, the ascending chain this of submodules of Z-submodules of V is not stationary and hence V is not Noetherian. So, now we still have to prove two things that it is not our Artinian, it is Artinian and these are the only subgroups. Alright, once you do these are the only subgroups Artinian is easy. So, let me indicate that first.

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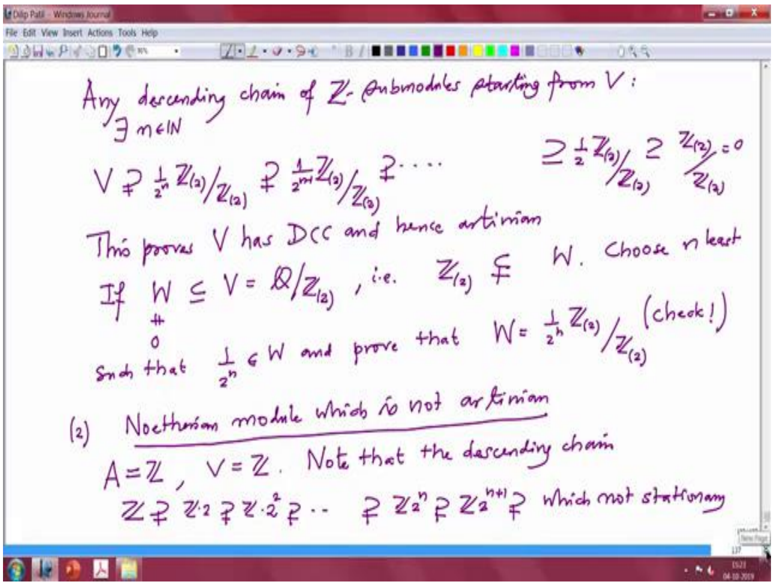

So, if these are the only subgroups, so we, how do you start the descending chain? At most you can, any descending chain of Z-submodule starting from V. So, if it does not start for V you start from V. So start from V. Then you want somebody which is properly contained that. But then the only submodules are half power n times half n Z2. It is this because these are the only submodule so this n in natural numbers. So, there exist n in natural numbers such that this.

But the moment you start with n now, you have no, not much room to decrease, then you can decrease only 1 by 2 power n minus 1. So, this will be 1 by 2 power n minus 1, Z2 mod Z2 and so on. So, ultimately it has to come down to half Zmod 2 not Zmod 2, Z bracket 2, Z bracket 2 and then 0, Z is 2, Z is 2 which is 0 module 0 submodule of that. So, therefore it, therefore V has a DC, so this proves V has DCC and hence Artinian.

Now, we have justify that any submodule looks like this. So, if you have any submodule W, if W is a submodule of V which is remember this is Q mod Z locate at 2 and assume it is nonzero. Then what do you do? You choose, See it, this W is a submodule these nonzero means, so that is Z2 is contained in the W. We can assume and this is not proper because this is nonzero and therefore, I can choose.

So therefore, this W should have a fraction whose denominator is odd or whose denominator is even. So, therefore, it has to contain some power of half and you choose, so that he has a least power of, so choose n least such that 1 by 2 power n belongs to W and prove that W must be half power n Z2 this Z2 and these I will leave it for you to check. So this is check, this is just a little playing with the fractions, the numbers and so on. So, we have an example of an Artinian module which is not matter.

And second one now you want to give an example of a Noetherian module, Noetherian module which is not Artinian. So, again the ring is Z and the module V is also Z. So, first of all note that the descending chain Z contained in Z2 these are even numbers, contained in Z2 square these are integer multiples of 2 square and so on. Z2 power n containing Z2 power n plus 1 and so on, at every stage it is not equal. Therefore, it is descending properly which is not stationary. That is it has no DCC. So, therefore, it is not Artinian.

