An Introduction to Smooth Manifolds
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Lecture-07
Smooth functions with Compact Support

Hello and welcome to the fifth lecture in the series. So, last time I had stopped at the construction of a C infinity function with compact support. This time let me actually carry out the construction.

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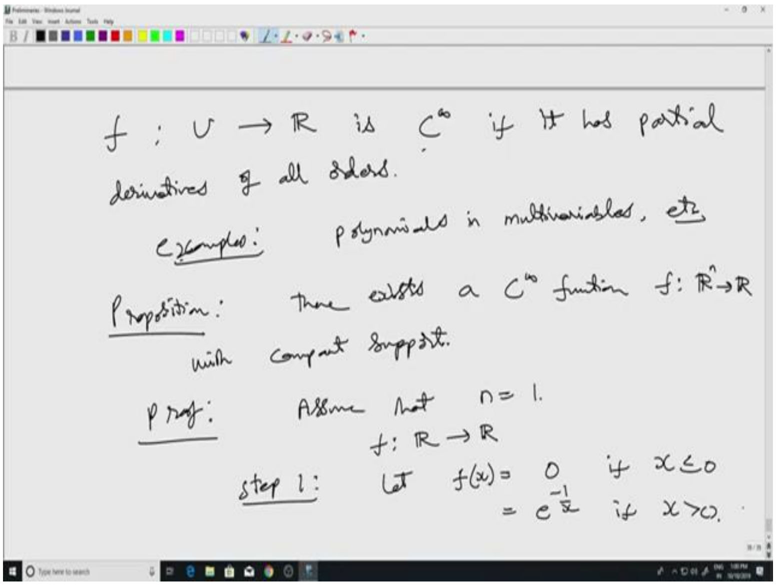

So it is a, there are several steps involved. So proof, assume that n equals 1. So, all I want is to function from R to R which is C infinity and has compact support. So now assume that n equals 1, so step 1, this is perhaps the main step in a sense. So, let us, so note that let f of x is equal to 0 if x less than or equal to 0, and is equal to e to the power - 1 over x if x is greater than 0.

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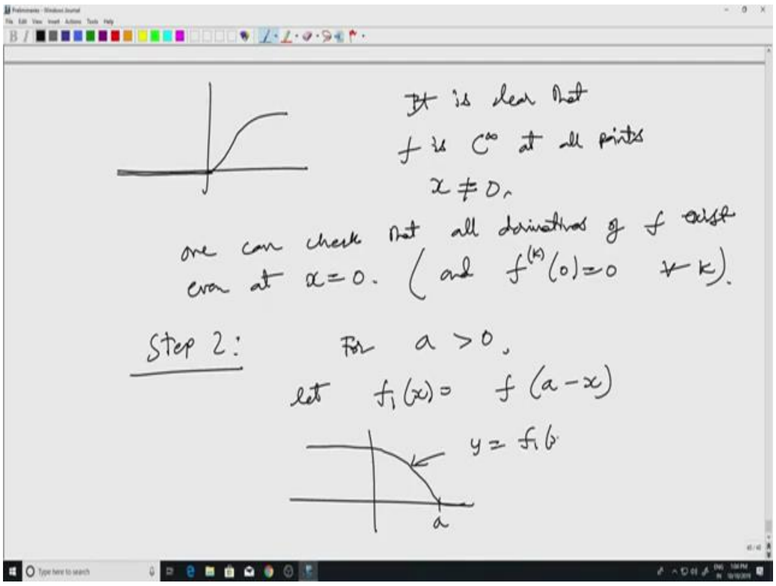

This is the building block for what will follow. So the graph of this function looks like this. Now this it is obvious that this f is C infinity, it is clear that f is C infinity at all points x not equal to 0, it is a small exercise to check that even at x equals 0, all derivatives of f exist. So one can check (not) all derivatives of f exist, even at x equals 0. And this essentially hinges on the fact that recall that the definition of f involved in exponential and as x goes to 0, this 1 over x will go to infinity, but there is a negative sign, so, this will go to - infinity. So, and we are exponentiating that, so you go to 0 very rapidly.

