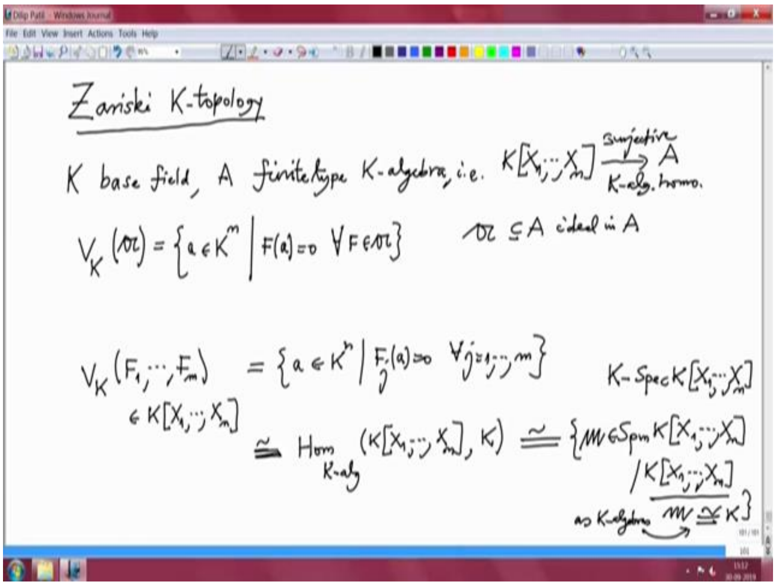
Introduction to Algebraic Geometry and Commutative Algebra.
Professor Dr. Dilip P. Patil.
Department of Mathematics.
Indian Institute of Science, Bengaluru.
Lecture-15.
K Zariski Topology.

(Refer Slide Time: 0:39)



Welcome to this lectures on Algebraic Geometry and Commutative Algebra. Today in this lecture I will do what is known as Zariski topology. I will be more precise and say Zariski K topology and this is the topology. I will recall what a topology means and we are going to, our notation is the following. So, K is fix base field and a is finite type K algebra and in this setup, in the last lecture, we have defined what is V K of an ideal A where A is ideal in A. And what is this? Let us recall quickly this is all those points a in K power m.

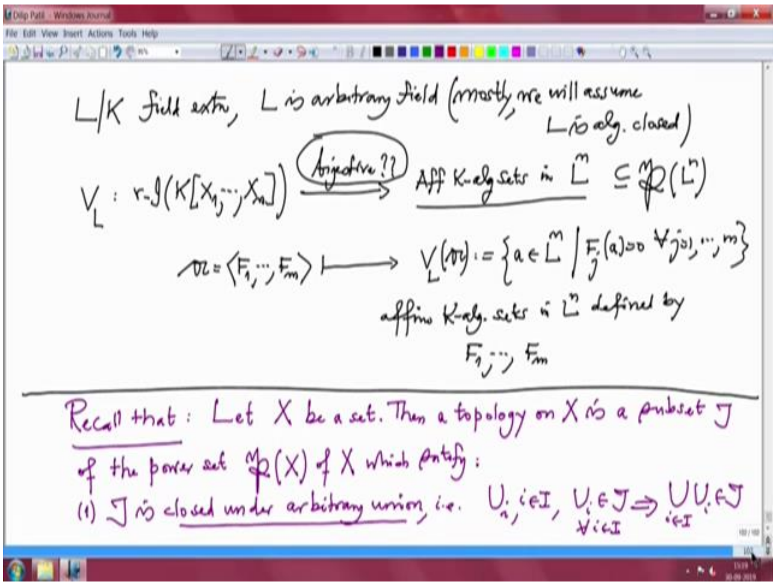
So, when you when this A is a finite type K algebra, that means A is a quotient of polynomial ring in n variables. So, that means there is a surjective map, surjective K algebra homomorphism from polynomial ring in n variables to K. This is surjective K algebra homomorphism. Initially we have defined this V K of polynomials and those are all those points in K power N, such that polynomial vanish there.

