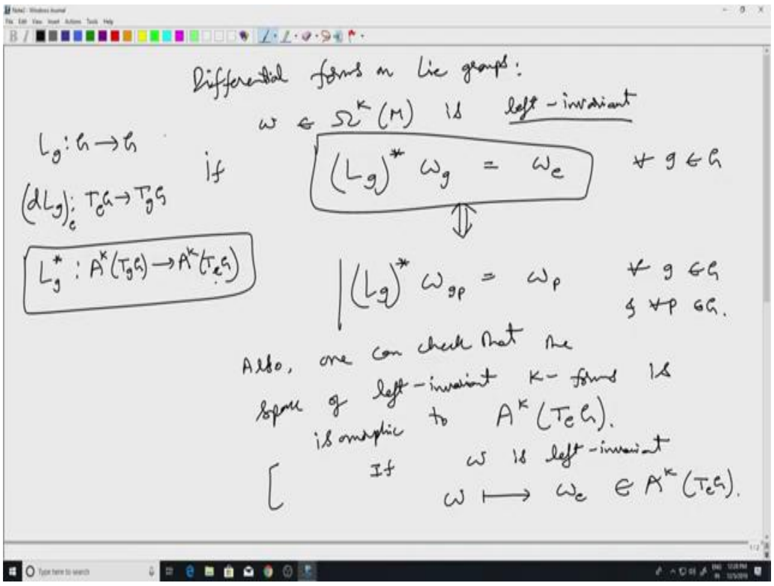
An introduction to smooth manifolds
Professor Harish Seshadri
Indian Institute of Science
Department of Mathematics
Orientation on Manifolds 1
Lecture 66

Hello and welcome to today's lecture. So, before I begin my discussion of, the plan is to talk about orientation. Before I do that, let me just give 1 more example of some special classes of differential forms.

(Refer Slide Time 0:52)



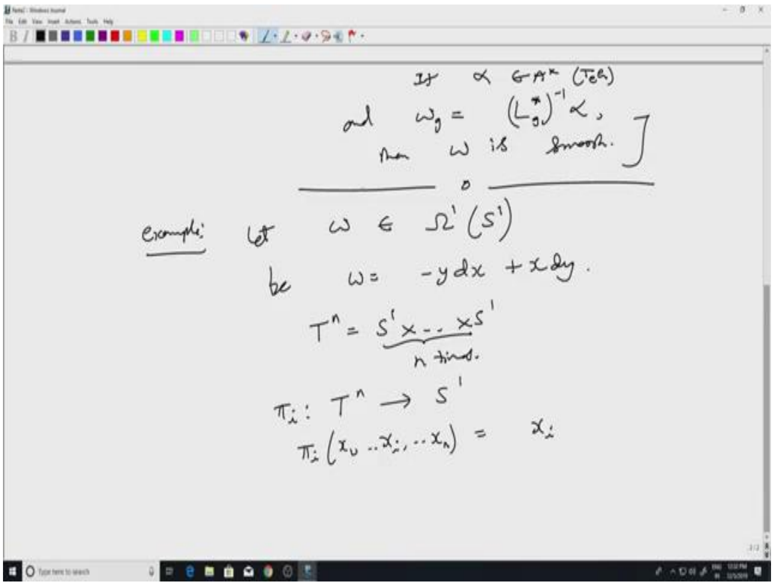
So, this is differential forms on lie groups. Whenever we are in the setting of lie groups, it is natural to look at whatever object we are considering. It is natural to consider objects which are preserved by this left translations or for that matter right translations as well. So let us say we make the following definition, Omega and Omega k M is left invariant. Then we say that to if well, so I have this the left translation diffeomorphism, if and the pullback of Lg star of Omega at the identity, but not quite identity. So, let us recall that Lg is going from g to g. So, therefore, the derivative of Lg, d Lg at identity is going from the tangent space here to tangent space here.

And if I put a, then if I look at the corresponding map on alternating k tensors that map so this is what we have been calling Lg star upper star. So, sorry not this one, the next one is what we have been calling Lg upper star. So, the corresponding map will be from the alternating k tensors on this to this, which is a star of this map which is what we have been calling Lg star, at well we can say at identity, let me just leave it like that. So this here, I should change it to g equal to Omega e. And so as this is more or less the same way we define left invariant vector fields except that the maps this the induced map, corresponding to left translation goes in the opposite direction.

