An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 09
Examples of Smooth Manifolds

Hi, so last time I had stopped, towards the end of the lecture. I had mentioned 3 conditions one has to check to ensure that some set is a manifold. So let us see how one actually does that with some simple examples. Later on, in the course we will have more non-trivial examples.

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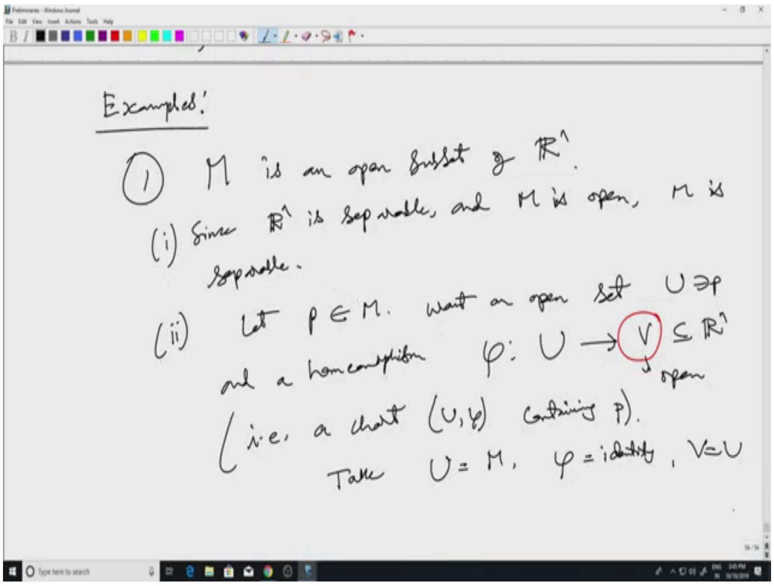

To begin with, let us take M is an open subset of R n. So, the set I start with is an open subset of 1. And I would like to claim that this is an n-manifold. So, this since it is an open subset of R n, actually just a subset will do. We can regard M as a metric space. Namely the Euclidean metric can be gives rise to a distance function here. And since R n is separable and M is open, M is separable as well. M is separable, so this takes care of the first condition. Second one is that locally Euclidean, locally Euclidean of dimension n. The obvious thing to do is we want to, so, let P belong to M, I would like to find an, want an open set U which contains P and homeomorphism phi from U to some V in R n, where V is again open.

So, in other words, I want a chart containing P, i.e. a chart, this pair, we have already called a chart, a chart U phi containing P. So, given any point, I should be able to construct this, but in our case m itself is an open subset. So, just take whatever P is take U to be M itself and phi equals identity and of course, V is also equal to, this V here, is also just take to be U itself. So this is completely trivial. So the locally Euclidean property is satisfied with just one chart in this case.

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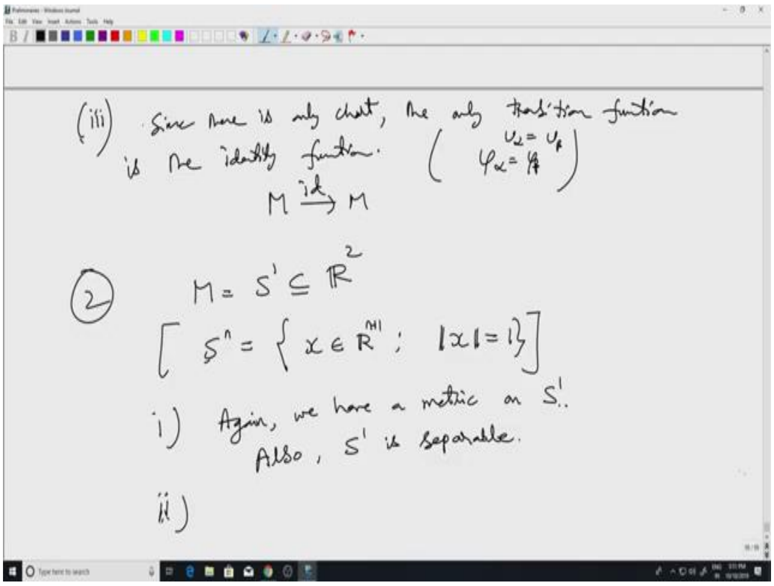

And since there is just one chart, the question of transition functions being smooth becomes quite trivial. So, the only time when the transition functions arise when you have U alpha intersection U beta is not empty. Since there is only one chart, the only transition function is the identity function because, well phi alpha equals phi beta, U alpha equals U beta, phi alpha equals phi beta. So when I take phi beta composed with phi alpha inverse I just get the identity.

