Introduction to Algebraic Geometry and Commutative Algebra
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Lecture-16
The map VL

Alright, so come back to this half of, later half of this today's lecture. In earlier half I have defined what is the Zariski topology and we now we will define, we will study some properties of this topological space. All right.

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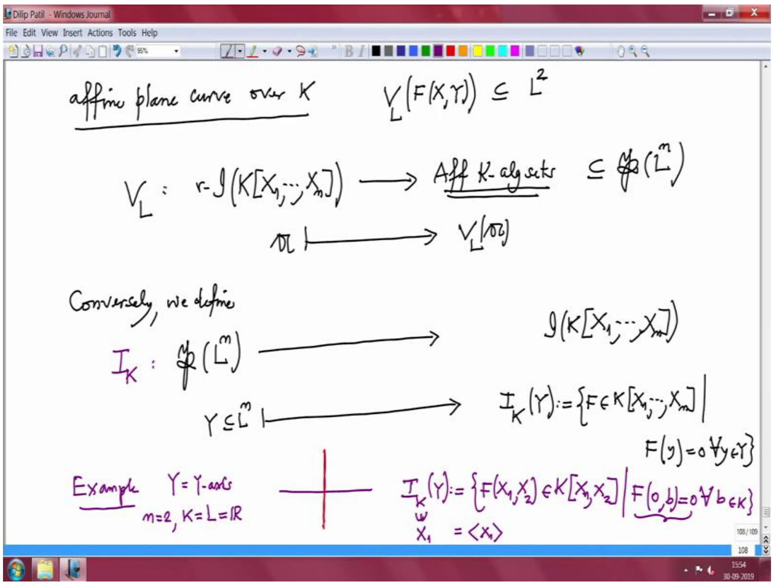

The first one is, so whenever I use for example, when I say plane curve, affine plane curve over K, this means it simply means your field is K and we are looking at VL of one polynomial and 2 variables. I will write it F XY. And this picture we are drawing it in L2. So these are called affine plane curves over K, K because we know that this f is called a defining equations and that has coefficients in K, therefore this over K.

And we have seen in some examples earlier that, when is this infinite and when the compliment is infinite, this I have done by using what is, what I have proved is identity theorem for polynomials. All right. Now, remember our goal is to study the, this topology with the help of ideals, this topology with the help of ideals. So, in other words, I want, I am looking for other way map. So VL, this is a map from r ideals of K X1 to Xn to A double f K algebraic sets.

We have defined this map. This is a subset of power set of L power n, any ideal A going to VL of A. Conversely, we define the map in the other direction. In fact, I will define a map from the power set of Ln to the ideals in the polynomial ring and what is the map? So take any Y, Y is the subset of, arbitrary subset of Ln and where do I map it? I will write it, IK just to remember that IK of Y. So this is by definition. So what is the definition?

These are all polynomials, F in K X1 to Xn such that if I take F of any y, this should be 0 for all y in y. Clear? I take all those polynomials which vanish on every element of y. So little example we write. So example, so let us take again y equal to y axis. That means I am in n equal to 2. And if you like k equal to L equal to R, in this case, usually, you have to take this case because we can draw a picture from which you can get an intuition.

So that means pictorially it is this and y axis. Now, what is this set? Let us write down the definition. So, this is IKY is precisely all those polynomials is now 2 variables, n is 2. So, these are all those polynomials F into variables X1, X2 such that this polynomial should vanish on every point in Y such that F of y, y is y axis, so that is here. So that means the X coordinate is 0.

That means a comma, a is 0. So, 0 comma B is 0 for all b in Y axis. So that is arbitrary K, arbitrary element. So these are precisely, but obviously which is the polynomial here, for example, I gave you few polynomials which have these property namely, one of them is x. So obviously, the polynomial x1 belongs here and any other polynomial when I put x equals 0, it is 0, that means there is no pure Y term.

