Introduction to Algebraic Geometry and Commutative Algebra
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Lecture 02
Definitions and Examples of Affine Algebraic Set

So, welcome back to this second half of this lecture. One more example I want to write it, this is just to remind You that You have studied similar stuff there in college but without, not in this notation and not in this thinking.

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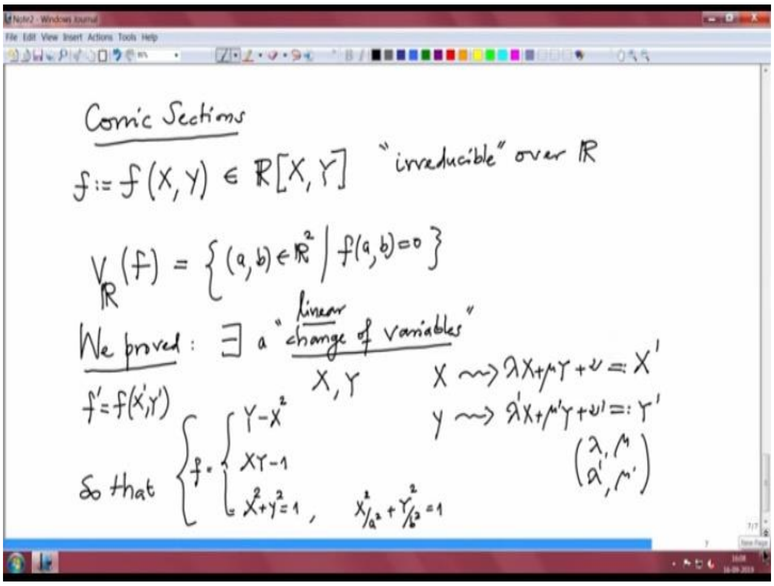

So, this is called conic sections. So if You go back to Your college days and remember You were studying a polynomial in 2 variables, f of X, Y. We were 1 polynomial, no, I called variables the letters not X1, X2 but X and Y. So, this is the polynomial in two variables X,Y or the field of real numbers and we were assuming that it is irreducible.

This word I think You know it, irreducible over R. That means, You cannot factorize these polynomial in 2 different factors. 2 different non-constant factors. So such a polynomial is called irreducible polynomial and actually it is not necessary to assume this irreducible but those, that time, some vocabulary was not studied properly in the college days. Therefore, they were making that assumption to make simplification simpler.

So what are we looking for? We are looking again for solution set or reals again. See, those days this, this is very old, this if you know this this goes back to the Greeks. So, this V R real solutions of this polynomial f. This is f. These are a,b, real numbers in R2, such that f of a,b is 0. Then, we prove, this is proved in college so we proved.

This proof was not so simple given in a college days because there were lot things we are coming and every time we are making shortcuts. Just simply because the concepts were not available at that time in the college curriculum.

So, what we did was, so there exists a change of variables. Now what, what does this mean? This means we can change this variables X and Y. Actually, strictly speaking, it was a linear change of variables. There exist a linear change of variables. So that means we change X to some lambda X plus mu Y plus constant. And Y to lambda prime X plus mu prime Y plus mu prime. So that when you say change of variable, so that, that means this mu, if you call this as X prime and this as Y prime.

From this X prime and Y prime, you should be able to recover back X and Y. That is equivalent to saying if you look at this matrix. So the matrix lambda, mu lambda prime, mu prime, this should be invertible matrix. So that we can get back your X and Y and why, how do we do this? I will not repeat that process here. But what was the need to do this so that these polynomial looks simpler.

So that f looks like f the new when I make the change of variable, nu polynomial f prime. F prime is f of X prime Y prime, will look like there are 3 cases. Y minus X square. So now the new variables also I call it X. So the new polynomial will look like Y minus X square or Xy minus 1 or X square plus Y square equal to 1.