In fact, so as it will go to 0 very rapidly and so rapidly that if you multiply e to the power - 1 over x by any power of x, or rather divided by any power of x still one can check that the numerator goes to 0 much faster than the denominator. So therefore the whole thing goes to 0. So it is an elementary exercise to check this. Now given this and of course, it will follow that in fact all derivatives, because f is identical is 0 for x less than 0, if derivatives were to exist, it would they be forced to be…

By looking at left hand derivatives, it would be immediately clear that all derivatives f k0 equal to 0 for all k. That is the main point is the existence of the derivative. Okay. Now, given this what I do next is step 2 for a greater than 0, let us take some real number. So, let f1 x equals f of a - x. So in other words, I just, so the graph of this f1 is going to look like this. This is a. So this is y equals f1 x.

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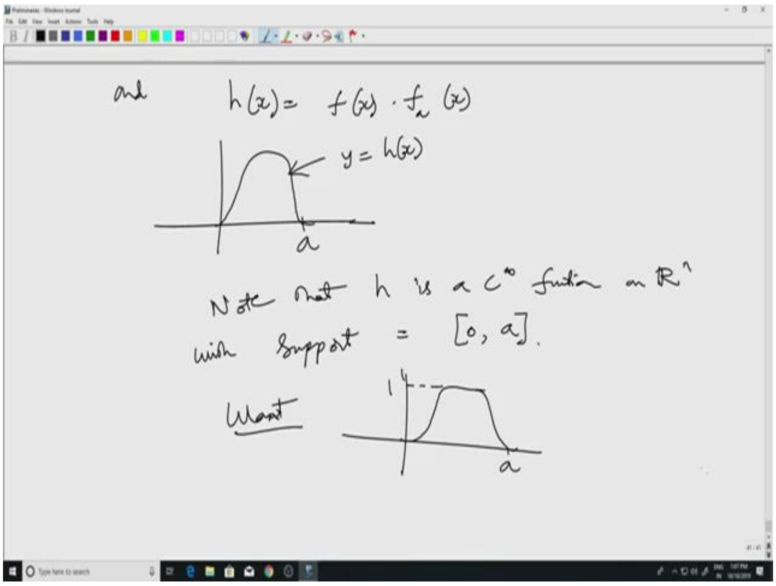

And h of x is equal to f of x times fa x. So I just multiply the 2 functions that I have obtained. So the graph of h will look like. Well, if x is less than 0, f of x is identically 0, if x is greater than a, fa x is identical is 0. But in between 0 and a, both of these functions are nonzero. So therefore, this will look like something like this, so this is y equal to h of x. And we have our C infinity function with compact support. Note that it is a C infinity function on Rn and in fact, we can exactly say what its support is.

With its support equal to the closed interval 0 to a. But we want to do something better than this, what I want to construct is I want a C infinity function which is actually identically 1 in a smaller sub interval of 0 to a and 0 outside 0 to a. So, in order to ensure that this h becomes identically 1. So what I want is, I want something like this, becomes flat at the top and then descends to 0. So let us say this is 1. So I want something like this, the C infinity function, the graph looks like this.

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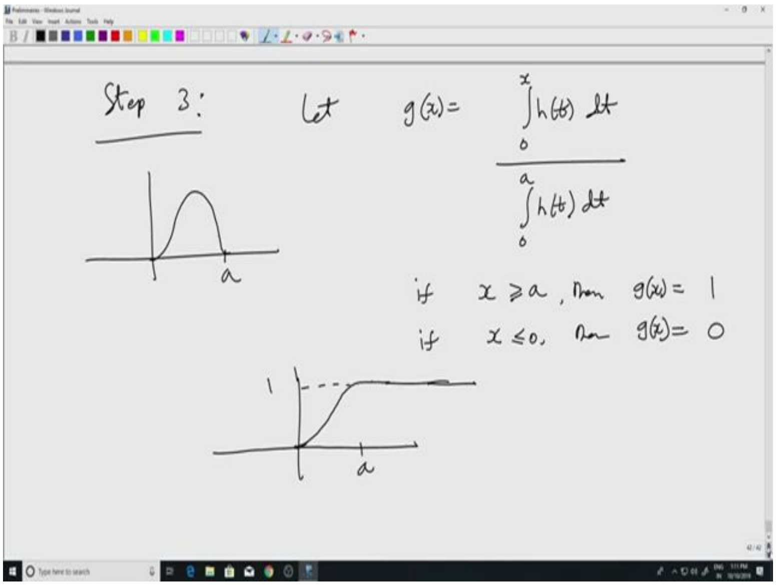

There is a nice trick to accomplish that. So that was step 2. Step 3, let g of x equal to 0 to x, h of t dt divided by 0 to a, h of t dt, where h is the function which we had constructed earlier in the last, so this was h. Now it is clear that if x is greater than or equal to 1, then g of x is, oh not 1 sorry, if x is greater than or equal to a, if x is greater than or equal to a, the upper integral 0 to x, I can split into 2 parts 1 is 0 to a and a to x. Well, if x is greater than a, in between a and x, my function h is going to be 0 anyway, so it does not contribute anything, so this would be identically 1.

