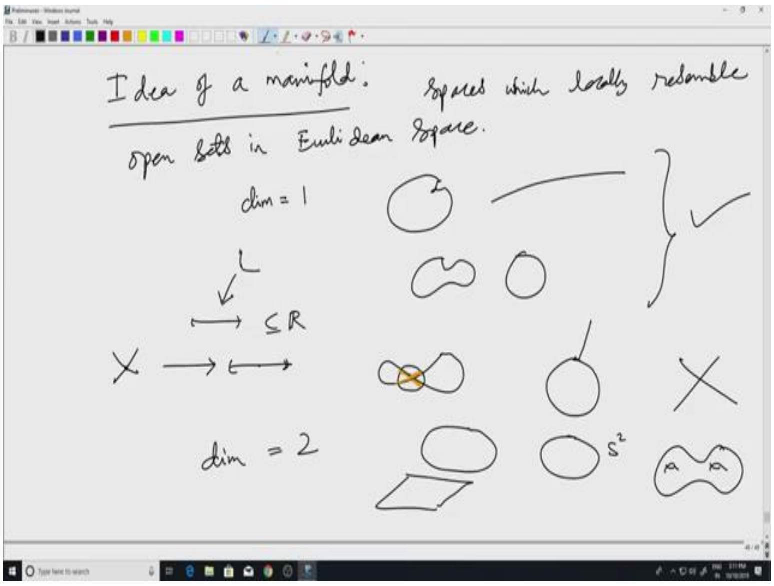
An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science Bengaluru
Lecture 08
Smooth Manifold

Hi, so today, in today's lecture, we will actually start with the main subject at hand, namely, smooth manifolds.

(Refer Slide Time: 0:42)

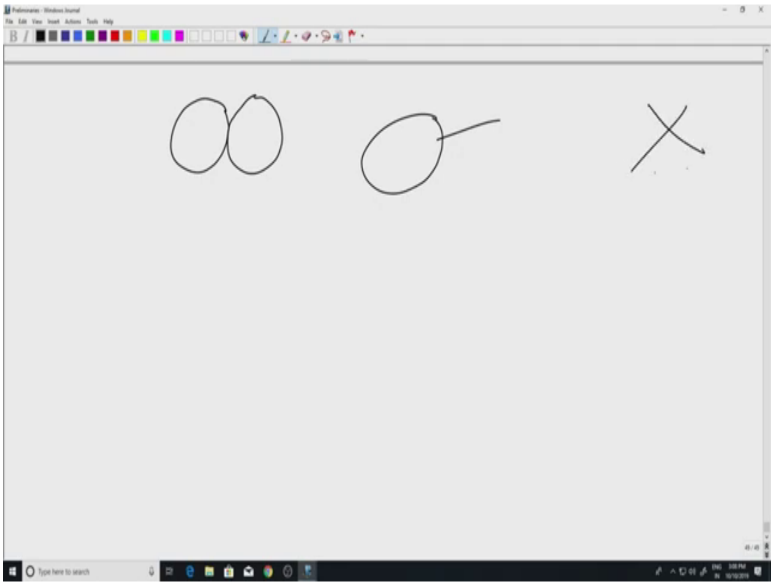


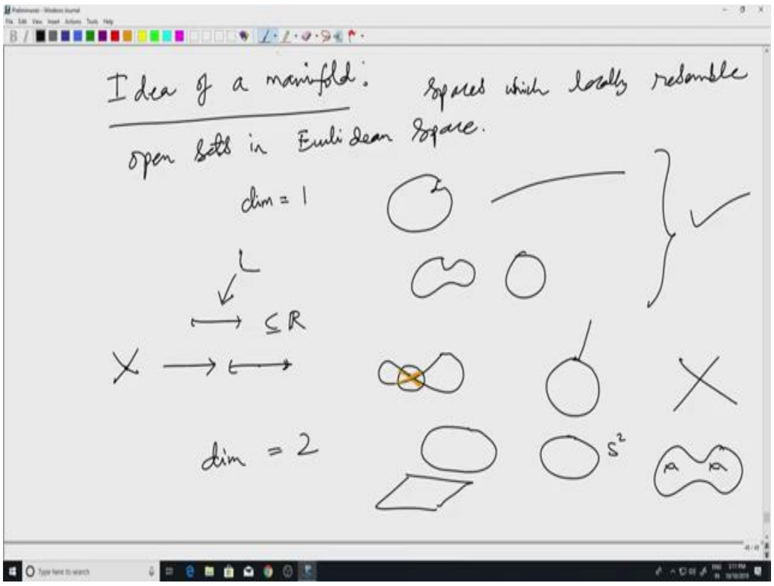
So, the idea of a manifold, broadly speaking is, so we want to talk about spaces. So, I will be somewhat vague to begin with spaces which locally resemble open sets in Euclidean space. Well, this is the statement is not meant to be taken literally, but just as a motivating remark. Now, so for instance we would like to allow things like so, in dimension, there is a concept of a dimension of a manifold.