It will look like this, one of this and now what was the advantage. One more case I forgot, namely, one more case. X square divided by a square plus Y square divided by b square equal to 1. So these are the 4 cases. Now what is the advantage of this? Now let us say how do, plotting is much easier now. So in the first case, unfortunately

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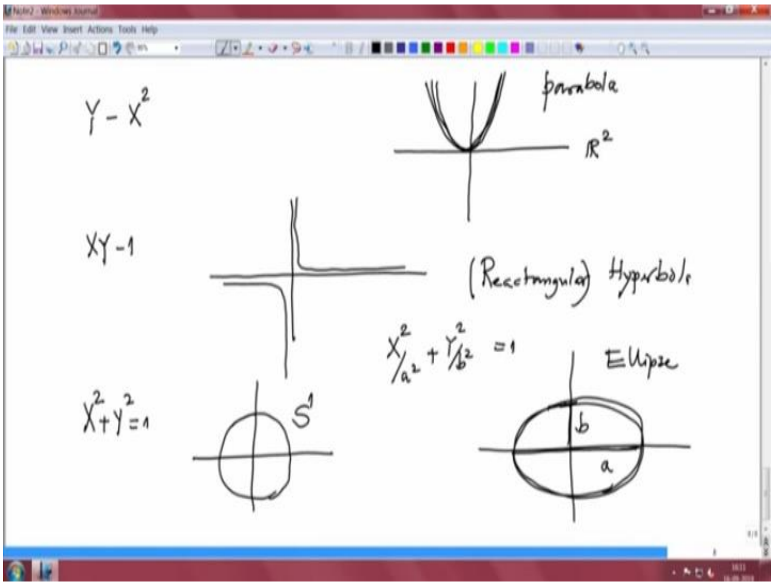

I have to go the next page, then in the first case this case, Y minus X square. What is the zero set? If You want to plot now in a real plane, this is R 2. So it is, Y is always positive so it is like this. This this is called a parabola. In the other case, Xy minus 1. Xy minus 1 case, it is, this is not touching here. This is called rectangular hyperbola and in these cases, X square plus Y square equal to 1.

This is a unit circle, this is a unit circle as well it is denoted by S1 and in the other case, X square by a square plus Y square by b square equal to 1. This is, this is the same like circle but it is the axis I have the different, this is called ellipse. The major and minor axis, a and b, here, so when so this is a. This is a and this is b.

And then we were studying these geometric figures by using algebra. So for example, we were, from this equation we were deciding what is a tangent. What is, what are the normal, their equations and so on. So, we were studying geometry of these pictures and that was already quite a bit but we were doing it more mechanically. There was no substantial theory and that was the reason why it was very limited to two variables and that also the degree I forgot. Let me go the earlier thing.

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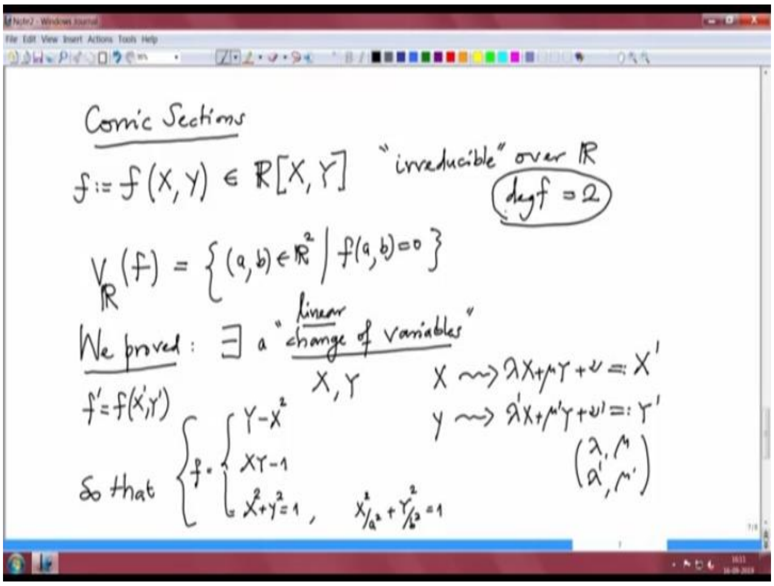

I forgot to say that degree of f is 2. This I forgot to make this assumption.