And if x is, on the other hand if x is less than or equal to 0 then g of x is, if x is less than 0, this integral 0 to x becomes - of integral, negative of the integral from x to 0. But on the negative side h is 0 anyway, so this is 0. So this new function g has the form, so this is for x less than this and so this is a, so it rises up maybe I should make it a bit more curved, and reaches 1.

So, by step 3, we seem to have somewhat regressed I mean, we had a C infinity function with compact support, but now, I no longer have compact support because, well, it continues to be 1 for any x greater than a. But on the other hand, I know that it becomes identically 1 after some time.

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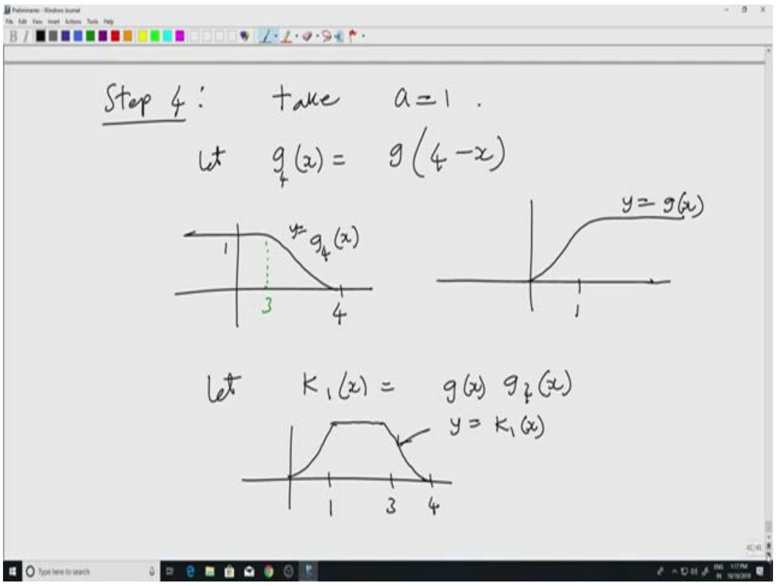

So, this is good enough to get both things identically 1 on some portion and compact support. Now, what I will do is, I had that a, I can continue to persist with a, but let me just take a equals 1 for convenience, take a equals 1 and let g 4, I will use the subscript 4, x equals g of 4 - x. So the graph of this g4 is going to look like this. So the point 4 and I have the point 3, at the point 3 the function starts descending, well okay, so, let me just redraw the whole thing. This is better. So, this is g4 x.

Just for reference, how I got this was that I mean by definition g of g1. When I took a equals 1, then g of x itself was of this form. This is why, so this is y equals g4 x, this was y equals g of x. So, the point is that, this becomes identically 1, if x is greater than or equal to 1. So here, whatever is inside the brackets here, so, right, so this 4 - x should be greater than or equal to 1 to ensure that this thing is identically 1, so 4 - x greater than 1 would amount to saying that x is less than or equal to 3.

So for Yeah, maybe I should change this a bit. So it is not quite, move it a bit here, so this is 3. It is not quite accurate, I mean, the scale is off but… So when for, so when x is less than or equal to 3, this becomes identically 1 and when x is greater than or equal to 4, this thing becomes negative, so it is 0, so it looks like this. Then what we do is again, repeat the same steps as in the earlier cases. Let k1 x equals g of x times g4 x, so I multiply these 2. Then we have a function which looks like this.

So the important points here are 1, 3, and 4. So I will climb up to 1, continue on this flat top and descend back to 4. So this is the graph of y equals k1 x. Now we are almost done, we have now have finally have a C infinity function, which is the compact support and which is identically 1 inside a smaller interval. But I will just do one more change, I will just shift it so that instead of the graph looking like this at these specific points, I want the center, the graph to be centered around the origin, so I will just do a translation.