In dimension 1, we can we allow, something like this, which is a circle which is slightly bent at one point or we can have just an open arc. It can even be disconnected. The union of these 2 pieces, we would like to, what we do not want to allow is, so all this, this okay. We wanted to be okay.

What we do not want is something like this or circle with a spike sticking out and this sort of 8. So we do not want to allow these 2. And if one, this is just in dimension 1 and similar considerations apply in higher dimensions as well. For instance, in dimension 2, I would like a sphere which is flattened or just a regular sphere S2 or I will describe what this is later on and or just a plain, all these are fine.

(Refer Slide Time: 3:31)





But what is not fine is something like this. So if you take two spheres in R 3 and they touch at one point or if I take a sphere and put a line on it. So these we do not want. Now, there is an easy way of sort of picking out the ones we want from the both in dimension 1 or any dimension for that matter.

Namely, if you just look at it closely, one notices that in the first 3 examples, whatever point one takes, even if I take a point here for instance, close to that point, the space I have just looks like a bent arc. So, we can define a bijection, continuous bijection from this arc into an open interval inside the real line and same thing can be done for the other two as well.

But in this other two examples, we want for instance, in the first example, if one looks at the point of where these 2 where the figure 8, the two circles meet. If I stay close to that point, stay close, meaning, so I will be picking out something like this across. Well across cannot be, there is no continuous bijection from a cross to an open interval, it is something more than a continuous bijection that we need. Namely, we demand that the universe should also be continuous.

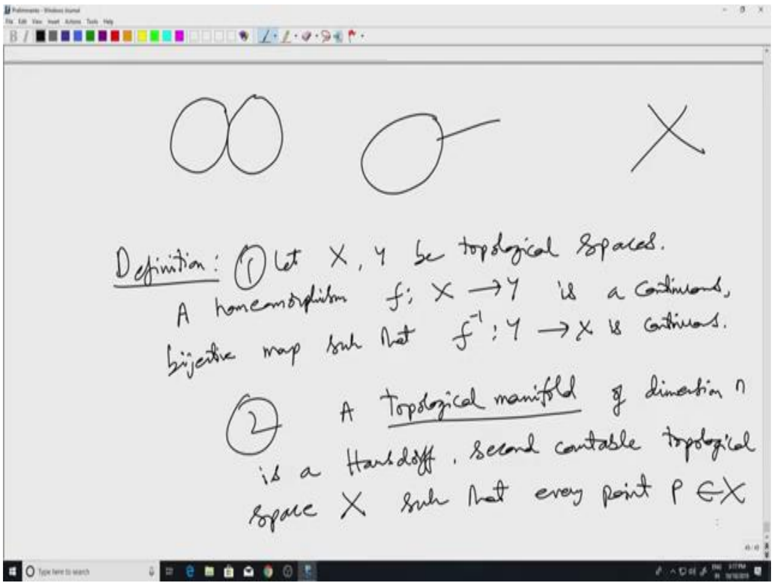
I will formally define that. But for the moment, let us just think of it as a way of deforming this to in open interval. In a bijective manner, this cannot be deformed to this, there is no way of getting rid of the cross, the point of intersection. Formally a proof can be easily given just by saying, by looking at, by considering the fact that the interval is connected, rather, if I that is, that will not help us much. But rather if I remove a point here, this problematic point from this space, I will get 4 pieces.