Even though I have identity as a base reference point here. Just like for left invariant vector fields, one can check that this is equivalent to saying that Lg star of Omega g p equal to Omega p for all, by the way here I should say for all g in g. And here for all g in g and for all p in g as well. So, p goes to g p under left translation so, the corresponding map on k forms will be in the opposite direction and it will give me this. And so, this is one thing, the second thing is that well so one can check that the space of left invariant here I say space since the set of left invariant forms of a given degree forms a vector space, we can add them or multiply by a scalar, one can check the space of left invariant k forms is isomorphic.

Essentially whatever happens either on the tangent space at identity should determine everything, or for that matter as this equation shows, whatever happens at one point should determine whatever happens on the whole group, but it is more convenient to work with the identity. Let space of left invariant k forms as isomorphic to Ak TeG. And this isomorphism is very similar to what we did for vector fields. So, if one starts with, if alpha belongs to Ak TeG or rather if Omega has left invariant. I start with the left invariant form and just map it to send it to its value at the identity, so in other words, an element of Ak TeG and this map is an isomorphism. So if I know it, and the proof is just exactly like this. And just if you are given something in element of Ak TeG, I can obtain an Omega g by putting this Lg star on this side by taking the inverse. And that one can check that it is an isomorphism.

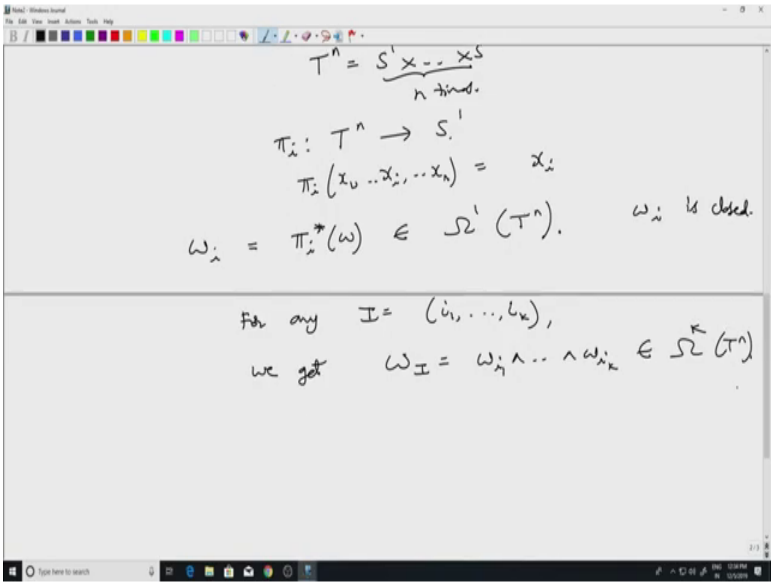
(Refer Slide Time 7:52)



But what is requires a bit of proof, which I did not prove even for vector fields is that when we do this, if I start with something, if alpha belongs to Ak TeG and Omega equal to, Omega g equal to Lg star inverse alpha, then Omega g , Omega is smooth. This is the thing which has to be checked as g changes, what we get as a smooth form on the lie group. So this part has to be checked, but again, this has to be checked even for the vector fields, which I did not go over. As reference, I would recommend John Lee's book on Smooth manifolds that has a short proof of for the vector field case and this case as well.

Well there is a, yeah this is a one thing, the other example that I wanted to talk about is let us look at, another example is let Omega be in the one form on the circle be Omega equals minus y dx plus x dy, whatever we had been talking about in the last lecture, the same form, but except that I pull it back to S1, that form was on R2 minus the origin. Now I have pulled it back to S1. So this form and we have seen that this is closed, but not exact on R2 minus 0 and so one can prove that it is not exact on S1 as well. But what I want to look at now is, the n dimensional Taurus n times. So, we have this projection maps Pi i and this is just Pi i of x1, xi, xn equal to xi.