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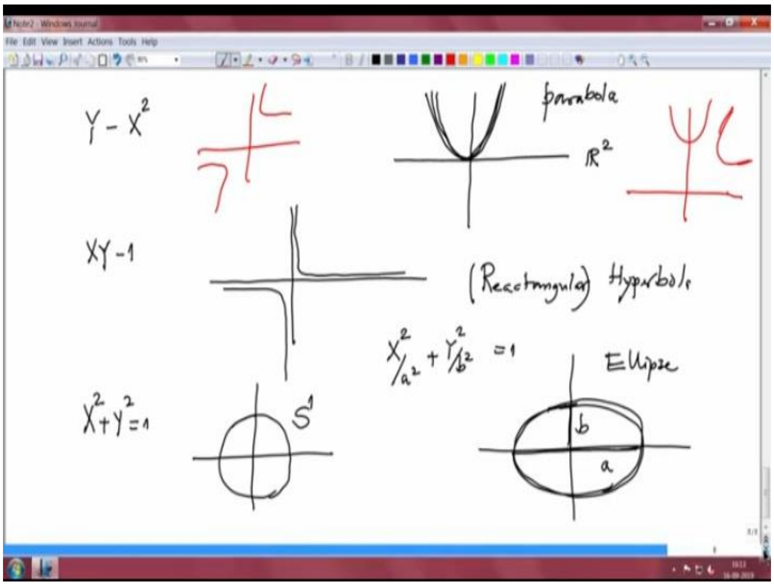

So, we are only studying degree 2 polynomials in 2 variables and already that, already that study of the geometric pictures, geometry of these pictures, that was already complicated because first of all, when given arbitrary equation, arbitrary polynomial in 2 variables of degree 2, we had some recipe to convert that polynomial to one of this. And already that was using some linear algebra and some, some calculation to decide that and then we were studying the normals, tangents and so on, the distance and so on.

Also, also we had studied from the pictures how to get the equation. So, suppose Your picture of the parabola was not like this. But suppose it is, this is not touching the X axis but it is something like this. Then what is equation or it is somewhere else. It is here. Then what was the equation? And similarly for circles, hyperbola. So, if hyperbola was not like this somewhere else, suppose it was like this then what are, what is the equation?

And the definitions of parabola, hyperbola, circle and ellipse, they were also given not using the polynomial, by using some invariants of this, like eccentricity, directrix and so on and so on. So, the study was not very uniform or study was not theoretically, it was not sound. So therefore, that is one reason. So, it needed more, more theoretical.

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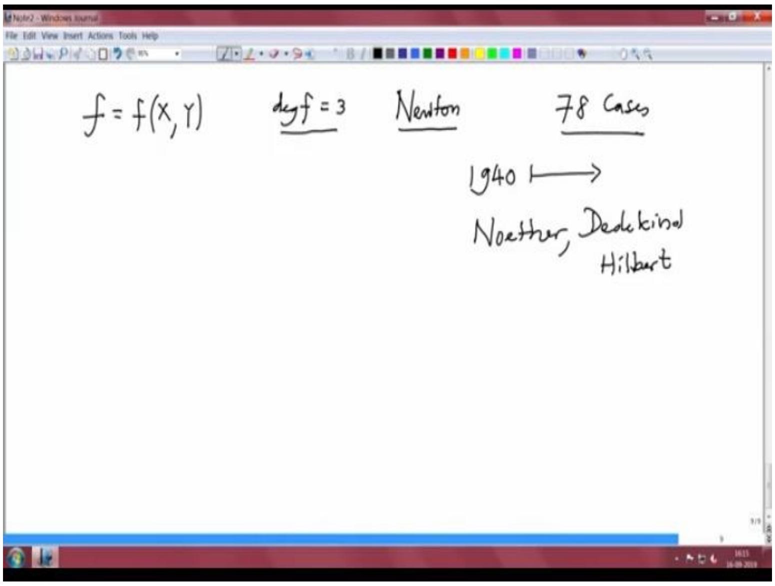

So if one tries to do it for, you can keep the polynomial in two variables only. f equal to f X,Y. So there, roughly speaking, there were 4 case in, when the degree f was 2 and when the degree f is 3 now, then it became much more complicated. There was an analysis available which was done by Newton and there were about 78 cases. So, that was too much complicated.

So, anyway, this was the day when seventeenth eighteenth century and so on. So subject needed a more revision and they needed more tools to study this and the tools only came in later half of the 20th century. That is, let us say after 1940 and this basically happened because commutative algebra was introduced during this time and also, it was introduced with the view in mind that studying geometry, it will help. That was one reason.

At the same time, a number theory also stuck like that. Stuck means the study was complicated and people have realized, people like Noether, Dedekind, Hillbert, etc. They realized that if you study commutative algebra that is also very useful for number theory. In fact, for number theory, the commutative algebra use was only so-called one directional case because they arrives only as curves. So, the algebraic geometry is even more larger than this subject.

So and in this course, our aim is to integrate both these courses. So large number of lectures I will have to digress what is called commutative algebra and for that, I will have to introduce little bit more general setup. So, what is that we are going to do it now?