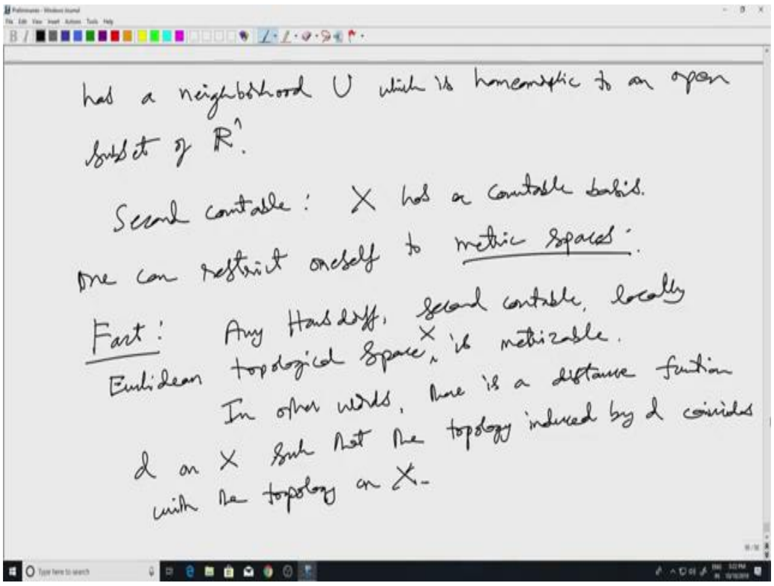
More formally, I will get 4 connected components. Now, this point has to go somewhere on the open interval, if there is such a map. And if I remove this point here, it would correspond to removing some point here. But I will be just left with 2 pieces, 2 connected components. And basically this more or less gives a proof that this cannot be homeomorphic to an open interval. And similar argument will apply for this space as well. So I would not go into that.

But now, the thing is, even here in dimension 2, the same considerations apply. So, how to formally define spaces which have this property that if you stay close to that point, we get a neighborhood, which is like an open subset of R n. So essentially, we use the concept of a homeomorphism.

But even before that the, I need to start with a concept of a space. Now, generally the most general notion of a space that one encounters in a first course is a topological space. So, I will assume that, it is one knows the definition of a topological space. Though, in this specific matter, one does not actually need that it turns out that metric spaces are enough. So, let me give the first the topological definition and then make a remark about metric spaces.

(Refer Slide Time 8:49)





So definition, so as I said, I need to first define the concept of a homomorphism. So let x and y be topological spaces, a homeomorphism f from x to y is a continuous bijective map such that f inverse from y to x is continuous. So, recall that we had talked about the few morphisms earlier between open subsets of R n. So, a diffeomorphism, for a diffeomorphism we had demanded that f itself is C 1 bijective and f inverse is also C 1. Of course, C 1 would imply continuity.

So, a diffeomorphism provides an example of a homomorphism between open sets. But that is a very restrictive type of homeomorphism. So, in the definition of a homeomorphism, there are no considerations of differentiability and so on. Now, again, one should be, there are examples where a map can be continuous bijective but inverse may not be continuous. So, one has to impose this condition. However, I do not want to dwell too much on this because soon we will be moving in talking about differentiable stuff.

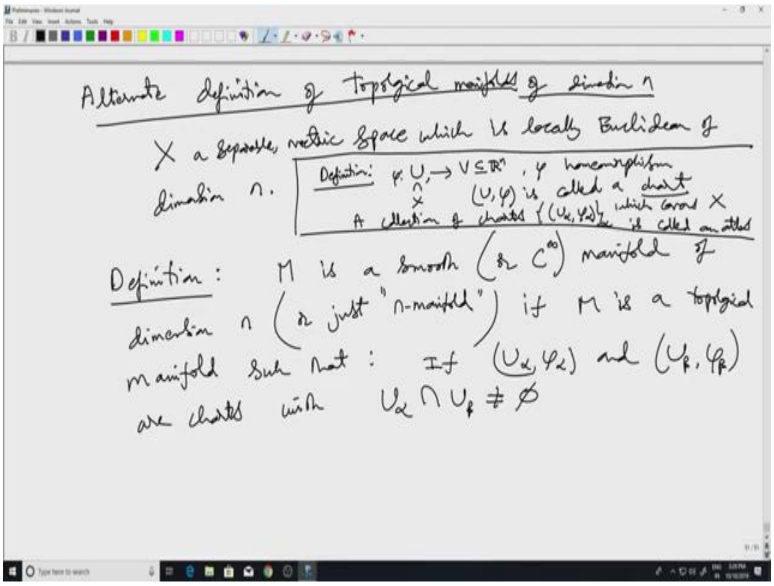
So, let us just keep, take this definition and then proceed to define the notion of a topological manifold. A topological manifold of dimension n is a Hausdorff, second countable topological space X such that, now this Hausdorff and second countable technical assumptions but the important thing is what I am about to write down now, such that every point P in X has a neighborhood U which is homeomorphic to an open subset of R n. This precisely, so this property, the fact the property that it has a neighborhood which is homeomorphic to an open subset of R n is usually referred to us. The, usually say that the space X is locally Euclidean.