So, this is same thing as f of a is 0 for all R polynomials f in A, that is what the definition was. And we wanted to, we have also said that so that means we have defined a map from, actually a let me recall that quickly. So we have defined the maps like this. First we defined this for VK was defined for polynomials F1 to Fm where these polynomials F1 to Fm are polynomials in n variables.

And this was by definition all those a in K power n such that Fj of a is 0 for all j 1 to m. But we have also identified this set, this is an identification we have given with homomorphisms K algebras from the polynomial ring in n variables to the base field K. This also we have identified with all those maximal ideals m in the polynomial ring such that the residue field at this M, this is isomorphic to K as K algebras. This isomorphism is as K algebras.

And they said we have been calling it a K spectrum of the polynomial ring. So strictly speaking, when we write this VK of a for ideal in this A, what you do is, if you want to look at the K spectrum of a, that is look at all those maximal ideals of a such that the residue field is isomorphic to K. And then we can identify this VK of a is all those points, so this should be for all polynomials capital A where it goes to the small a.

(Refer Slide Time: 6:11)



And then we have defined, we have defined a map. So we are thinking this map, we again, now more generally I have actually defined for L any field extension L over K field extension, where we have K is a base field and this L is arbitrary field. And in due course we are going to mostly assume, we will assume L is algebraically close. But when I start doing that, I will mention it. So, today L is arbitrary field extension of the given field K alright.

So, let me write, so let me first, we have defined this VL, this is a map from radical, so the notation I use was small r Rad, small r, not Rad sorry, this was small r I of the polynomial ring in n variables to what I call it affine A double f K algebraic sets in L power m. This is a set, this is a set and this is a subset of the power set of L power m. And in the last lecture I gave examples to show that this map, every element, every subset may not be an algebraic, affine algebraic set.

And this map was any ideal or any ideal generated by A which is supposedly generated by F1 to Fm. This is not, we have not yet proved this, I will prove it that every ideal here is finitely generated, assume that for the time being. And this ideal where does it go? This go to VL of ideal A which is by definition all those elements a in Ln such that Fj of a is 0 for all j from 1 to M. These are called affine algebraic K sets defined by the equations F1 to Fm. Let me write once at least, affine K algebraic sets in L power m defined by F1 to FM.

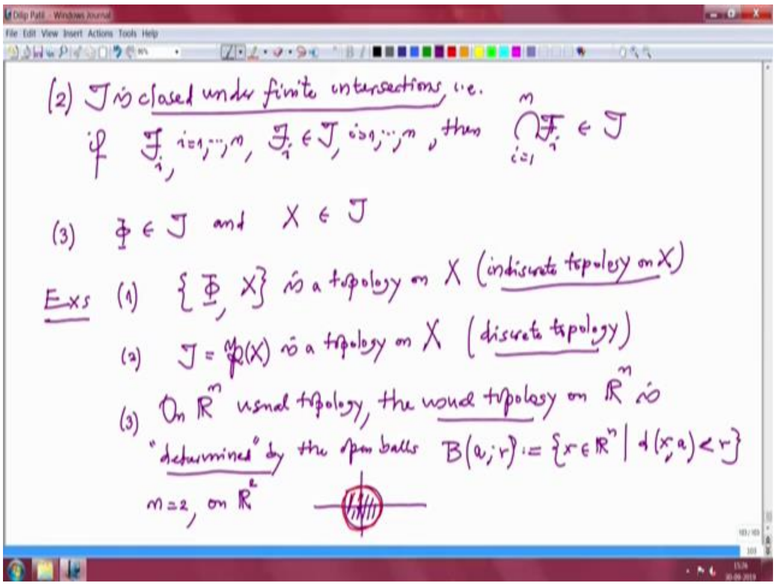
Now, there are so many things that, this depends on the these variables and coordinates and this I am going to soon, after a couple of lectures, I we go completely coordinate free, it will completely depend on the rings. But just to understand the classical way what people were doing earlier and how it matches with the modern algebraic geometry we have to do this. So, this map, our main concern to discuss that when this real map is a bijective, that is our main concern.