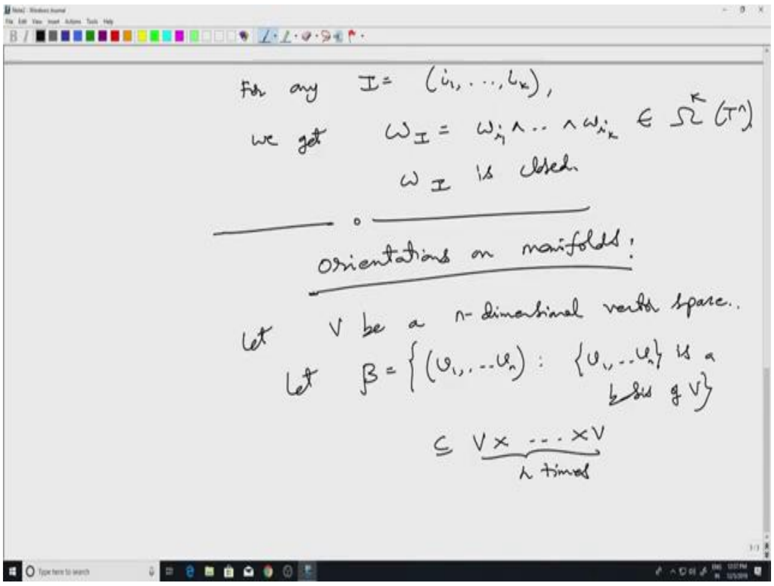
(Refer Slide Time 11:03)



So, here each x i is an element of S1, this xi does not denote Euclidean coordinates, but this Cartesian product, this xi has come from the Cartesian product. Now, the point is that I have this fixed one form on S1, I can pull it back to the n dimensional Taurus and get a form here. Depending on which projection I take, I get different forms well, using these one forms I can for any multi index i equal to i1 ik, we get a K form out of this the usual way Omega I is, so, this I will call Omega i, Omega capital I is Omega i1, Omega ik and this will be in Omega k T n.

And one can check, we have already proved that d of a product is the Leibnitz rule is satisfied therefore, one can check since all of these moreover, the fact that Omega is closed will imply that the pullback is also closed, since we know that d and the pullback operation commute, pull back of a closed form will be closed therefore, I get this, Omega I is closed and I also know that which product of closed forms is closed by the Leibnitz rule so, therefore, this is closed, Omega I is closed. Well, the significance of the forms is becomes evident in topology. So, whenever we have closed forms, they have topological significance and these specific forms it turns out are basically capture all the topological content of the n dimensional Taurus.

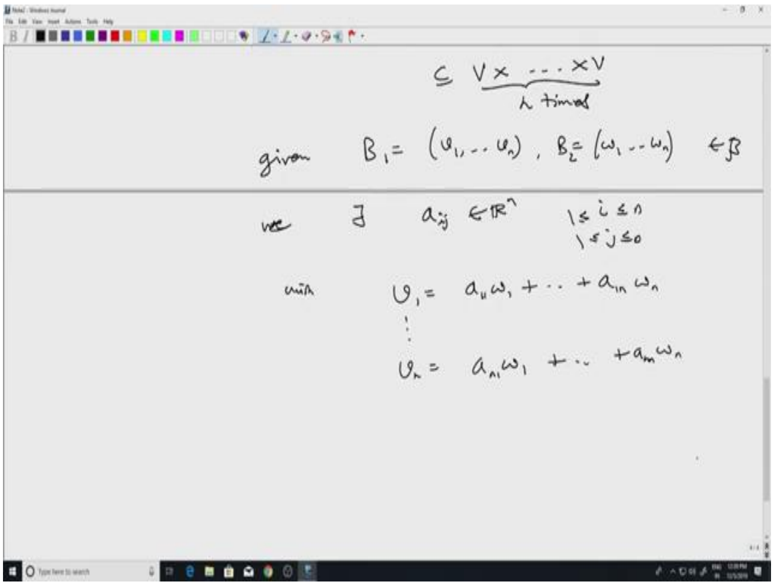
(Refer Slide Time 13:14)



So now, at this point, I would like to move on to a different aspect of manifolds, this is the concept of orientations, orientations on manifolds. And I will have to end the course after discussion of this topic. The natural way of proceeding would have been to talk about integration on manifolds, which is where the real role of differential forms become evident. We use differential forms even for orientation, but the real significance or the real necessity becomes clear when we talk about integration however, I will not be doing that.