Now, this as I said, one has to put these conditions hausdorff and second countable. So, let me quickly tell you what second countable is the same thing as saying X has a countable basis so for its topology. As I said, it is a theorem that in fact, if one starts with a topological manifold. In fact, one can restrict ourselves. One can restrict oneself to metric spaces because of the following fact. Any hausdorff, second countable locally Euclidean, again I should emphasize that locally Euclidean just means that every point has a neighborhood U which is homeomorphic to an open subset of R n.

Second countable locally Euclidean topological space is metrizable. In other words, let us give it a name X. In other words, there is a distance function d on X such that the topology induced by d. Every time one has a distance function it induces say topology the metric topology namely the distance function defines open balls in X and use that as a basis and you will get a topology. Note that the topology induce such that the topology induced by d coincides topology.

So, we started with a topological space. So, there was a topology to begin with, what we are saying is we can put a distance function such that the topology induced by the distance function is the same as the original topology. So, in short because of this fact, I can just write and notice that the moment you have a metric space, metrics the topologies automatically Hausdorff. It turns out that second countable as the same thing as separable, so for a metric space.

(Refer Slide Time: 17:19)



So if one is not too comfortable with topological spaces, alternate definition of one can work with the following alternate definition of topological manifold of dimension n. So I do not, I no longer I would I start with the metric space, X a metric space. I no longer have to specify Hausdorff, it is a metric space. So actually here I will just add X, a separable metric space. In other words, it has accountable dense subset, a separable metric space which is locally Euclidean of dimension.

So, this is equivalent to the original result because of the fact that I mentioned earlier. However, even though topological manifolds are interesting in their own right and they are studied extensively in geometric topology. What we are interested in as we would like to do calculus on manifolds, so in other words, we would like to talk about differentiable functions, vector fields, flows and so on tangent spaces, et cetera.

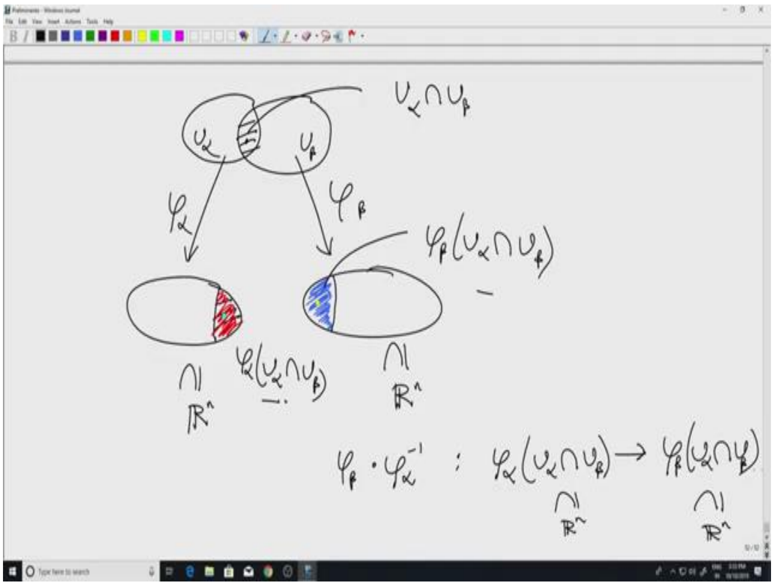
So we need the notion of a, notion of differentiability. That does not come with just this. So, I will make another definition, which at first sight is not clear why it is do the things are defined like this, but I will explain shortly, the need for this definition. So, I will say that M is a smooth or C infinity manifold of dimension n, just instead of writing manifold of dimension n one normally writes just n-manifold. M is a smooth or C infinity manifold, I will use the word smooth and C infinity interchangeably.