So, I will just write that, just put a question mark, when is it bijective, that is what our main concern in couple of lectures? So, that will give us possibility to deal, to connect algebra with the geometry that is what the main aim in this course is. All right now, right now, these affine algebraic sets are only this algebraic description is there. But to bring in more geometry and topology, I need to define topology on this. So, let me recall now what is a topology?

So, recall that, so let be, let x be a set, capital X be a set. So, what is the topology on X? Then a topology on X is a subset of the power set of, power set is as you saw, it is denoted by PX, which satisfy the following properties. Number 1, arbitrary union, so topology on X is a subset, let me give a name tau, tau is a subset of the power set and it satisfies some properties. And what are those properties?

So, tau is close under arbitrary union. That means, if I have subsets Ui's, Ui is indexed by arbitrary set i. If this Ui's belong to tau. Remember tau is a subset of the power set, so it contains some subsets. Then union Ui also belongs to tau, this is true for all i in I, then the union is also there. This means it is closed under arbitrary union.

(Refer Slide Time: 13:24)



Second one, tau is closed under finite intersection. That is if Fi is, Fi is not good, if gi, if you need some subset, so VI, no I do not know it is VI, script fi, i is from 1 to N, Fi belong to tau for all i then the intersection scooped Fi, I is from 1 to n also belong to tau. This means it is closed under finite intersections. Okay, then third one, empty set belongs to tau and the whole set X also belongs to tau.

If these conditions are satisfied, then we say that tau is a topology on the set X. And on the same set there can be many topologies, let us see some quick examples, so quick examples. Number 1 if I take the collection just 2 elements phi and X, this is clearly a topology on, this is a topology on X, this has only 2 elements and this topology is also called indiscrete topology.

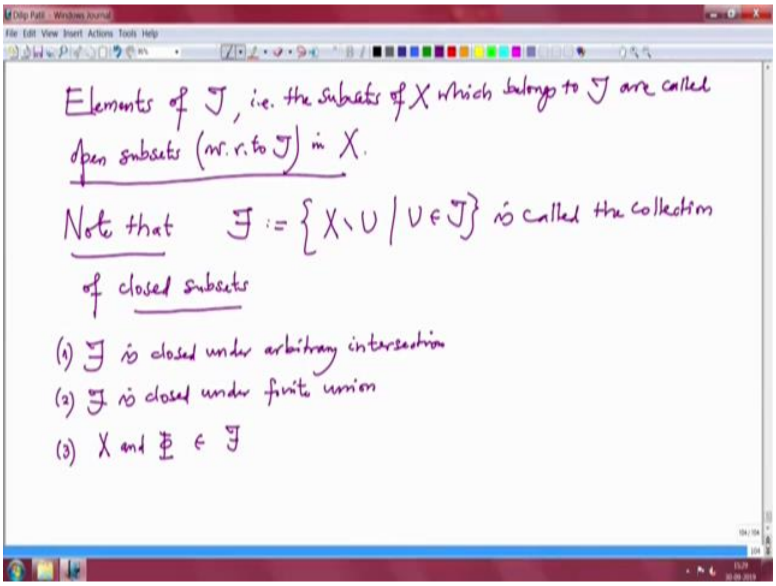
So, this is one, second, these are 2 extreme examples. On the other side if I take tau is a whole power set, this is a topology on X, this is called discrete topology and this is called indiscrete on X. So, on every set there are definitely 2 topologies. Now, let us take 1 more example, on R, there is a topology called usual topology, this topology is used to study what is called real analysis.

And what is that topology? In this topology you describe what are the elements in this tau. So, then the usual topology on R is, I will just say determined by the open… actually Y R n, R power n by the open balls. What are the open balls that is usual notation is B, center is a and radius is r. And this is by definition all those X in Rn, such that the distance from X and a, is fixed a is less equal to, strictly less than r.

So, if I want to draw the picture, let us say in n equal to 2, the picture is n equal to 2, n equal to 2, that is on R2. The open balls are, if I have to draw the pictures, there the, this is a centre a and this is the disk. Do not take the boundary, the boundary is omitted from that and take only the inside thing, only inside thing. This is an open disk, the disk because we are in R 2. If you are in R3, then it will be like a ball without a boundary, only the inside of the ball.