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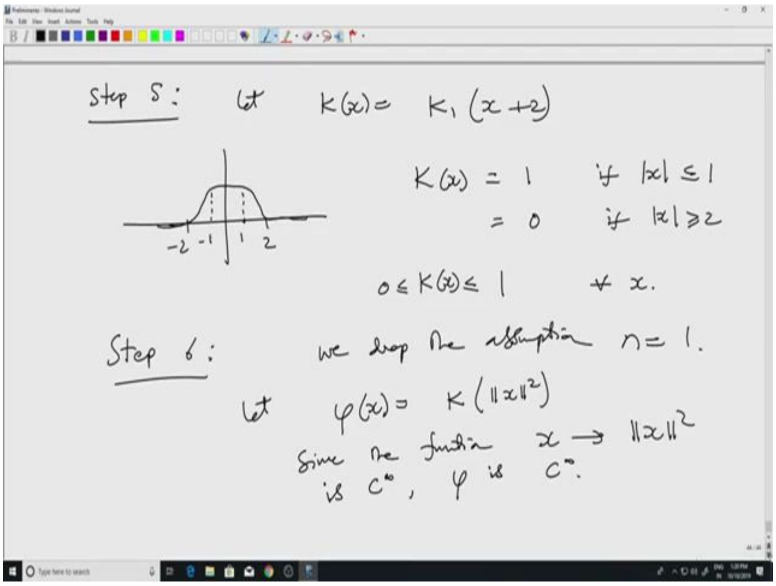

So step 5, let k of x the equals k1 x + 2. So now the graph is going to look like this. So this point would be - 1, this point would be 1, and oh, sorry, not quite this. Actually, this point would be 1 and this would be - 1, this would be 2 and this would be - 2. So, in other words, k of x has the looks like this. Then k of x is 1 if mod x less than or equal to 1, is equal to 0, mod x greater than or equal to 2.

And in between, when a mod x lies between 1 and 2, we, k of x will be greater than or equal to 0 and less than or equal to 1, for all x. The last step is, I no longer general we drop the assumption n equals 1. So let us take any n and let phi of x equal to, so whatever k I had, I will just take k of norm x squared. Since the function x going to norm x squared, well, this is norm x squared is just sum of the squares of the coordinates of x. So it is obviously a C infinity function, a C infinity. And this k is also C infinity by construction, the composition is C infinity, phi is C infinity.

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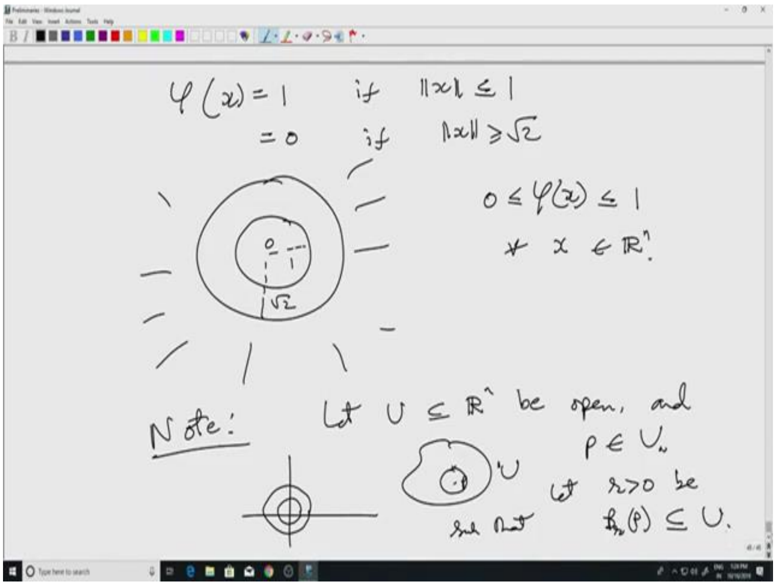

And let us see what the values of your phi of x equal to 1, if norm x is less than or equal to 1, and phi of x equal to 0 if norm x is greater than or equal to, since I took norm of x squared, I will have to put a square root of 2 here. So I have 2 concentric balls, one of radius 1. So, this is the origin, the other one is of radius square root of 2 and my function phi is identically 1 inside the smaller ball and outside the larger ball it is identically 0. In this transition in the sort of annulus between these 2 balls, phi of x is, lies between 0 and 1. In fact, that is the case for all x.

One final remark is that, so this completes the construction of the C infinity function. Actually, yeah, we have done something more, we have constructed a C function with compact support inside a ball and which is identically 1 inside a smaller ball. Note this was done for this specific ball centered at the origin, for if one starts with any open set then I can, what I can do is I can use this phi to define a similar C infinity function on U.