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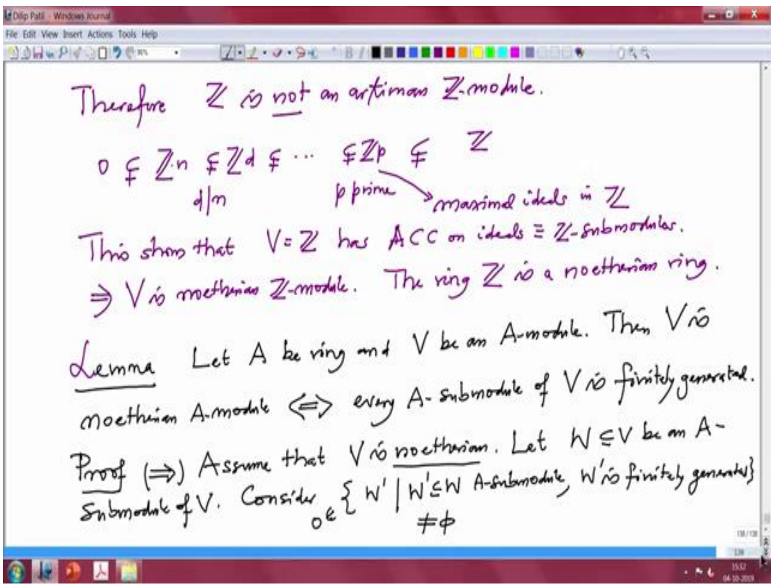

Therefore Z is not an Artinian Z Module. As the course progresses you will also see how do you prove this much easier way? So, now I want to show that it is Noetherian. That means what? Each, each ascending chain has becomes a stationary. So, now how do you get an ascending chain? You start with 0 and then you want submodule of Z but submodules of Z are precisely the ideals, right? So, then the next will be and we know all ideals in Z are principle.

So, this will be generated by some natural number n, so it will be Z times n. Now, if I want a bigger one that means it has to be the form Z times d where d divides n and this is contained here means n is multiple have d that when d divides n, so further if I want inclusion, then d has to have divisors, so ultimately there will be a stage where you will reach prime Z p, where p is prime.

So, if I want proper chain like that, it, it can go on till at most the number of divisors of n it cannot go on for a long time. And now we have noted that these, these ideals are maximal, maximal ideal in Z. Therefore, if I want a submodule or ideal which, which is bigger than this the only possibility Z, so, therefore, with the chain will stop here, so utmost it will have number of divisors of his, the nonzero term there that is this the ideal

So, therefore, in any case, therefore this shows that V equal to Z has ACC on ideals, which are also same thing as Z submodules. So, therefore V is Noetherian Z-module. So, in this we have proved, that means in other words we have proved that the ring, the ring Z is a Noetherian ring. Recall our definition of a Noetherian ring. A ring is called Noetherian if, a ring is called Noetherian if the A-module is Noetherian module that is for the definition. Similarly for Artinian.

Alright, so, this proves there is no relation between the Artinian modules and Noetherian modules but I will prove it soon that Artinian rings are Noetherian, but Noetherian rings need not be Artinian that Z the ring of integers is an Noetherian ring but it is not an Artinian because it is not Artinian Z mod. So, we will continue, I want to collect now some few statements about the rings and how do you conclude sometimes on the rings the, how do you conclude from the modules, the ring is, a ring has some special property.

So, in other words, I want to study the category of A modules more to conclude results about the rings this I will take up later but right now, let us prove a simple lemma which will be useful for, alright the lemma is let as usual A be a ring commutative always and V be an A module, then V is Noetherian A module if and only if every is A submodule of V is finitely generated.

So this is useful observation because remember what we approved above. We approved that Z is a Noetherian module or Z, but if you would have use this it would have been very simple because you know the submodules of Z are basically the ideals and ideals are actually principle therefore finitely generated. So, it would immediately follow from this. So proof, first, let us move this way.

So, I am assuming, assume that V is Noetherian and I want to prove now that every submodule of V is finitely generated, so let W be a submodule, an A-submodule of V and I want to prove W is finitely generated. Now consider the subset, see what is V Noetherian means what, that means the arbitrary family of some modules in V have a maximal element with respect to the inclusion that was the, one of the equivalent definition for the Noetherian modules.