And what do I mean by determined, that means the elements of the tau are precisely unions of such disk. So, this determine, this one will come, soon I will make it more clearer. Similarly, on complex numbers we can do it. And this one is usually used, this is the basis of actually the real analysis, complex analysis when we say open set, close set, etc. Okay last thing, the elements of this tau are called open sets.

(Refer Slide Time: 19:45)

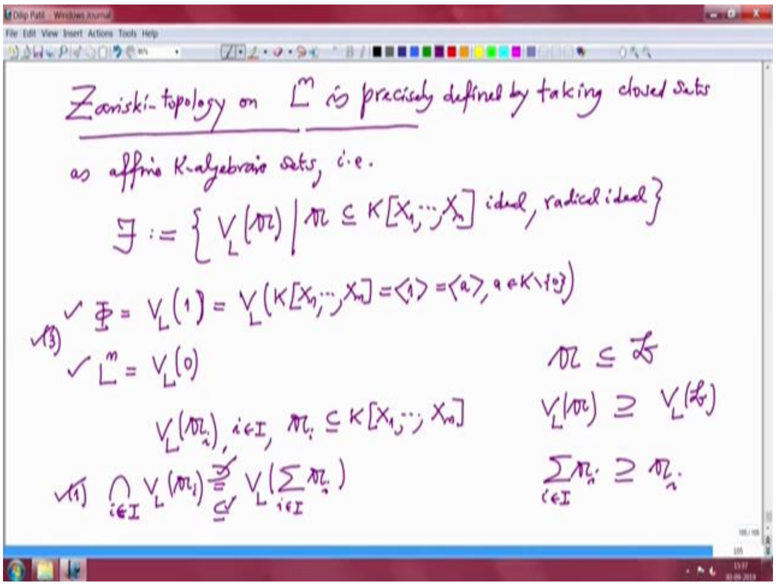


Elements of tau well, elements of tau are subsets of the given set X. That is, the subsets of X which belong to tau are called open subsets. Actually, strictly, I should write with respect to tau in X. Because if somebody gives a different tau, then open sets are different. Okay. Now know that if I want to give a topology on a set X, I will give the compliments. So the compliments, so that is script f, this is all those Y in tau, not in tau, all those compliments X - U, where U varies in this is called the closed sets, the collection of closed subsets.

So, in other words closed subsets are the compliment of the open sets. And therefore, open sets are compliments of the closed sets. And what will be the corresponding properties of the tau, this F, now the union will become compliment of the union will become intersection, so therefore intersection of those. So, that means this F is closed, so the first property will become F is closed under arbitrary intersection, that will be the corresponding to 1.

And second will become F is closed under finite union and obviously, third one that empty set will become X and X will become empty set. So, these are elements in the f. So, if you want to give a topology on it a set either you declare what are the open sets, but then they should satisfy the 3 properties or you give a collection of closed sets, that is they should satisfy these 3 properties and then we get a topology on the set X, where precisely the close cents are the given ones. Or precisely the open sets are the given ones.

(Refer Slide Time: 23:20)



So, now, I want to give an example of a topology which is usually called a Zariski topology. So, a Zariski topology. So, on L power n is precisely defined by the, I will give you a collection of closed sets defined by taking close sets, closed sets as affine K algebraic sets. So what does that mean, that is F is taken V L of an ideal a, where a is an ideal in the polynomial ring.

And we may assume the radical ideal, because saw the 0 set that V L of it does not depend on the ideal a but it depends only on the radical ideal a. So this, now to show that they satisfy the properties of the closed sets, we have to check this 4 things. Namely, it is closed under arbitrary intersection, it is closed under finite union and empty set and whole sets are elements there.

So, the first I will take third one. So, what is empty set, empty set is precisely VL of unit ideal or the polynomial 1, constant polynomial 1, there is no zero, that means it is empty set and this is also same thing as VL of the ideal Polynomial ring. This is a unit ideal in that polynomial ring. So, it is this, this ideal is generated by the polynomial 1 or for that matter any other content nonzero a, where a is in K, a is a nonzero constant, because they are all units.