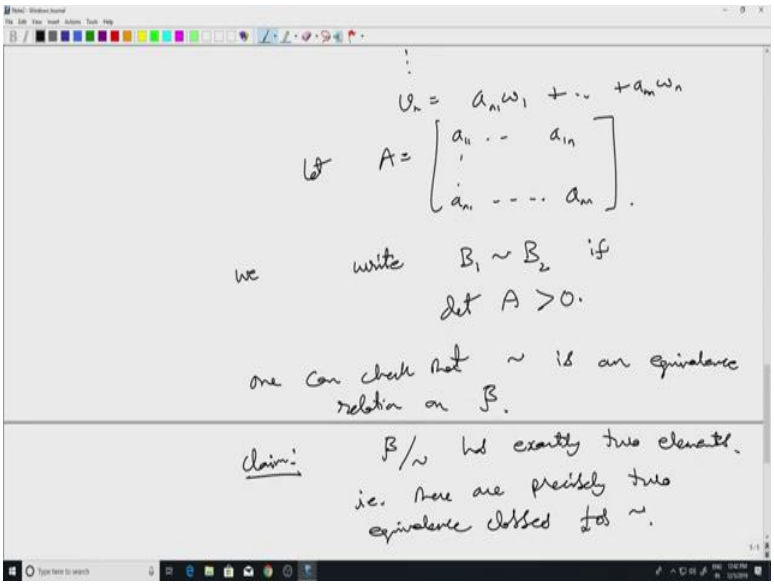
And once we talk about integration of forms then the big theorem would there would be Stokes theorem on manifolds. There one would have to deal with manifolds with boundary and so on. But, let me just talk briefly about the concept of an oriented manifold as well. What one really needs to start with is the concept of an oriented vector space, let V be a n dimensional vector space. Now so, let this beta be the set of all intervals of this v1, v2, vn, since I have used this round bracket so this, it is an ordered and tuple, set of all such things such that this an ordered set is a basis of V. In other words, beta is just the collection of basis of V, but we are looking at ordered basis. So, formally, this is just a subset of V cross V n times.

(Refer Slide Time 16:22)



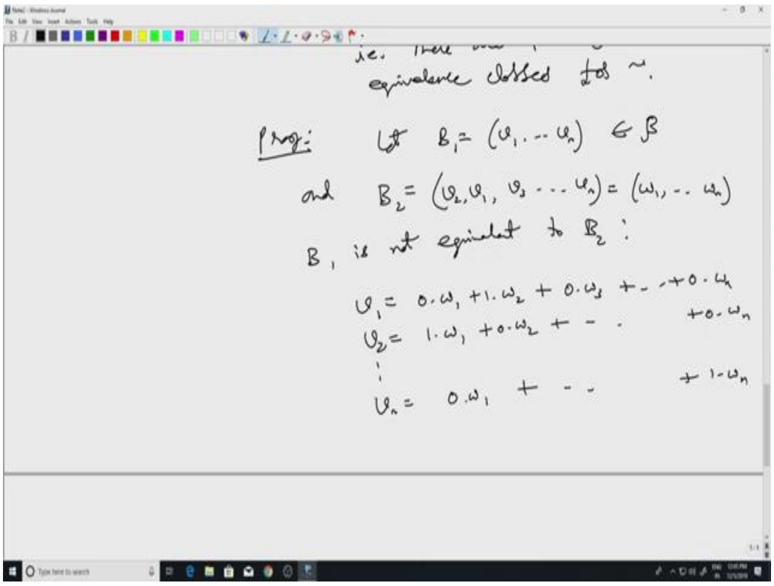
Now in this, the set of all basis of V, we can ask when 2 basis are sort have the, give rise to the same notion of orientation. Now, of course, we do not have a notion of an orientation yet, but, so here is what 1 would like to do. So, given B1 equal to v1, vn, B2 equals w1, wn in this beta. So in other words, given 2 basis, we can, there exists aij in Rn, i between 1 and n, j between 1 and n, with, so I can just expand v1 in terms of the w's, a11 w1 plus a1n wn, etcetera. vn is an1 w1 plus ann and wn.