Of course the identity function, this will be from M to M itself. Identity and one is done. So, now for something more interesting, let us start with M equals S1, for this, let us recall that Sn, the standard notation Sn, a superscript n is by definition, the set of all points in R n such that norm x equals 1. This is the unit sphere with center, the origin in R n. Here I have taken n equals 1, sorry this is not R n, this is R n plus 1. Sn is the unit sphere in R n plus 1. So S1 is the unit sphere in R2, which is just the circle, unit circle in R2. So, we would like to claim that S1 or more generally Sn is a smooth manifold of dimension n.

So, let us begin by constructing some obvious looking charts on S1. But as I said, even before we construct charts, we have to verify that this is actually second countable and so on Hausdorff. Again, since it is given as a subset of R2, we have a natural metric. Again, we have a metric on S1. And so the point is that once we have a metric, we do not have to worry about Hausdorff. We and the only thing we have to do is check separability. Separability also is something one can see directly on the, it is just a matter of constructing a countable dense subset of S1.

And this can be seen for instance by starting with the rational numbers in R and mapping R to S1 by cos theta, sin theta. That is just one way of doing it but there are other ways also. So I will not go stress on this too much. So let us, I just remark that S1 is separable as a metric space. What I want to focus on is, I would like to construct charts on S1, the locally Euclidean of dimension 1.

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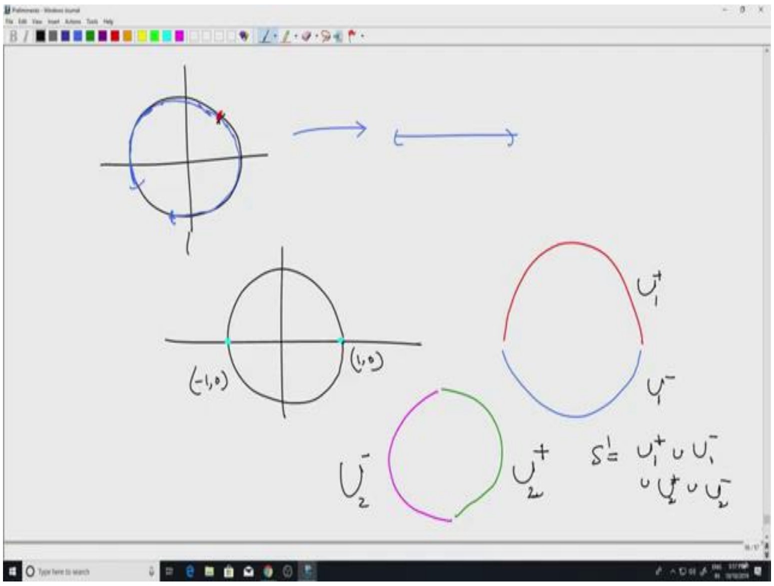

So how do we construct charts? So, after all what is a chart? A chart is something, it is given a point, we would like to find an open subset of the space around that point containing that point, which can be mapped bijectively to an open subset of R n. In this case, for instance, we want to map it to an interval in R.

Now it is, if I start with any point, let us say some point here, then it is quite clear that as, as long as I do not include the whole circle, any arc containing this given point, any arc which contains this point, so the art can be as big as one wants. So for instance, I can have an arc of the circle which goes all the way till here. Intuitively, it is clear that this arc can be mapped bijective in a homeomorphic manner to an open interval in R.

So, one can think of it like unwrapping, not unwrapping, rather straightening a piece of a string. So, imagine a string has been put in the shape of a circle. So, I just straighten out the string that is a homeomorphism onto an open interval. In fact, there is a, this can be done as I show next. And, but there is a instead of taking a large arc like this what happens if I take, I will just take semi circles. So, I will divide the circle, entire circle into 4 pieces. The first piece consists of the upper semicircle. I need open sets, so I cannot include the endpoints here. This is one open set, which I will call U1. Then I look at the lower semicircle. Well, I will call it U 1 plus this one I will call it U 1 minus.

Now, if I just take this two, the point is that they will not cover the whole circle. I need every point in the circle in M to be in one of these charts. So, the problem with just these two is that of course, this point and this point, which are actually just minus 1, 0; 1, 0 these two points are not in either U1 plus or U1 minus. So just to all, take care of these two points, I will have to throw in 2 more arcs. So what I will do is, I will take the left, this may not be so visible, so let me take a darker color.