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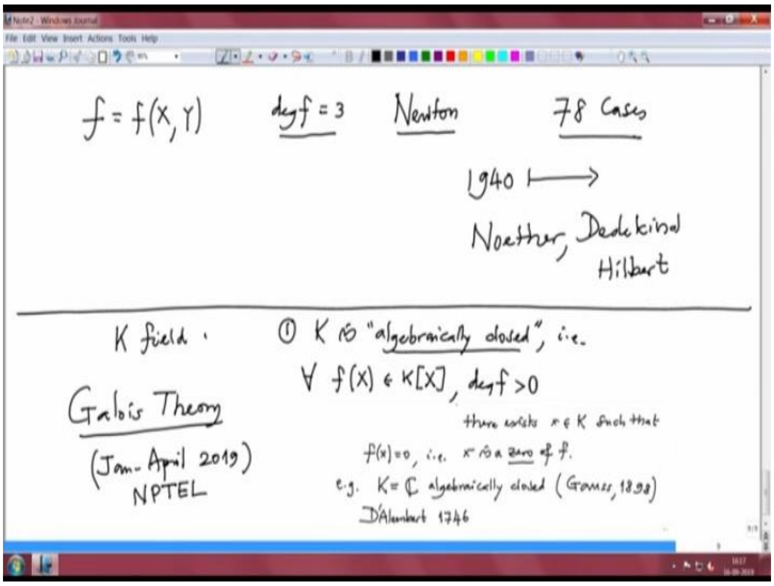

Let me now set up a formal notation which we will use. So, to start with K is a field. And as you will see if I take polynomials over a field, it may not have any solution in with coordinates in K. Then what do we do in these cases? So, for example we saw K equal to R and then X square plus 1 then there is no real solution. So, in order to avoid this, one remedy is to assume, 1 possibility is assume the field K is what is called algebraically closed. What did that mean?

Let us recall that, that means, so this means so that is, if I any polynomial f in 1 variable with coefficients in k. For every polynomial, there exists a solution in K. There exists small x in K such that f of x is 0. So such a thing also I will keep calling, that is, X is a 0 of f. So that is for example, for example, a typical example of algebraically closed field is k equal to C. This is not so easy to prove. This is, it's called Theorem of Gauss. Gauss has given first complete proof in 1898. But actually it was stated by D' Alembert in 1746. So, if you go to see a French book, they will say it is a Theory of D' Alembert. If you go to see German book, they will say it's a Theorem of Gauss. If you see English book, they will say Theorem of D' Alembert-Gauss.

I do not know whether I will prove this fact in this course but certainly if you see my course on Galois Theory in January to April this Year on NPTEL. This will have all the proofs of this nature. Because it was basically a course on solutions of polynomials in 1 variable.

How to solve polynomials in 1 variable? Or what does one mean by solvable by radicals etc. So that was, I will, strongly recommend for those who want to see the proofs of this. Algebraically closed, C is algebraically closed, it is certainly it's there. It is not so trivial but complete proof is there. So that was one remedy I suggested so that problem of, if you have 1 variable polynomial then the 0 set is definitely non-empty. So that we have some, some set to work with. It is not empty set.

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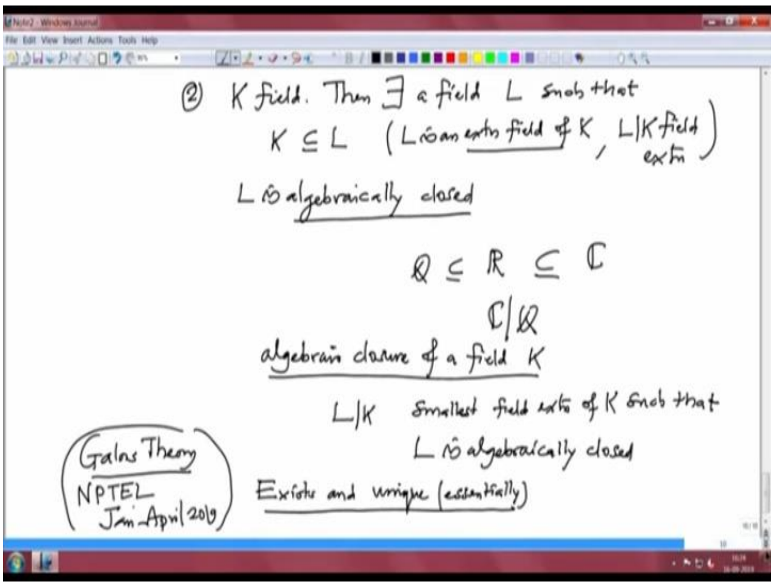

So this is one possibility. Another possibility, second possibility to overcome the remedy is, if I have a field K then there exist a field L such that K is a field of L. This means that, this I will keep saying that L is an extension field of K or I will also keep writing L or K the field extension.