(Refer Slide Time 17:59)



If I do this, then that amounts to saying. So, now let us look at this matrix A, let A equals this matrix a11 a1n, the change of basis matrix ann. So, we say that these 2 basis B1 and B2 are equivalent, we write, rather than we say that let me just say we write, B1 is equivalent to B2 if determinant of this matrix is positive. So now, then a small check, one can check that this notation Tilde is actually in equivalence relation on beta. So in other words, reflexivity, symmetry and transitivity hold for this relation Tilde. So let us look at the set of equivalence classes of this as exactly 2 elements i.e. there are precisely 2 equivalence classes for tilde.

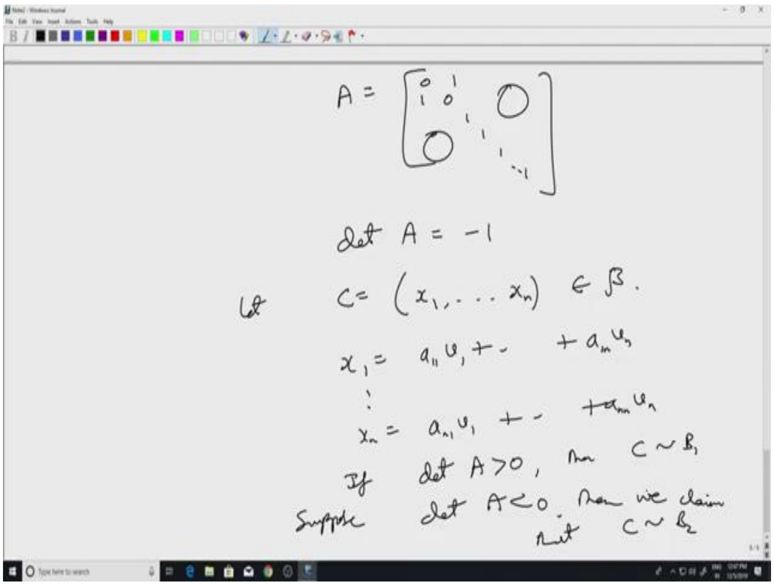
(Refer Slide Time 20:36)

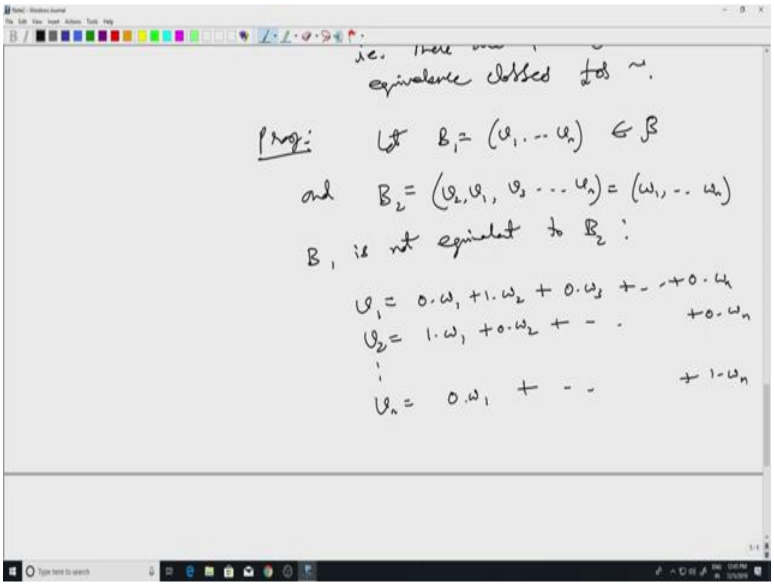


And the proof is very short. So to say that there are precisely 2 equivalence classes for this, all I have to do is come up with 2 basis such that any other basis is equivalent to one of these 2. And of course, these two should not be equivalent to each other. So let me start with any basis, let B1 equals v1, vn Beta, and the second basis all I do is I just interchange v2 and v1 and keep the remaining vectors the same, v2, v1, v3 onwards this in the same order as for B1. And of course, this is the same as a set this is the same as this so this still a basis, just the ordering has changed.