In my preliminaries lectures, I was focusing mainly on C1 functions. Hence forth, when I say smooth I mean C infinity, infinitely differentiable dimension n if M is a topological manifold such that, now in order to state what is going to come here, I need to make a small definition here. So when I have this locally Euclidean of dimension n, I have a chart and so I have a open set. So, I need to make one more definition, so I will try to sneak it in here. The in the definition of local Euclidian I have this open set U and a homeomorphism phi from U to some open subset of R n.

So, this U is in X and phi is a homeomorphism. So, this pair is called a chart and if a collection of charts U alpha, phi alpha index by some alpha which covers X is called an Atlas. So, this notion of Atlas is not that important for us or rather what the, however, we will be using charts very frequently. So, a chart is just a open set and a homeomorphism on to an open subset of R n.

And a collection of charts which covers X, the covers X meaning that every point should be in one of these charts. Such a collection is called an Atlas. Now with these, these 2 definitions, what I want is that such that if every time 2 charts intersect, I want something to happen. If U alpha, phi alpha and U beta, phi beta are charts with U alpha intersection U beta not empty. So the picture is like this. I will continue on the next page.

(Refer Slide Time: 24:28)

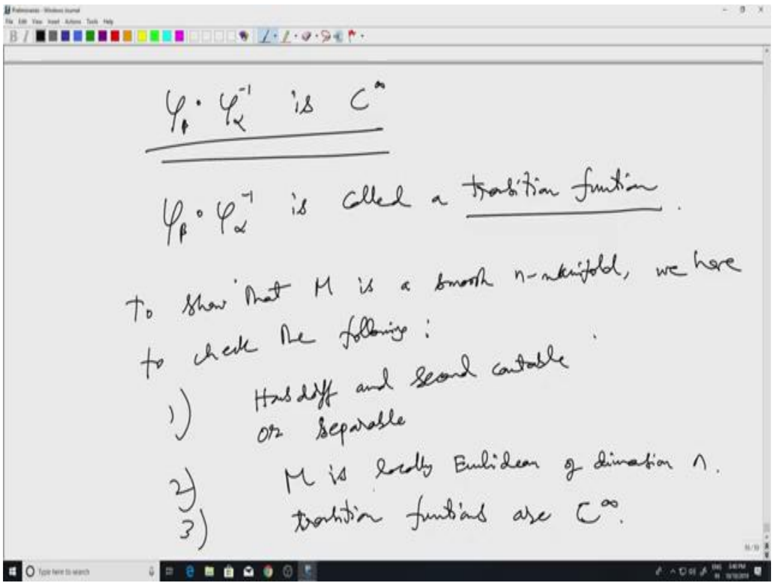


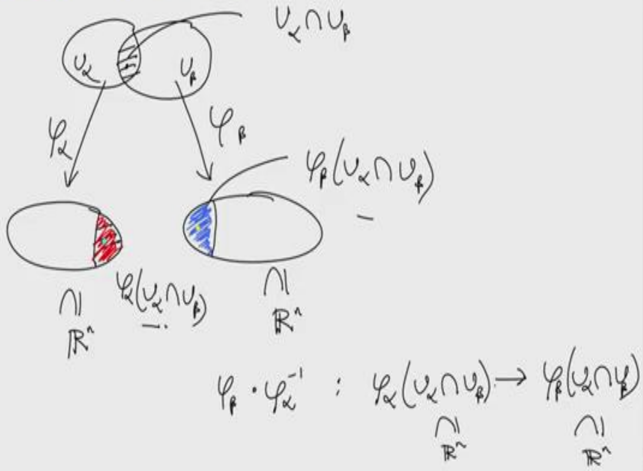
So, I have well, so this is U alpha, this is U beta and they have some intersection. Well, I also have this homeomorphism and another homeomorphism here. Now, this homeomorphism phi alpha will map U alpha to some open subset of here, these are in R n. Now let us look at this intersection. This is U alpha intersection U beta, under this homeomorphism phi alpha, this gets mapped to some other (open), notice that phi is, phi alpha is a homeomorphism.