This simply means L is a bigger field. That means the plus and dot in K are extended to L. The same plus and dot operations are there. They are more, they are on the bigger side. So such a thing is called a field extension. So, this says if K is a field then I can always find a bigger field such that L is algebraically closed.

So, for example, for R we already will see, this is algebraically close. So, C or R. So for if K is R then I can take L equal to C. If I have Q, well C is also an extension of Q. So I could also take C or Q. But these may not be the smallest algebraically closed fields which contains Q. So there is also a concept called algebraic closure of a field K.

That simply means it's a field extension. The smallest field extension of K such that L is algebraically closed. Even such a thing exist, so exist and essentially unique, unique essentially I will write here, essentially. This is again a theorem which again I will recommend you to see my course on Galois Theory. This is proved there, NPTEL January to April 2019.

So the second remedy is, you consider a pair, not only K and fix a bigger field L. So the bigger field is algebraically closed and with this now we can make nice definitions and then we will play with those some time.

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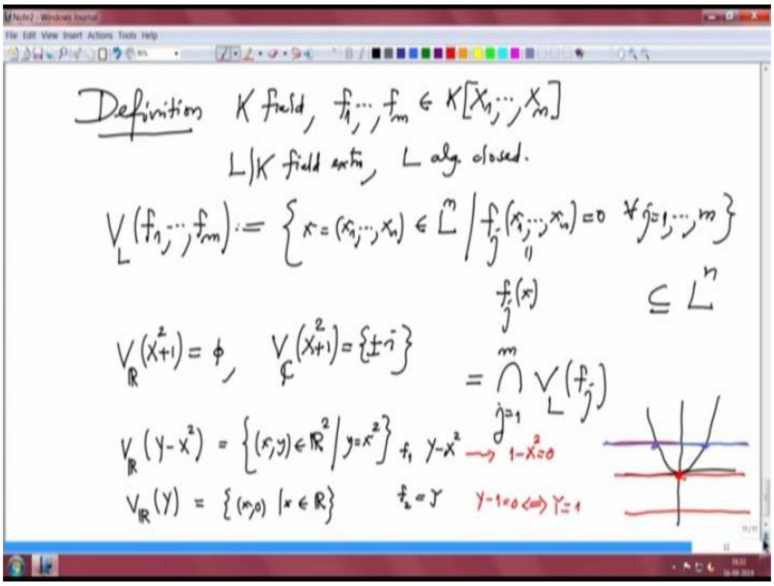

So now definition. So K is a field, arbitrary field and f 1, 2, fm are polynomials with coefficients in K. So that means they are polynomials in the polynomial ring or K in n indeterminates and then we are looking for a solution set but not with coordinates in K but coordinates in L and so L or K field extension, L algebraically close. Then we are looking for this notation VL f1 to fm. This is by definition all those tuples X. X1 to Xn.

Not in K power n but L power n such that all these polynomials vanish at X for all J 1 to m. This also I will just write fj of X. X is a tuple. So this is a subset of L power n. So for example, you know the difference, note the difference. If I write VR of X square plus 1, this is empty set and VC of X square plus 1, this is plus minus imaginary i. So this is the difference. This is because we do not come to empty situation. We have to go on with (())(25:26).

Now first note is this common solutions in L power n. That is, also by obvious, earlier also I remarked, this is the common solution therefore it is intersection over J of fj, VL fj. So here also in principle we could understand these set solutions at better if we know for 1 variable, not 1 variable, for one polynomial in several variables. But the problem is also to understand intersection. So, understanding intersection I will just say the following.

See, suppose I would have taken VR. What do I mean by understanding? So suppose, I will illustrate real first. So VR of let us say y minus x square. Now this is a parabola. So this is all those tuples x and y in R2 such that, y equal to x square and suppose I take, now we will say, we will draw the pictures. So this, this is my parabola.

And I have I have considered this parabola and now I consider another one. VR of y. This is the tuple. These are which tuples? This is Y is 0. So this means it is X-axis. So this is all those strictly because we are looking in R2. So, this is X comma 0 as X varies in real numbers because Y is 0.