So empty set is there. How about the whole said, that means L power n, okay so L power n, I want to write this as 0 set of some ideal or some finitely many polynomials with coefficients in K. But well that is VL of 0 Polynomial or 0 ideal. So, obviously, ideal is 0 so, it vanishes on every point. Already, the only polynomial in that is 0 and 0 polynomial vanish on every element in L power n. So, this is clear. So, we have checked the whole set and these are algebraic K sets.

Now, the first one that is it is closed under arbitrary intersection. Closed under arbitrary intersection, what do I have to show? If I take affine algebraic K sets, so they look like this, VL of Ai, this is arbitrary family i in I and an Ai's are ideals in the polynomial ring K X1 to Xn. And then if you take the arbitrary intersection that is this i from i in I this, this is again, this should be again VL of somebody, because affine algebraic case, it is precisely VL of some ideal.

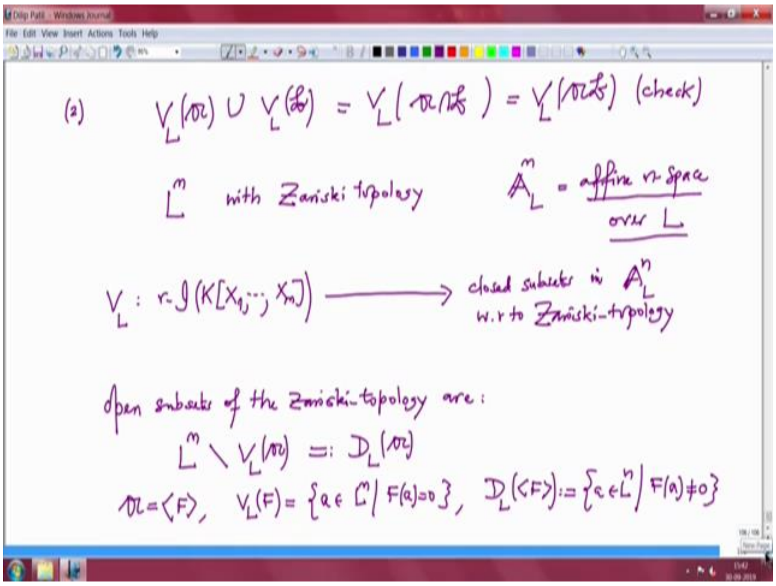
So, what will that ideal be? The right candidate for that is summation i in I Ai. Again recall that when I write this notation, this is a notation for the smallest ideal which contains all these given ideals Ais. Or equivalently this is the ideal generated by all the elements in or any element in this ideal is precisely the finite combination of the elements from the corresponding Ais and the coefficients allow them to be arbitrary polynomials in K X1 to Xn.

So, we have to take this, once we checked this, then our first condition will be checked. But note that these affine algebraic sets have the property that it is clear from this, the smaller the ideal bigger the affine algebraic set. So, what I am saying is, if a is contained in the ideal B, then VL of a and VL of B, what is the relation that this is inclusionary versus. So, therefore, obviously, all this ideal, some ideal contains all of them.

Therefore, the some ideal Ai i in I contains all the Ai's. Therefore, when I play V, this will be containing all VLs. So, that means, this inclusion is clear, this is clear. Now, to show that an element here is also an element here, that means, what do I have to show, if you have a point here is L power n, which vanishes on every polynomial in every ideal Ai, then it also vanishes on this. But on every Polynomial here, but any polynomial here is a finite combination of the polynomials with these corresponding Ai's and the polynomial coefficients.

But then each combination will also vanish, so therefore, this is also clear. So, this was the property 1, so, we have checked 1, this was 3, so, we have checked 3, now we only have to check 2. And what was 2, 2 is it is closed under finite union.