So, its continuous and continuous inverse, so which, that implies that phi alpha takes open subsets in X rather open subsets of U alpha to open subsets of R n. So, in particular, this goes to some open subset, which is phi alpha, U alpha intersection U beta that is this part. So, let us just, that is this and this goes to another open set, the same thing goes to another open set under phi beta. Of course, this is phi beta of U alpha intersection U beta. So, I have 2 open subsets of R n now, phi alpha, U alpha this one as well as this as well as this.

And what I can do is, I can start with a point inside this open set here. I can start with some point here and apply phi alpha inverse, I will end up with something here then apply phi beta then I end up with the corresponding point in this. So let us write that down. So, phi alpha inverse composed with phi beta will map is we can regard as a map, U alpha intersection beta mapping to phi beta, U alpha intersection U beta. The point is that these are both now both in R n, open subsets of R n.

(Refer Slide Time: 28:17)





Alternate definition of topological manifolds of dimension n

X a separable, metric space which is locally Euclidean of dimension n.

> Definition: $\varphi : U \to V \subseteq \mathbb{R}^n$, $U \subseteq X$, φ homeomorphism, (U, φ) is called a chart.
> A collection of charts $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ which covers X is called an atlas

Definition: M is a smooth (or C^∞) manifold of dimension n (or just "n-manifold") if M is a topological manifold such that: If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts with $U_\alpha \cap U_\beta \neq \emptyset$

$U_\alpha \cap U_\beta$

One can restrict oneself to metric spaces.

Fact: Any Hausdorff, second countable, locally Euclidean topological space X is metrizable.

In other words, there is a distance function d on X such that the topology induced by d coincides with the topology on X.

Alternate definition of topological manifolds of dimension n

X a separable, metric space which is locally Euclidean of dimension n.

> Definition: $\varphi : U \to V \subseteq \mathbb{R}^n$, $U \subseteq X$, φ homeomorphism, (U, φ) is called a chart.
> A collection of charts $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ which covers X is called an atlas

I still have not defined the condition is left, I said such that something happens. Well, the condition is that this map, phi alpha inverse phi beta is C infinity. Notice that it does not make sense to say that phi alpha itself is C infinity or phi beta is C infinity because the notion of a differentiable function make sense only when the domain is in open subset of R n. The domain of phi alpha, domain is an open subset of R n and the target is some Euclidean space. But phi alpha is, its domain is an open subset of some metric space or topological space.

Similarly, phi beta, so to talk about differentiability of phi alpha, phi beta, in our usual sense, is a well, it does not quite make sense. But, this phi alpha inverse phi beta is a map from between open subsets. And one, it is perfectly fine to talk about C infinity. This has a name, so this is called a transition function. So, these transition functions they are defined when whenever you have 2 charts, which have a non-empty intersection, you get a transition function.

And everything hinges, so the on these, the whole notion of a smooth manifold is crucially involves, I mean this transition functions. Now, why does one bother to define it in such a complicated way the notion of a smooth manifold? But before that, let me give some examples of smooth manifolds. A trivial example is M is just an open subset of R n. In this case, so remember that to show that something is a smooth manifold what we have to show? We have to show two things, one as, well actually three things.

One is, it is a separable metric space is one thing, either that or Hausdorff second countable. Those are usually quite straight forward to check. The one non-trivial thing is locally Euclidean. That is the second condition. The third thing is that this transition functions are C infinity. So, in fact, let me just before I start with the examples, let me just write down that what we have to show? Suppose we are given some set M is a smooth n-manifold, we have to show, we have to check the following.

First one, if you do not have a metric, natural metric, if M is given as a topological space, we have to show that it is Hausdorff and second countable or if it comes with a metric, we have to show that it is separable, it has a countable (()) (den) subset. Second thing is M is a topological, M is locally Euclidean, M is locally Euclidean of dimension n. This is, the second step is the main step.

So what this means, again, to remind ourselves what this means, we have to start with some point in any point in M and we have to put it inside an open set and construct a homeomorphism of that open set to an open subset of R n. Well, that is not enough, then once we do that, we get a whole bunch of charts and an Atlas. Then we have to check that transition functions are C infinity. So these are all independent steps. One has to verify. So, we will stop here and I will begin with examples next time. So we will check that these 3 conditions are satisfied in some simple cases. Thank you.