So I will take the left semicircle, the right semicircle, so this I will call U2 minus and this is U2 plus. So altogether I have decomposed S1, a union of force arcs, U1 plus union, U1 minus union, U2 plus union U2 minus. And, of course the, from the, the very fact that S1 has been written as a union of these 4 sets, by definition means that every point of S1 is contained in one of these 4 arcs.

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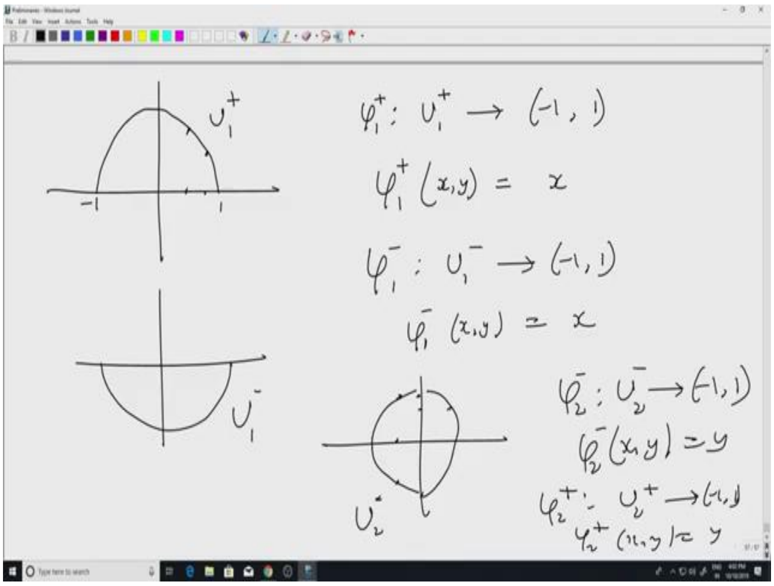

Now, but the, we need, a chart is not just an open set, we also need homeomorphisms. So let us start with U1 plus. Let us see what a homeomorphism can be? This is not quite a good picture. So this is U1 plus. So, I would like to basically associate, so given any point here, on this arc, I would like to associate a point in R. So what I will do is, the obvious thing to do is, if I start with any point, I can just look at the x coordinate of that point. So, to every point on this arc, there is a unique x coordinate and that is going to be my map.

So, I define phi 1 plus from U1 plus 2 and the range of x values. So, this is minus 1 and this is 1. So, minus 1 to 1. The map is defined to be phi 1 plus of x,y. Remember that since S1 is a subset of R2, every point of S1 has given by a pair of it has two coordinates x, y and I just define it to be x. Similarly, when I work with U2 rather no this is not U2. This is still U1 but with the minus sign, when I have phi 1 minus from U1 minus 2, I do the same thing, minus 1 to 1, phi 1 minus x, y equal to x.

So, this map phi 1 plus is going to map the upper hemi circle to this open interval minus 1, 1. And this does the same from this to this and it is as far the other two, well, so this was U2 minus. So I define phi 2 minus from U2 minus. Again, so this time, what I do is, if I do the same thing as I did, by mapping to the x coordinate, I would not get a biject 1 to 1 map because this point here, and this point here, for instance, will get map to the same point.

So anything above and directly below will get map to the same, will have the same x coordinate. But we just look at the y coordinates now, so anything here will get, I can map to this point. So and the range of Y values is minus 1, 1 again. So phi to minus of x, y is y. And similarly, phi 2 plus of x, y as y, so even the right side, I just take a point, look at the y coordinate. That is going to be my homeomorphic.

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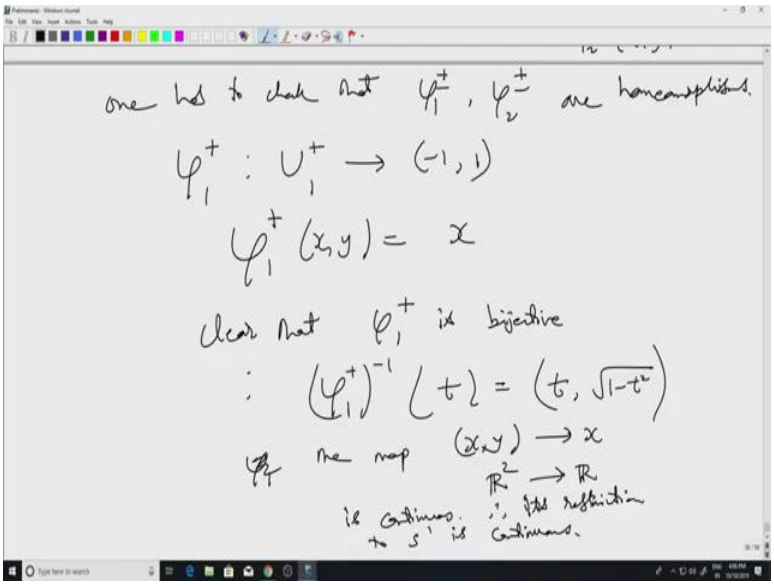