See this condition means there is F does not contain any pure monomial in y, because if it is there then when I put B there, it will not be 0 for every. So therefore, this ideal will precisely be generated by x1 because all monomials in this, all the polynomials in this ideal will have X1 present in every term. There is no term which is not present, where X1 is not present. So, therefore, this is precisely the ideal generated by X1. So one can calculate like that. All right. Now, once you have a topology, so another thing I want to use, so we have this. We have defined the map. This map, the name I will give is IK.

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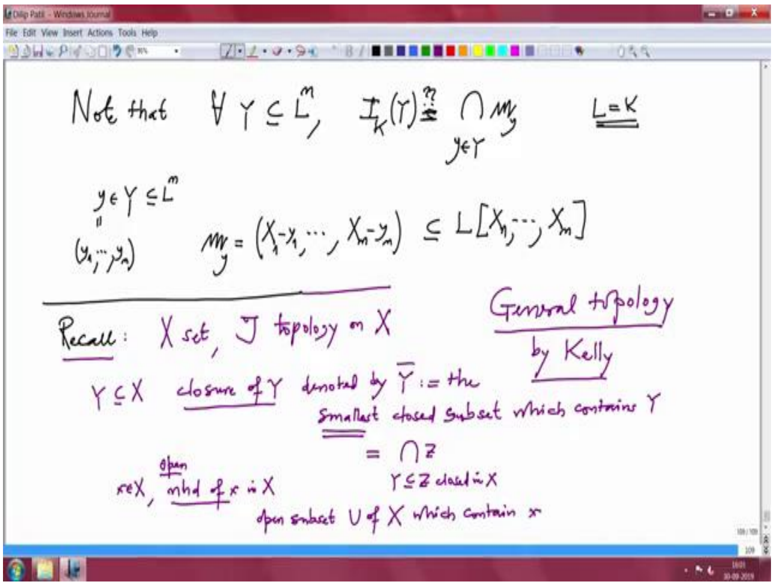

And note that by definition of ideal, by definition this, so I will write on the next page. Note that for every subset of y in L power n, IKY is precisely the intersection y in y. So what is, I will write until, so this is my, okay. Now let us recall where we have y is in L power n, y is in L power m. So elements of y are tuples, small y in Y are tuple, y1 to yn.

So therefore and this is, my is precisely remember, my is precisely the ideals generated by x1 minus y1 xn minus yn and these vanish on, this is the maximal ideal which vanish on the point y1 to yn and everybody vanishes. So it is inter section, but the point to note here is a priori this is an ideal where the coefficients are in L. This is an ideal in L X1 to Xn but we want an ideal in IK.

So we have to, this is, this equality is not quite correct. So I would like you to think about this equality till at least couple of lectures, but definitely it is correct when l equal to k because in that case, this one. So ultimately our aim is to, when we assume that the base will be algebraically close, then these problems will not come, but then these problems will keep coming and then we have to either modify our definition or find remedy which will work in arbitrary case.

So these will come with time and understanding. Okay, another issues I want to raise is when I say if I have a topological space recall. These are all, because I am not going to do topological thing extensively. So, I will recall from first goes on a topology if you would have done it that will be good. Otherwise we will have to recall these things from the the topological books, some basic books on topology.

And I would suggest you should look at the General Topology book by Kelly. This is one of the good book and it is the first course, you can find all the definitions which I have used there. So what is the close? Closed sets I already defined. They are the compliments of the open sets and whenever you want to give a apology and a set, you either give open sets or give a closed sets, that is equivalent because you can get closed sets on the open set by taking their compliments and you can get open sets on the closed set by taking their compliments.

So what is the closure of a subset? So X is a set and let us say tau is a topology on X. What is the closure? So, if I have a subset y of X. What is the closure of Y? Closure of y denoted by y bar and this is by definition, the smallest closed set, closed subset which contains y. That means what, whenever you use the smallest, that means it is the intersection of all those close subsets which contain y.