So if I want to draw the picture together with this, so it is this one. If I would have taken other line than the X- axis, parallel, then it would be this. So I should have use the different colours. So what is common zeroes of these two first? These two is only this point. And these two, if I have used the blue colour for this, what will be the common zeroes. It is this and it is this. There are two now and here it is only one. Now this, algebraically you do not see this, but geometrically when you see, it is seen. So how do you prove this?

Obviously you cannot just, the picture does not give the rigorous proof. Picture only gives the intuition. So, when you try to write the proof, you have to write an algebraic proof. And that is what one mean by understanding this intersection. So understanding this intersection can be very complicated. Now, this is happening because this y equal to 0 is a is a tangent line. So it intersect only one point and that point should be actually counted twice. So, how do you prove such a thing? Rigorously one proves, I will just indicate the proof quite quickly.

Because, so rigorously how do we prove? We have this one equation, y minus x square. This is our F1 and F2 is y. And what do we do? You eliminate y between these two equations. Then what do we get? That means you take y equal to x square and put it in this. So here you do not see much but if you let us take x equal to 1. So do not take not x equal to 1, this is y equal to 1. Instead of this, I would prefer to take that gives a better illustration, y minus 1.

So what is the, how do you solve these two together? You put y equal to 1 in this. This is y minus 1 equal to 0, that is same thing as checking putting y equal to 1. And when You put this equation, this y equal to 1 in this equation, this get equation get transformed to 1 minus x square equal to 0. You are looking for the solutions here. So therefore there are two solutions. Otherwise, what do you do? You put y is equal to 0 in this equation and then you get x square equal to 0. But x square equal to 0 means x is 0 but that will come twice. So we have to check that the quadratic equation that has how many, how many zeroes.

First of all, if I would have taken this line, x equals to minus 1. Then there is no solution common solution as you can see. But you cannot write the proof. This is not rigorous proof. So, real rigorous proof is eliminate one variable and get a equation in only one variable.

But in because the degrees are 2 when You eliminate 1 variable, You will get a quadratic equation in one of the variable and then we know from the study of quadratic equation either it has no 0 or one 0 with multiplicity 2 or two different zeroes. That means, each will have multiplicity 1 and therefore, this picture gives you idea how to prove and then conversely.

This is a typical interplay between algebra and geometry but this was very, very simple case and we have to go long way. So we need much, much better machinery. So I defined this set and we want to study this and what is our machinery. We are only to start with a polynomials in m variables over a field K. That is, only machinery we have so far.

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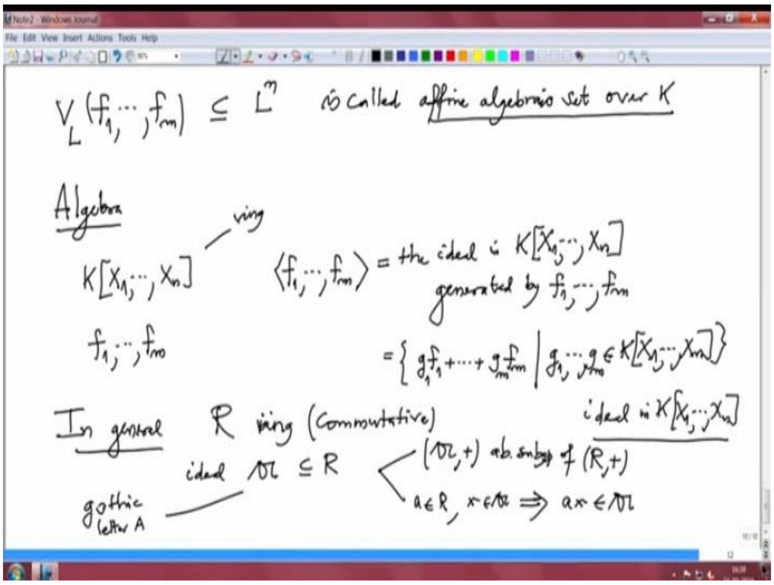

So going back this VL, this V VL of f1 to fm, this is a subset of L power m. This is called is called affine algebraic set over K. Over K means simply to remember that the polynomials have coefficients in K but this solution set is in L power m. Now we want to come back in some way like algebra. So what is algebra in order to study this and what is (to) what is the geometry involved to study this. This is what we have to make it very clear. So algebra involved is, so we started with polynomial ring in n variables over K and we started with m polynomials.