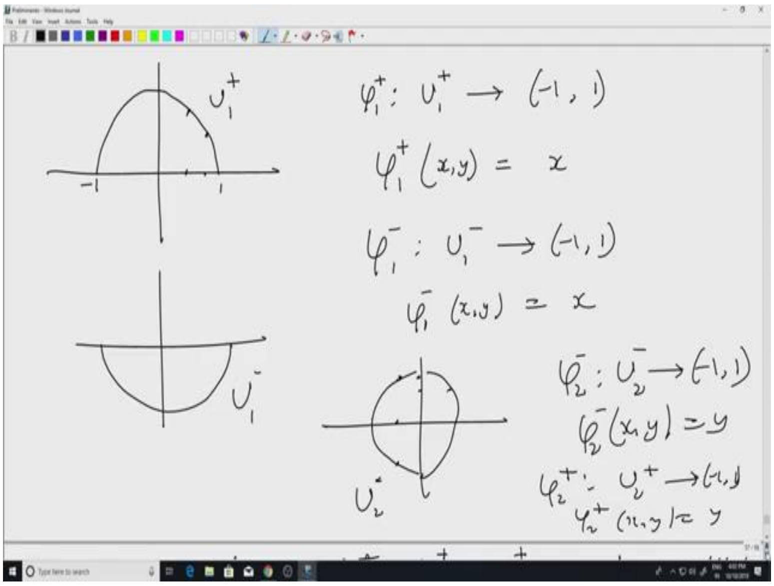

So, well, I have defined maps. It remains to that phi 1 plus or minus phi 2 plus or minus these 4 maps are homeomorphisms. Fortunately, we will not have to do this kind of very detailed calculations subsequently, but at least once we can go over all the details. Now, what does it let us try, let us do it for phi 1 plus, phi 1 plus is a map from U1 plus 2 minus 1, 1. Given by phi 1 plus x y, it is x. It is quite, it is clear that phi 1 plus is bijective. In fact, one can explicitly write down the inverse map, phi 1 plus the inverse acting on some real number t is just, so I am given the x coordinate, I would like to recover the unique point on the upper hemi circle corresponding to the x coordinate.

So we know that, that has the y coordinate has to be square root of 1 minus t square, the positive square root of 1 minus t square will give me the. So, I have explicitly constructed a inverse, the moment you have an inverse, of course, the existence of an inverse is equivalent to this map being bijective. So, we have the bijective part is taken care of and also why is, why is are these two 1, then one has to take phi 1 plus and phi 1 plus inverse are continuous.

So, let me quickly say why that is a case. One easy way to see that is the map x, y going to x, this is a map from R2 to R is obviously continuous. Therefore, its restriction to S1, the point here is that the topology on S1 can be regarded as coming from the metric induced by R2 or one can just look at the topology on R2, one take the subspace topology on this, it is the same thing. So therefore, if I have a map on the bigger space, it is restriction to S1 is continuous.

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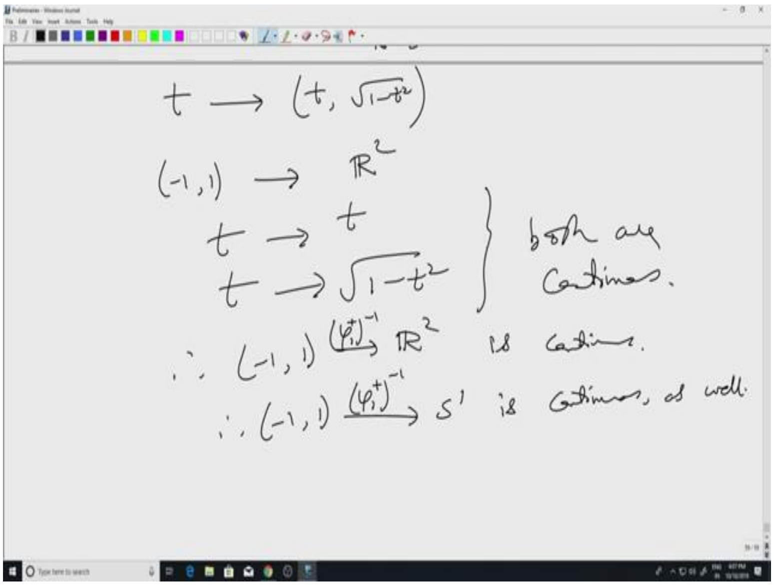