So y is contained in z. This is intersection of z, this is close in x. So obviously this is, all these are closed. Therefore, it is closed under arbitrary intersection and therefore, this is a closed set and it contains y. Therefore, this is the smallest closed set which contain y. Similarly, what is an interior? Or what is the neighbourhood at a point? So a subset, let y be a subset in X and small y be an element in y, so neighbourhood of y, so why did I write y unnecessarily? This y is not needed. So, I will use given element x, x in x, neighbourhood of x.

This is by definition, let me use the word open neighbourhood of x. Open neighbourhood of a point in X is precisely an open set, open subset, u of x which contain x. So not all open set but open set which contain the given element x is called an open neighbourhood of x. And okay so these are, when do you call a point x to be a interior point of x? So interior point of, now, I have to say interior point of a subset and we will write down some examples.

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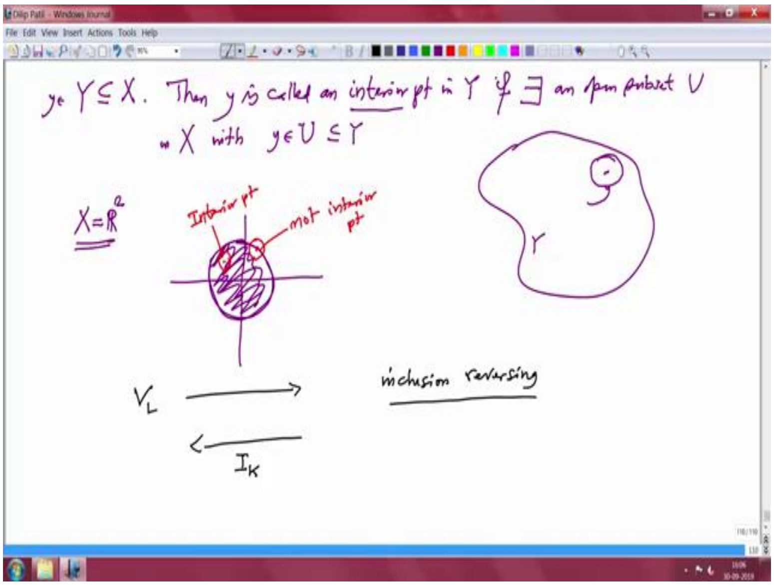

So suppose y is a subset and y is an element here. So then y is called an interior point in Y if there exist an open subset u in x with the y belonging to u, with u is contained in y. So to draw the picture, so here, this is y, this is small y, this is a subset y. So, if I can find a small open set which completely lies in Y then you call it an interior point. So, for example, if I take this disk, suppose Y is my closed disk, closed means it contain the boundary also, including this.

And I am taking x equal to R2. So and if I take a point on the boundary here, no matter what small neighbourhood I take, it will go out of this y. So these boundary points are not interior points, but inside points are interior points because I can take a small radius open disk which is completely contained here. So this is not interior point and this is interior point. So, I do not want to recall too much a topology at once because we will use it and we will recall it when we need it.

So, now going back so we have these two maps, VL in one direction and the other direction, we have defined a map ik. And both these maps are what is called inclusion reversing. What does that mean? That means, if there is inclusion here between the ideals, then the VL will reverse the inclusion. Similarly, see ik is defined for any subset. So if one subset is smaller, then ik will be bigger because ik precisely all those polynomials which vanish, all polynomials vanish on the given subset, that is precisely ik.

Now, I want to study the properties of these two maps. I mean, when will these maps will be the inverses of each other? And in the best classical situation we will prove that these maps are inverses of each other. That is what our aim is, but before that, I will need to study a little bit more algebra. And the first one, for example, I would like to recall that we have kept pending whether the ideals in the polynomial ring over a field are finitely generated.

That means are they given by finitely many polynomials. This is now very important because whenever we want to check something is a affine algebraic set, we only have to check for finitely many polynomials. So for that, I would like to study now, what are called Noetherian rings and also I will study the Noetherian modules. So this is one of the important topics. So, I will start at least and we will study that first and we will come back to this topology and algebraic geometry once again. So, I will go back to study.