In other words, I would want the support of this new function to be inside U rather than near the origin. This U might be quite far from the origin. So, U is some set here. Well, this phi that I had constructed, all the action took place around the origin. But if I want to move the, using this I want to get some similar picture inside U, what all I do is let us just take some point here and P belongs to U. So, let me say, let this be open and P belong to U. Since this is open I can find a ball with center P of some radius r, such that the ball of radius P, which I denote by Br P is contained in U.

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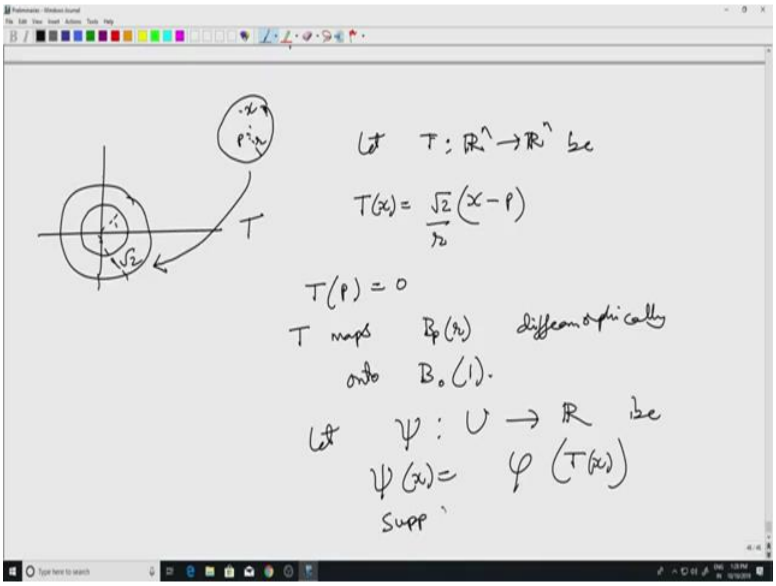

So, this is P, this is r. Now, obviously, so and here is our ball of radius 1 and square root of 2. So phi was supported inside this, I want to move this ball here and define a function here. So, what I do is let me just take, if I want to use this, this ball centred at the origin to define a function based at t, well, I will just map this ball of radius r to the ball of radius 2. So in other words, let T from Rn to Rn be T of x equals, first I will do x - p, so that p goes to the origin and then I want to do some scaling as well.

So that r goes to, so when x - p equals r, I want norm of x, norm of Tx to be square root of 2. So I just multiply by square root of 2 and divide by r. So consider this map. So what this does is T takes P to the origin and T maps the ball of radius r diffeomorphically, I mean T is just a linear map plus a constant, what is called in other words a fine linear map. This r is a constant, p is a constant. So all I am doing is multiplying x by a constant and adding another constant vector.

So it is quite obvious that T is C 1 and its inverse will also be. Yeah, and the point is that T is also a linear isomorphism of Rm. So its inverse is also a linear isomorphism and it is smooth. That whole thing can all, but all I care about is it is a diffeomorphism T maps Bp are diffeomorphically on to the ball of radius 1 centered at the origin. And let C from U to Rb, C of x is phi of t of x. So I use this map T. If I start with some x here, I will use T to come here and then compose with my map phi. So what this accomplishes is that I will get a C infinity function support of C.

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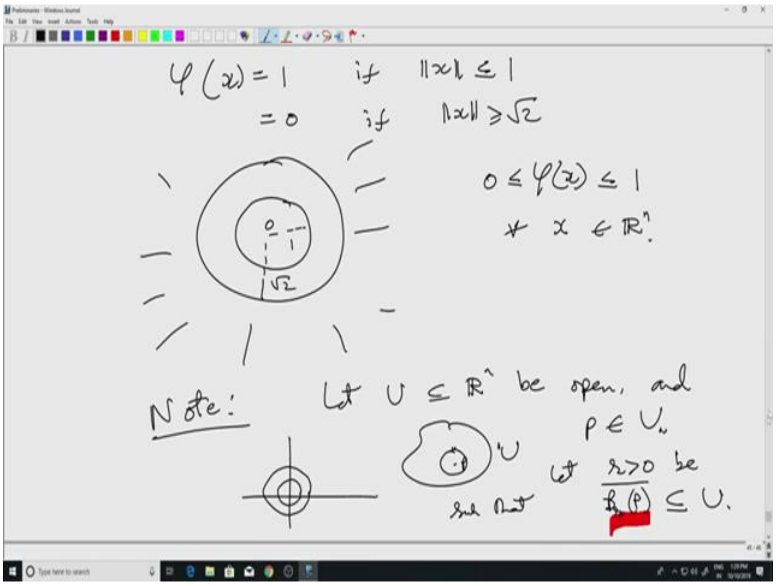