Now, as for the inverse map, the inverse map I wrote down already, it is t going to t square root of 1 minus t square. Well, this I can just regard it as a map from minus 1, 1 to R2. Now when I have a map from, say an interval into R2 and I want to check it is continuous. It is a same thing as checking the continuity of each component, component function. The first component as t goes to t, the second one is t goes to square root of 1 minus t square, t going to t, t going to square root of 1minus t square, both are continuous.

Therefore, from here to here, phi 1 plus inverse is continuous. And if I regard again because of subspace topology and so on. So, therefore, if I regard it as a map into S1, this is continuous as well. So, this proves that what we have shown so far is that it is a topological n- manifold.

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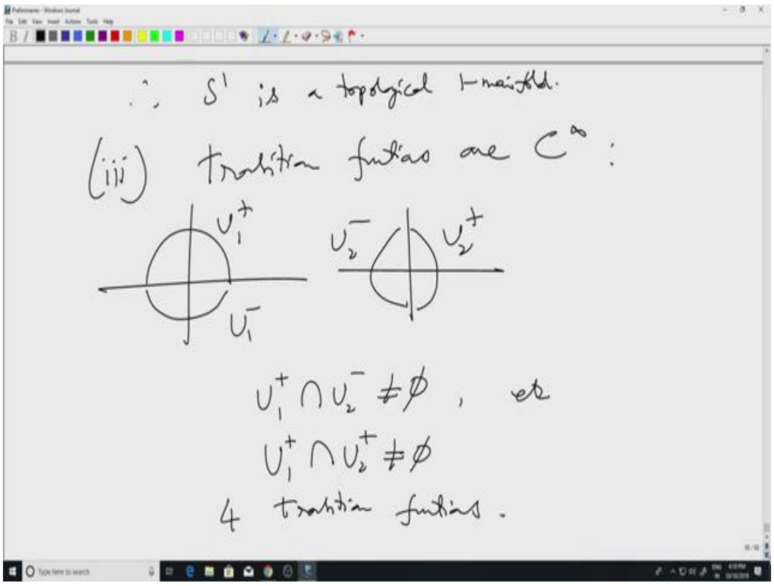

Therefore, S1 as a topological 1 manifold. Now the, let us now we have the task of the last step would be transition functions or C infinity. Now first of all, let us just quickly observe what possible transition functions can arise. This is 1, so U1 plus, U1 minus, U2 minus, U2 plus. So, I have 4 charts. Now, transition functions are defined only when 2 charts have non-empty intersection, U1 plus and U1 minus do not intersect. Similarly, these 2 do not intersect.

But U1 plus intersects with U2 minus, U1 plus also intersects with U2 plus. And similarly, U1 minus intersects with U2 minus U2 plus. So all together, et cetera, so all together we will get, so there are four possible intersections. So we will get 4 transition functions. Now let us quickly see what I just checked one of them.

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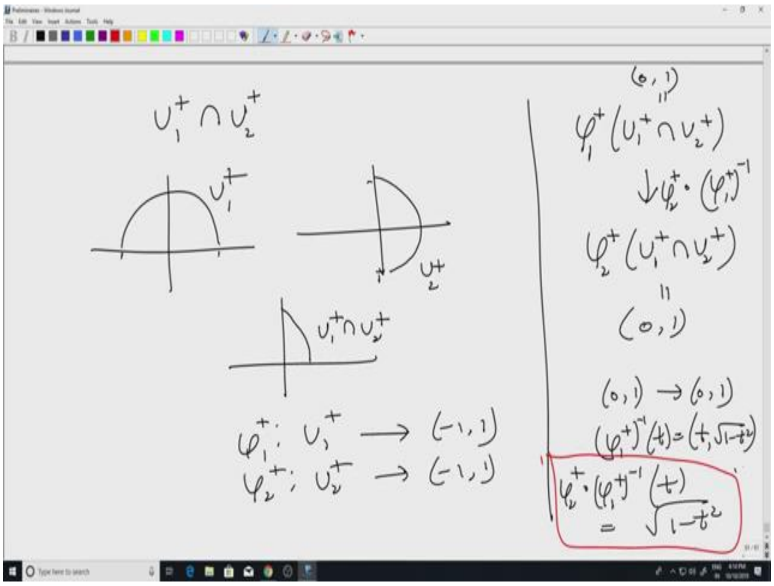