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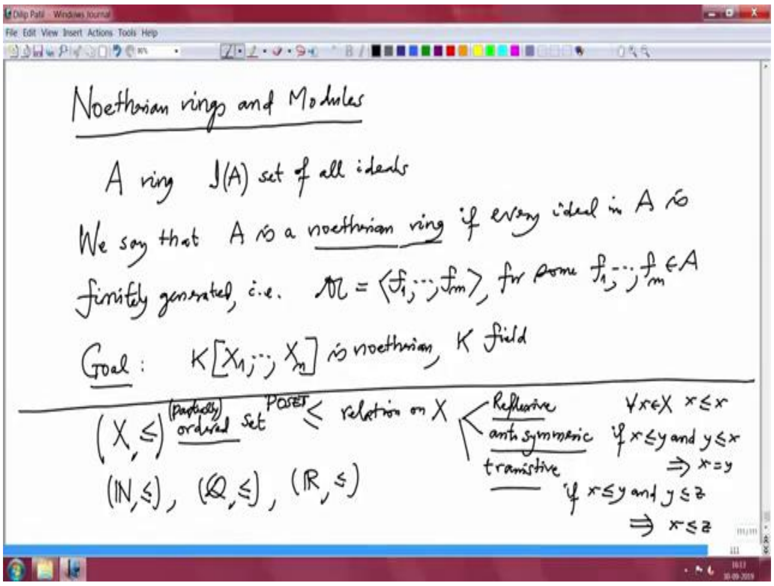

Now, the next topic of studies, Noetherian rings and models. So, now we will keep it pending for our notation that rings, so now I will switch back to the earlier arbitrary notation for our commutative ring A. A is ring and IA is the set of ideals. Then, when do you say the ring is Noetherian? Precisely when every ideal is finitely generated, so we say that A is a Noetherian dream if every ideal in A is finitely generated.

That is every ideal A is the form, we will write like this f1 to fm for sum f1 to fm A and our aim, goal is to prove that polynomial ring over a field infinitely many variable is Noetherian where K is a field. Well, but now, first of all this, I want to improve this definition in the same that it will work for a module also. In a module there is no ideal. So when do you say a module is Noetherian module?

So one possibility is to say that every sub-model of m finitely generated, that is one possibility. But then when our course will go on, more and more algebra will come in and then I will have to say, when the ring is Artinian, so I will have to give a equivalent formulations of this ideal finally generated, if every ideal is finitely generated or no. All right.

So this is what I want to give equivalent condition. Then also when I want to prove theorems about Noetherian rings, Noetherian modules, etc, I will have to prove a theorem twice, but I want to set up a notation and definition in such a way that many theorems about Noetherian, Artinian, I can prove it together. All right. For that I will first recall what is a Noetherian? So first let us recall this was a little bit confusion in earlier also.

So, I would recall formally when I have an ordered set x less equal to, ordered set, what does that mean? That means this is a relation on the set x, which satisfies three properties and what will be the three properties? Reflexivity, anti-symmetry and transitive. So this means x x, means x is less equal to x for all x. Similarly, if x is less equal to y and y is less equal to x, then x equal to y, this is anti-symmetry, this is reflexivity. This is true for all X and Y where x is less equal to and y is less equal to.

Transitivity is x, if x is less equal to y and y is less equal to z, then x is less equal to z, that is the transitivity. And typical example of a ordered set is n with the natural order or q where this order is now extended from n or r. All these are ordered sets. And but c, there is no order. Remember c does not have order. So this is our ordered set. Normally, most of the books, we will refer to it as a partially ordered set.

And also computers science guy will write posset. But remember, I am not going to use this word partially, I will only keep saying, ordered set. And when do you say the set is totally ordered? That means any two elements are comparable. Given x and y, x is either less equal to y or y is less equal to x.