Now I want to remind you, there is something called ideal. This you have noted that this ring and if I say ideal generated by f1 to fm. This is the ideal in K X1 to Xn, generated by the polynomial f1 to fm. What does this mean? This means you take all K X1, X2, Xn linear combinations of this polynomial. So, that means g1f1 plus, plus, plus, plus, plus g m f m, this collection of such linear sums where allow g1 to gm, not only constants but allow them to be polynomials again in the same variables and this collection forms an ideal in this ring. In K Y1 to Yn, not Y, X1 to Xn.

Now let us quickly recall what is an ideal. So in general, if I have a ring R, R is a ring. Whenever I say ring, always commutative in this course. If I want to say anytime non-commutative ring, then I will specify but if I forget, ring for us is always commutative ring and what is an ideal? Ideal, that is, I will write this gothic A.

This is a subset of R such that, there is no underline there, so this is this is a gothic, this is gothic letter which corresponds to the letter ‘A’. So therefore, I keep calling ‘A’, this is because of the Dedekind. Dedekind was the first to introduce ideals and studied ideals and so on.

So what is an ideal? First of all, under addition it’s abelian group, abelian subgroup of R plus and it has a property that if I take any element a in R and any element X in A then a Times X should be in A. Then such a thing is called an ideal in the ring R. See it is more than a sub-ring. Sub-ring would just mean that under addition it is abelian group, under multiplication it is sub monoid and so on. So this is an ideal and note that it’s very much from the definition.

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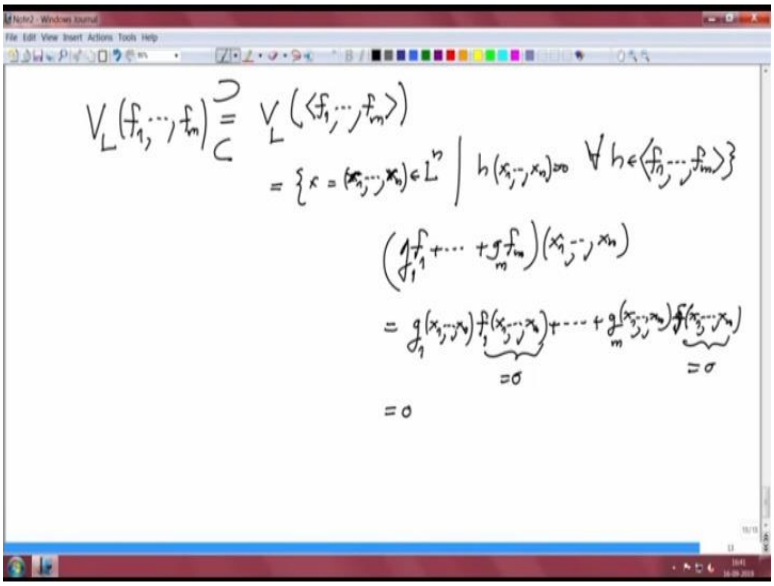

It is clear that this set, this solution set does not depend on does not depend on these polynomials but it only depends on the ideals generated by the polynomials. So this is same thing as ideals generated by this and what does one mean by this. This right hand side is, this means this is the set of all x, x1 to xn in L power n. These are small x so one has to be careful of writing the notations, small x and capital X. So this is small x, that is why I write this, a tag little tag, so this small d.

So all those points such that h of x1 to xn is 0 for all h in the ideal f1 to fn. This is very clear because, you see, because if all polynomials in the ideal vanish then all polynomials fj is vanish. So this definitely clear, conversely, if you have all these polynomials f1 to fm vanish at that x then the linear combination of f1 to fm with coefficients in the polynomials will also vanish because you see, if I want to evaluate such a combination at x1 to xn, that is same thing is as g1 x1 to xm times f1 x1 to xn plus gm x1 to xn Times f x1 to xn.

Such a combination, each one of them is 0. We are assuming that X is here then each one of them is 0 therefore this combination is 0. So therefore, we have proved this inclusion also.

So, in other words, this we can more generally define the affine algebraic set by defining it for any ideal and then we will continue this study in the next lecture and I will have to digress some more algebra because algebra gives more precision.

Geometry is more intuitive but it is less precise. So while writing the proofs we will be more precise by using algebra only. So, I will stop here and we will continue in the next lecture. Thank You.