First of all, B1 is not equivalent to B2 well, if I write the change of basis matrix for B. So v1 would be 0 times v1, sorry, let us call this to be consistent with our earlier notation, w1, wn. So it is 0 times w1 plus 1 times w2 plus remaining things are all 0. And v2 would be 1 times w1 plus 0 times w2 and everything else is 0. And as for the remaining stuff, they are all so 1 times, so for the i th term, I will just get an i for the wn, etcetera.

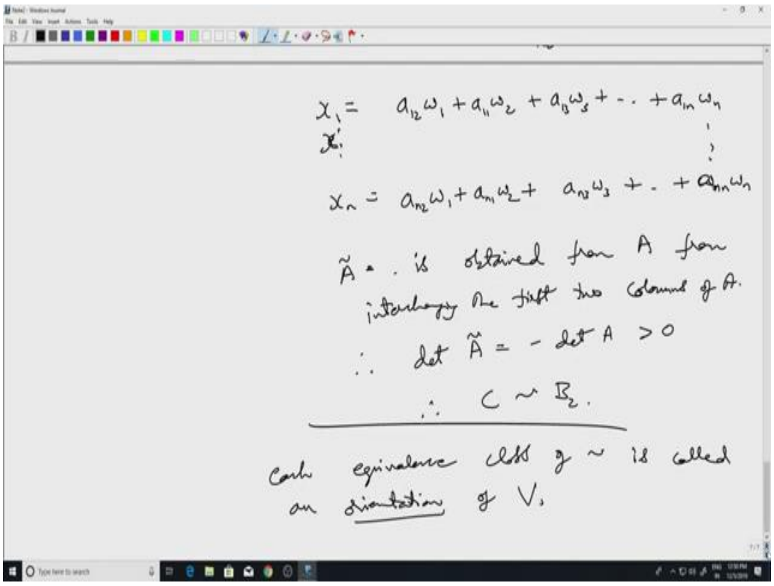
(Refer Slide Time 23:08)





So in other words, this matrix A will be 0 1 1 0. And once elsewhere on the diagonal and 0s of diagonal apart from these two, the first so that A will be just minus 1. And so this one B1 and B2 are not equivalent, now anything else is equivalent to one of these two. And the reason is, let C equals, I will take any other basis, let me call it x1, xn in beta. Well as usual, I can write C in terms of this, let me start by writing C this basis in terms of the basis B1. So, a11 v1 etcetera., a1n vn, xn equal to an1 v1 plus ann vn. Now and if this suppose if that A is positive, then C is equivalent to B1 by definition. Suppose, that A is negative then we would like to claim that C is actually equivalent to B2 that would prove what we want that there are exactly 2 equivalence classes.

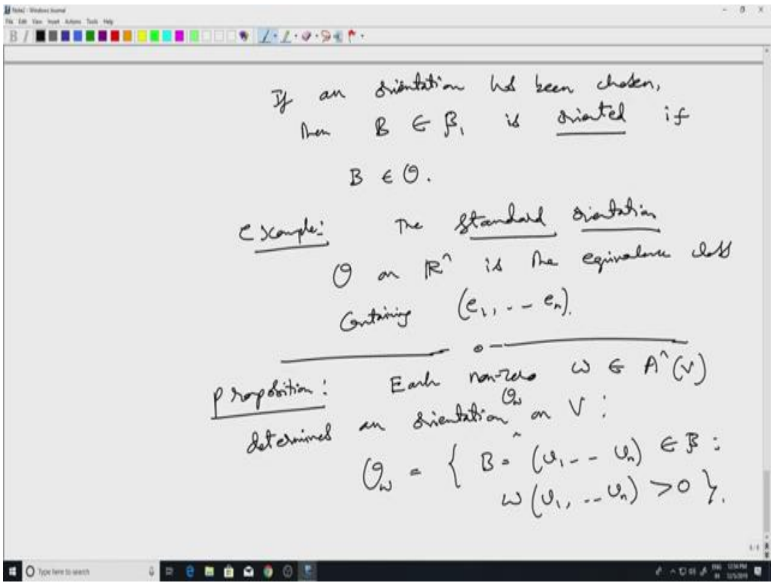
(Refer Slide Time 25:20)