So let us take U1 plus intersection U2 plus. So, U1 plus was here and then U2 plus was this. Their intersection actually is just going to be this, the open arc U1 plus intersection U2 plus. This is however, the transition function is not defined on U1 plus (intersect) it is rather the transition function here phi 1 plus was mapping this two, this portion and the transition function was mapping to minus 1 to 1. So if I look at this, so both phi 1 plus maps U1 plus 2, minus 1, 1, phi 2 plus does the same thing of course it is a different map.

Now this intersection, so the recall that the transition function was defined from phi 1 plus of U1 plus intersection, U2 plus 2, phi 2 plus of the same thing. Now, this phi 1 plus of U plus intersection U2 plus in other words of this portion, this is just, so phi 1 is the map phi 1 plus was just looking at x coordinate. So x coordinates for this arc, it varies from 0 to 1. Similarly, this was looking at y coordinates, y coordinates, although varies from 0 to 1, so 0 to 1.

And this was phi 1 plus inverse composed with phi 2 plus. So I have a map from 0, 1 to 0, 1. I have to see what it is. Well, I already have a formula for phi 1 plus inverse, phi 1 plus inverse t was t square root of 1 minus t square. And phi 2 plus was just looking at the y coordinate. So therefore the transition function, phi 1 plus inverse composed with phi 2 plus acting on. So transition function is from this open interval to this, I will input as t. So, when I will first do phi 1 plus inverse, I get this, then I look at the y coordinate, therefore I get this.

So, I will put this in. So this is the explicit formula for one of the 4 transition functions. All of them will resemble this, one of them, two of them will, another one will be exactly they have the same formula. The other two will have negative signs in front of this. So, this is what and the point is that this map square root of 1 minus t squared is infinitely differentiable function from 0, 1 to 0, 1. If I included 0, then this would no longer be infinitely differentiable. But I am in the open interval, so I am fine. So, this is a C infinity function.

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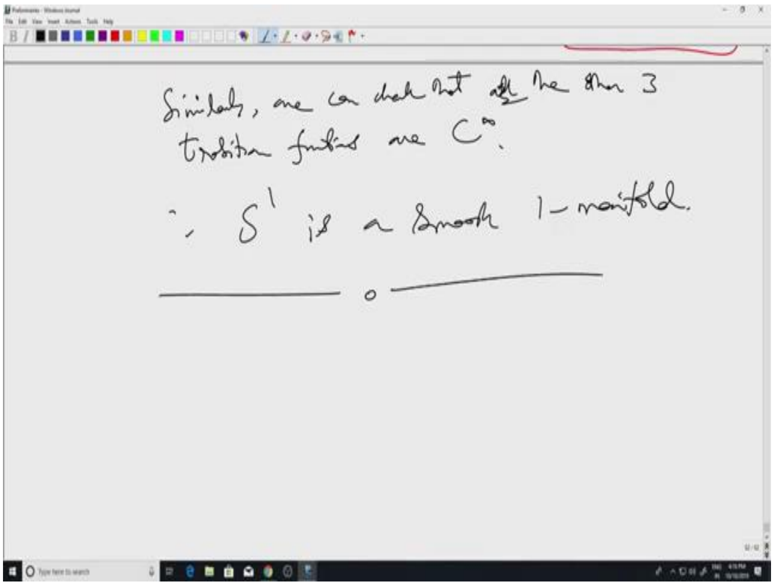

Similarly, one can check that, similarly one can check that all the other 3 transition functions are C infinity. So, that proves that, therefore, S1 is a smooth 1 manifold. Now, this is extremely cumbersome if one had to do this through if one proving that something is a manifold involved all these steps, it would be a very tedious task. But fortunately there are there is a powerful theorem, which will give us last classes of manifolds without having to explicitly construct charts.

But even for the case of S1, it is possible to cover just use two charts and just check one transition function. So, I will mention that, it will work for all Sn. So, in my next lecture, I will return to the case of spheres and show you the construction of a, some nicer chart. So, all right thank you.