Well, actually to be a bit more careful, I will have to assume that be such that. So this ball Bpr that I started with, in this so this ball that I started with here, I want it, so that let us assume that its closure is actually inside U. So, I mean all I can or another way of putting it is I start with a ball like this and instead of working with R I work with R by 2 for instance. So that will do the job.

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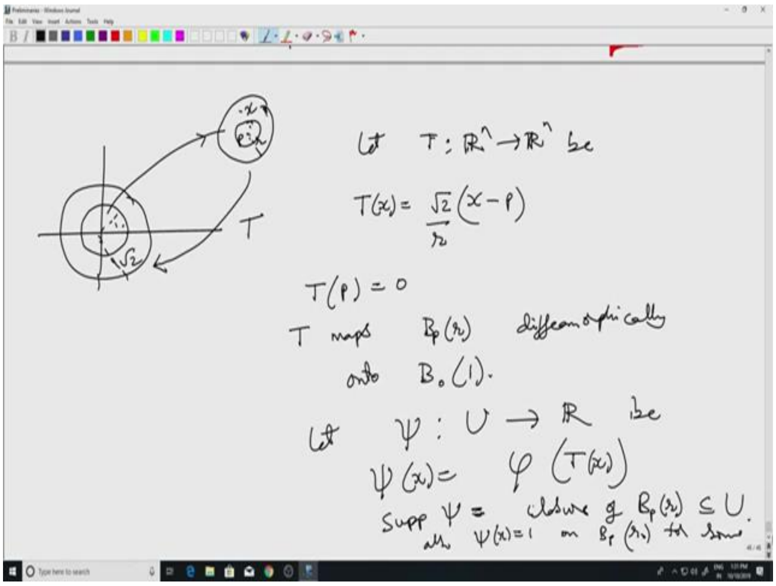

So, with that condition in mind if I take support of C will be the closure of this ball and which we assumed in the previous slide is contained in U. And moreover, this other ball, the ball of radius 1 here gets mapped to also, the C of x is equal to 1 on the ball of the radius… Well, you just have to figure out under this map. So under the inverse map, where does this go to?

This gets, this ball of radius 1 will get map to a smaller ball here on BP, so I want to write it down. So I guess it would be, yeah, so on BP, r0 for some that, the actual value is irrelevant.

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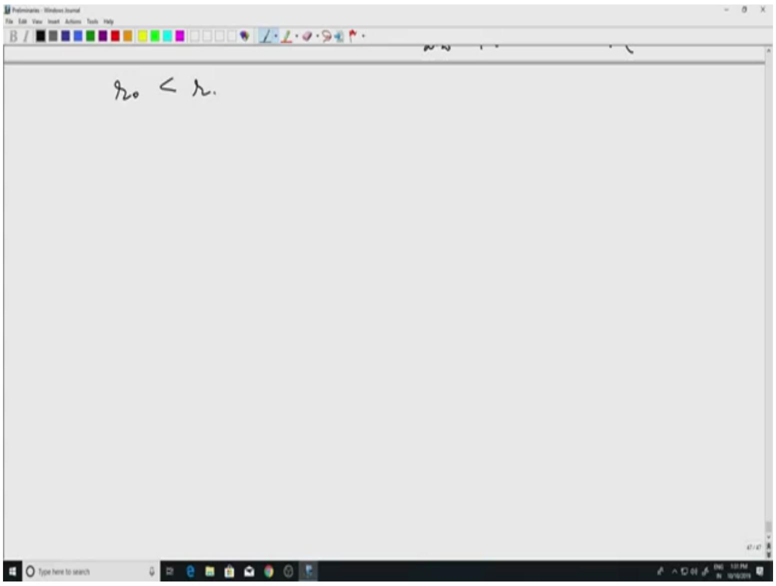

Some r0 less than r. So that concludes the preliminaries needed for, to begin our study of smooth manifolds. In the next lecture, we will start with the definitions and some examples of manifolds. Thank you.