So, now consider a subset of submodules of W that is W prime, W prime, W prime is submodule of W. This is a submodule and W prime is finitely generated. Of course, this family is obviously nonempty because. So, I should have said in the beginning we can assume this W is nonzero. So, we can assume this W in nonzero. So, W definitely has nonzero element and I simply take a submodule generated by any element of W.

So, if W belongs to W and W is nonzero. I do not even have to the zero submodule is here 0, I do not even have to assume this. So, let us not assume unnecessary things. So clearly 0, submodule is here. And I do not even need to assume that. So, zero submodule is always finitely generated, zero submodule is generated by the empty set or zero element. So, it is finitely generated, and it is there. Therefore this family of submodules is non empty and because of Noetherianness of V, this will have a maximal element.

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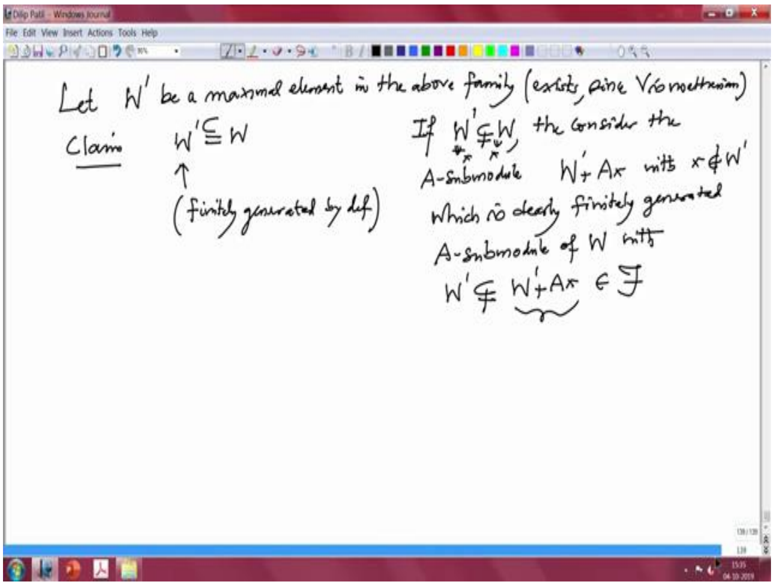

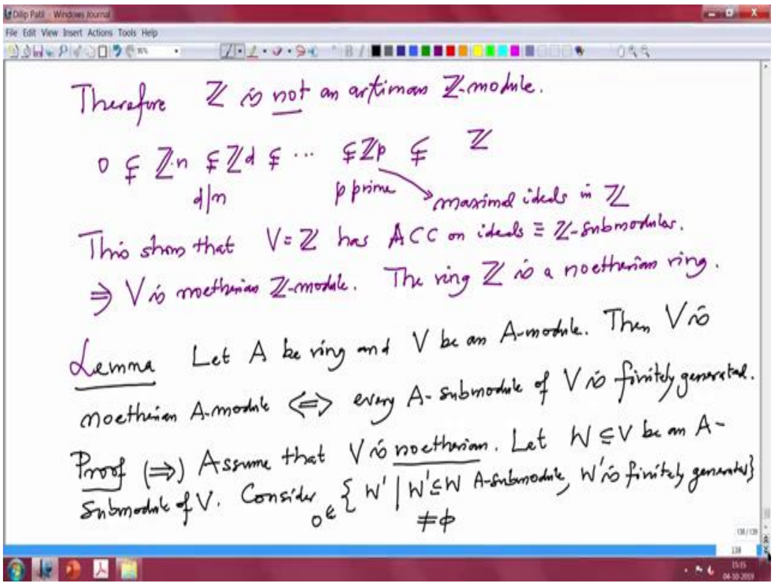

So, let us take that maximal element. So let W prime be a maximal element in the above family, these exist since V is Noetherian. Now, I claim we prove that claim W prime equal to W and because it is of maximal element in that family in by definition W prime is finitely generated and if I prove W prime equal to W, W will be finitely generated. So, this is obvious, this is by the choice of this W prime and this W prime is finally generated that we know by definition.