So, this whatever I wrote here, this expansion, let me rewrite in terms of the w's, which is basis comprising which makes up B2. So x1 equals, now for B2, v1 and v2 are switched, so, I would have to write it like this a12 w1 plus a11 w2 and the remaining terms are the same. So, that would be a13 w3 etcetera., a1n wn, and likewise x. And this will be done for all the terms xn is an2 w1 plus an1 w2 and then an3 onwards, it would be the same ann wn. In other words, the matrix, new matrix A Tilde, such that which relates the basis C with the basis B is given by just all I have done is, I have interchanged the, is obtained from the matrix A, all the columns are the same, except the first 2 columns have been interchanged.

Here it is a12 an2 is the first column here. The second column is a11 all the way up to a1, while for A it was the other way, obtained from a by interchanging the first 2 columns of A therefore, that A Tilde is equal to the negative of that A and we already assume that A is negative, so this is positive so therefore, C is equivalent to B2. So that we have two equivalence classes, each equivalence class so that completes that equivalence class of Tilde is called an orientation of V.

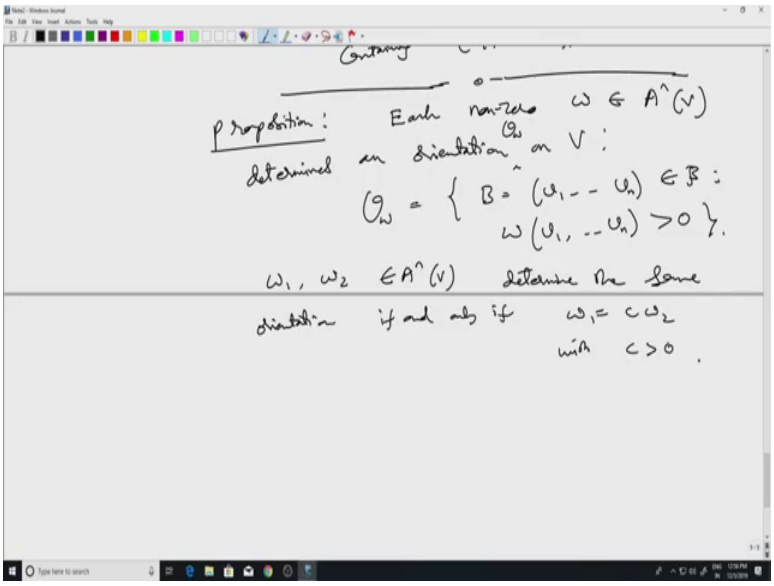
(Refer Slide Time 28:49)



So, let us use the script O to denote an equivalence class. If an orientation has been chosen, then if you take any basis is said to be oriented if this B belongs to beta. So, B lies in that equivalence class which we have chosen then we call it oriented. So, the simplest example to keep in mind is example, the standard orientation O on Rn is the equivalence class containing the standard basis e1, e2 to en. So in other words, we say that a basis of Rn is oriented if that basis is related to even up to en with the change of basis matrix having positive determinant.

Now, there is a very nice way of relating this concept of orientation through differential n forms on V. Proposition each non-zero Omega in A n V alternating n forms on V determines an orientation on V. Let us denote this by O subscript Omega, so O subscript Omega is the set of all basis B equal to v1 up to vn in Beta such that Omega evaluated on v1 up to vn is positive.

(Refer Slide Time 31:40)



And if you have 2 different Omega 1 and Omega 2 in A n v non-zero forms determine the same orientation if and only if Omega 1, they are just positive multiples of each other that is crucial, the sign of C is the crucial thing here with C greater than 0. So in short, the choice of a n form non vanishing n form will give us an orientation. And it is a well and we can say when exactly we, if it starts with a different form, then we can say that we will get a different orientation only if the 2 forms are, we know that, first of all, we know that A n v is 1 dimensional, so I need to n forms are multiples of each other scalar multiples, but they determine the same orientation if and only if there are positive multiples of each other. So I will talk about this next time and also about how this can be used to give a couple of different characterizations of orientations on manifolds. Okay, thank you.