(Refer Slide Time: 30:50)



Alright, for that is what is enough to check is. So, suppose this is I am checking second property. So it is enough to check that intersection of 2 elements, 2VL a and intersection VL b, this also should be VL of somebody, that means it is union, not intersection, union. This is also VL of somebody but what could that be? That is obviously, you can guess that is intersection of 2 ideal or this is also same thing as VL of the product ideal.

And again here you use the earlier argument to check that, the smaller the ideal bigger the VL. So, I would just leave it for you to check this equality, they are not difficult. You take an element here and show it is here. And note the fact that A times B is the smallest, A times B era ideal generated by the product of the polynomials. So, with this you get a topology on L power n and this topology is called the Zariski K topology, because we are defining our ideals are in the polynomial ring over K.

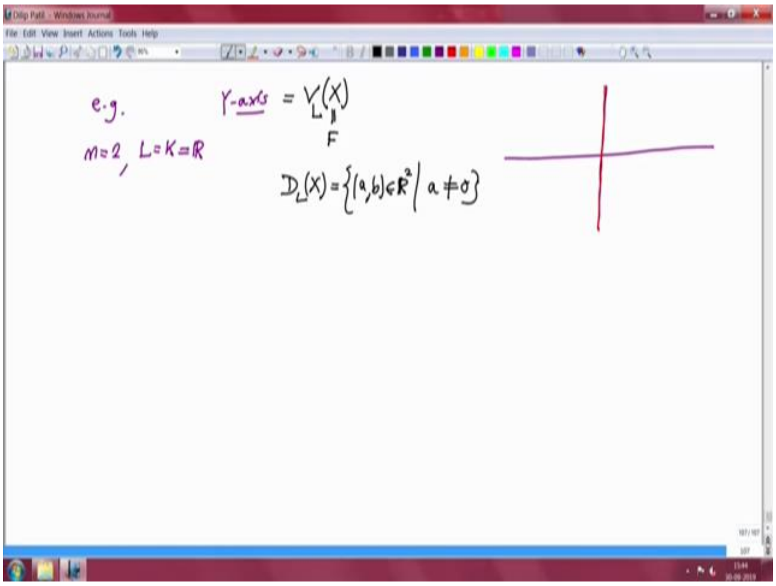
So, now we are going to equip L power n with Zariski topology. And that is also, one uses, if you see as you algebraic geometry books, they will use such a notation. This is affine n dimensional, affine algebraic, affine space, this is affine n space over L. So it is not merely a set, it is a topological space also. So our map VL is now map from the radical ideals in the polynomial ring to these closed sets.

So closed subsets in An L with respect to Zariski topology. And now our problem is to study some basic properties of this topology. For example, this is what we will do it in next couple of lectures. So, what are the open sets, these are the closed sets, closed sets we have defined affine algebraic K sets.

Open subsets of the Zariski topology are precisely, now, the compliment, so L power n compliment VL of an ideal, radical ideals, but this I want to give some notation for this. So, this is D L of the ideal A. So, this is what all those points which is not here. That means, there is at least one polynomial where it does not vanish. So, in particular let us take to understand, let us take ideal A.

Suppose it was generated by a single polynomial, principal ideal. Then what is VL of a, VL of f, this is the 0 set of f. This is a in L power n such that F of a is 0 and. What is D L of this principle ideal f, this is precisely all those elements a in Ln such that F of a is not 0. So, for example, it is interesting to understand this.

(Refer Slide Time: 35:53)



So, for example, let me just take F equal to say y axis. So, that is what, so I am taking n equal to 2 and if you like L equal to k equal to let R, so that we can draw the picture. So, this and I have taken this y axis and we want to know whether it is close, it is open and so on. So, what is VL of, so when I say Y axis, this means it is VL of X and X is 0 you get precisely y axis. So, I should not write here, F here. So, in this notation this is our S in the earlier notation.

And what is the compliment, DL of X, this is, remove this Y axis from the plane. So this is the can write down in the notation. This are all those points a b, a, b in R2 and you are not allowed y axis, that means, that means a can never be 0, so a is nonzero. And we will deal with such examples more and more to get more and more acquaintance. And now we will also study a little bit basic properties of this Zariski topology. We will continue after the break. Thank you.