Now, so, suppose it is not equal, if W prime is properly contained in W, then consider the submodule, A-submodule W prime plus cyclic submodule generated by x with x not in W because of this I choose and element X here which is not here and consider this, this is some of 2 submodule, this is (())(27:16) submodule which contain both. Now, this one is finitely generated that is obvious because it degenerated by generating set of W prime along with x, which is clearly finitely generated A-submodule of W which contain with W prime properly contained in W prime plus Ax.

But then by definition of that above family all finitely generated submodules of W belong there. So, this will also belong to that family F, let us is call it F. So, I will go back and this family I am denoting by F. Alright it belonged to F but this was the maximal element under inclusion there. So, these contradicts the maximality of, the maximality of, of W prime in F, therefore we, therefore we proved our claim.

So, this proves that W equal to W prime is finally generated that is what we wanted to prove. Now, I have to glue the converse, converse is if every submodule is finitely generated then I want to prove V is Noetherian. And that means what I have to prove that every ascending chain condition on V is stationary so start with ascending chain. So, suppose, we are supporting now every A submodule of V is finitely generated and let start with an ascending chain.

So that I will call it a U naught containing U containing Un, containing, containing Un, containing this and is a ascending chain. Chain in SAV inclusion, this ordered set. And U is the union and from 0 or m in N Un. This is clearly is submodule of V because of this, because it is a ascending chain. So U clearly, U is an A-submodule of V.

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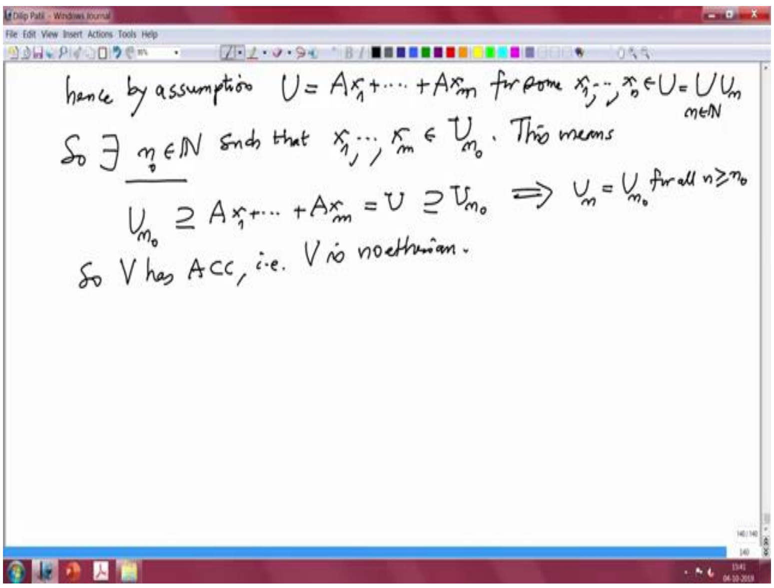

And hence by assumption U is of the form finitely generated. So, finitely many elements we generate U, so that is, U is a finite a linear combination of these X1 to Xn for some X1 to Xn in U and remember U are the union, Um, m belongs to N and this is a chain, ascending chain. So, all these guys, they exist, so there exist n large, n in N or U n naught in N such that let me call it not n but m here. Such that all these X1 to Xn they already belong to Un naught.

They belong to the union so they belong, say let us say X1, X2 belong to 1, Un1 and Un2 here but to the maximum over there and then so, therefore, keep doing this. So, there exist n naught in N such that all of them belong there, but this means, so this means Un, Un naught already contains Ax1 is submodule generated by X1 to Xn because this is a sub modular it is these guys are in Un naught basis the smallest submodule which contain that therefore this but this is U and U contains Um naught because U is a union therefore equality here.

So, therefore U equal to Un naught is finitely generated, no, is yeah, not finitely generated but, so already up above after that Un equal to Un naught for all n bigger equal to n naught because it was already there and so. So that proves so V as ACC that is V is Noetherian. Alright so now in the, I will stop this and in the next half, I will deduce some consequences from this observation this lemma. So we will continue after a short break. Thank you.