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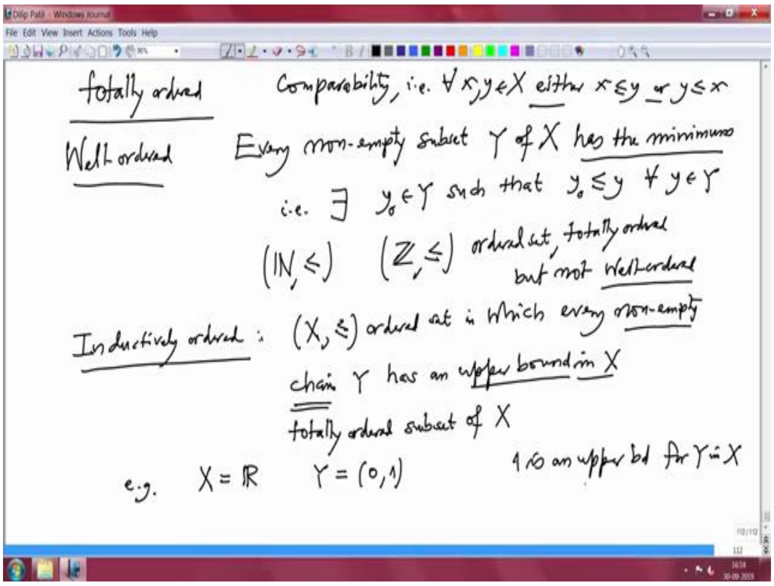

So now, what is well ordered? So totally ordered we have defined. Totally ordered means comparability. That means, for all x, y in X, either x is less equal to y or y is less equal to x, either one. Well order that means every non-empty subset y of x has the minimum. That means, there exist an element y naught in y such that y naught equal is less equal to y for every y in y.

So, typically n, this one of the axioms says that n is well ordered. That means any non-empty subset of n has the minimum. This is also called minimal principle or well ordering principle. Now, if you see if you look at this ordered set, z less equal to, this is ordered said. This is also totally ordered, but not without it. Simply because if you take the negative integers, that does not minimum.

You can keep going on the negative set as long as one want. So it is not well ordered. There are not well ordered set or even Q or R, they are all not well ordered sets. They are totally ordered, not well ordered. And remember, when we use Zorn's lemma also, I have used the term, inductively ordered. What was the inductively ordered? That means, ordered set in which every, if every chain, non-empty chain has an upper bound, then you call that set to be inductively ordered.

So this means x ordered, ordered set in which every non-empty chain, chain y has an upper bound. What is a non-empty? Chain is what? Chain is a totally ordered subset of x. This is a totally ordered subset of x. And this upper bound means it is an element which is bigger than equal to every element in y, but that element may not be in y.

So for example, if you take so example, if you take x equal to r with usual order, and if you take y equal to the open interval 01, 1 is an upper bound. 1 is an upper bound for Y in x. So upper bound in x you should say. So say like upper bound, we have a lower bound also. 0 is a lower bound or minus 1 is a lower bound and so on. And see most of this language is very clearer if you use it correctly.

Now, I want to define two kinds of sets, two kinds of more ordered set. So what was well ordered? Well-ordered was every non empty subset has a minimum. Now, I want to define what is Noetherian ordered and Artinian ordered. A Noetherian ordered be every non-empty subset has maximal element or maximal element, not, I am not saying the maximum, but a maximum limit.

Similarly, I will say Artinian is ordered set in which every non empty subset has a minimal element. Now there could be many. And now, I will study these basic properties of this Noetherian and Artinian in the beginning of the next lecture and we will use this language to prove that the polynomials ring over a Noetherian is again a Noetherian. In particular for a field, if k is a field, k polynomial in one variable is Noetherian and keep repeated use of the theorem.

I said we will conclude that polynomial ring over a Noetherian ring infinitely many variables is also Noetherian. And then we will study also modules of the same time. So with this I will conclude this lecture and then we will continue our next lecture, starting from the study of Noetherian and Artinian ordered sets and use this terminology to prove theorems about Noetherian modules and Artinian models and you will see, we only have to prove only one theorem and other theorem will come like a duel.

So with this language our job of proving theorems in Noetherian and Artinian reduced to the half. So that is the advantage of this language okay. So with this I will stop and we will continue in the next lecture. Thank you.