An introduction to smooth manifolds
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Department of Mathematics
Special Classes of Forms
Lecture 65

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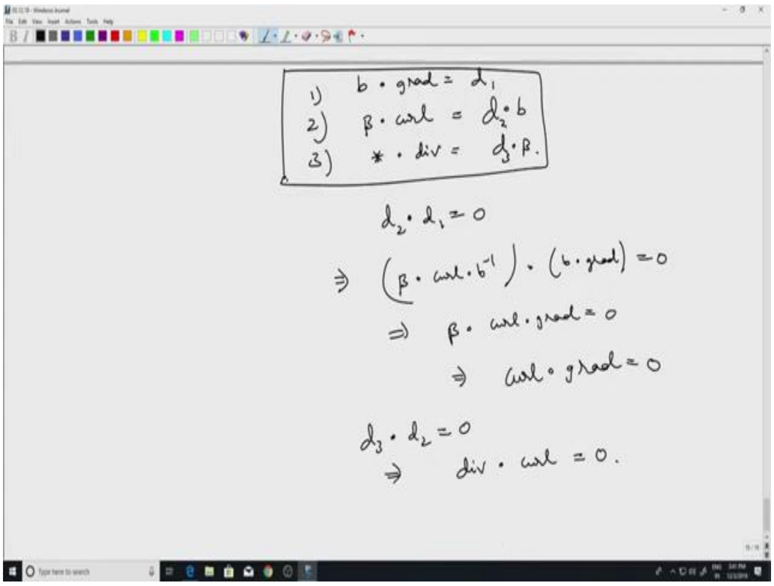

Hello and welcome to this lecture. So, let me make another important remark. This what I wrote in towards end of last class was I tried to relate. I did not prove it though, by the way this requires proof that this is not that difficult but it requires proof that these three equations hold. And I said that d 2 compose with d 1 is 0 implies curl composed with grad 0 and likewise, d 3 composed with d 2 is 0. Similar calculation shows that this implies divergence composed with curl as 0, divergence composed with curl as 0, let us see.

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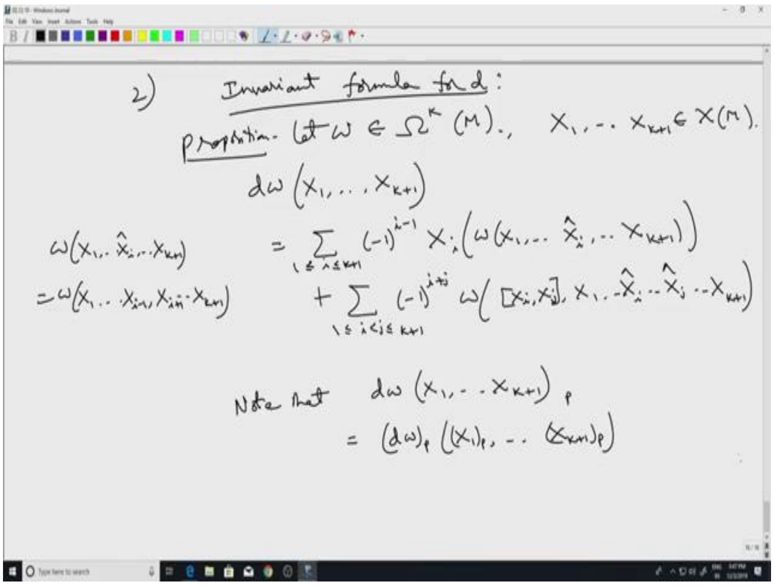

So now the other remark I want to make is that when one has an invariant formula for d. Now what does one mean by invariant in the setting? Is I just mean something which does not use local coordinates as part of the definition, so and it goes like this. I have already mentioned the special case of this formula for d acting on one form. But in fact that is true for any k form. So, if I start with Omega M capital Omega K so I will sate it as a proposition, let, since I already have a definition of d, the claim is that d can be recaptured in a this way, d Omega acting on X1, X2 , Xk. So here I will take X1, X2, Xk plus 1 are all vector fields on M.

So this, yeah the proposition is that this d Omega whose definition we already know in terms of local coordinates can be written as minus 1 to the power i minus 1 Xi Omega X1 Xi hat Xk plus 1 plus so this summation is over i, here i less than j between 1 and k plus 1, here between 1 and k plus 1 again. And this time it is minus 1 raise to i plus j Omega of the lie bracket of Xi Xj X1 Xi hat Xj hat Xk plus 1. Let me explain what these things mean. Well, first of all, when I write a hat over a vector field, it just means that that entry is omitted. So essentially, this term Omega X1 is just shorthand for writing Omega X1, and go all the way up to Xi minus 1, but then I jumped to Xi plus 1 like this.

And similarly, this thing the second term as well. I omit this Xi and Xj so, that is one thing. The second thing is this, these are Xi's are all vector fields and so when I, Omega of X1 up to Xk plus 1 after I omit Xi, I will get k vector field so Omega will act on that. So this is a function C infinity function, Omega of X1 up to Xk plus 1 is a C infinity function and I am just taking the derivation action of the vector field on a function. And well that is it, there is not much more to say about this except that there is one interesting thing going on here, which is the following. Note that if I evaluate the whole thing at a point, at a point P, this by definition is d Omega at , X1 at P, X k plus 1 at P.

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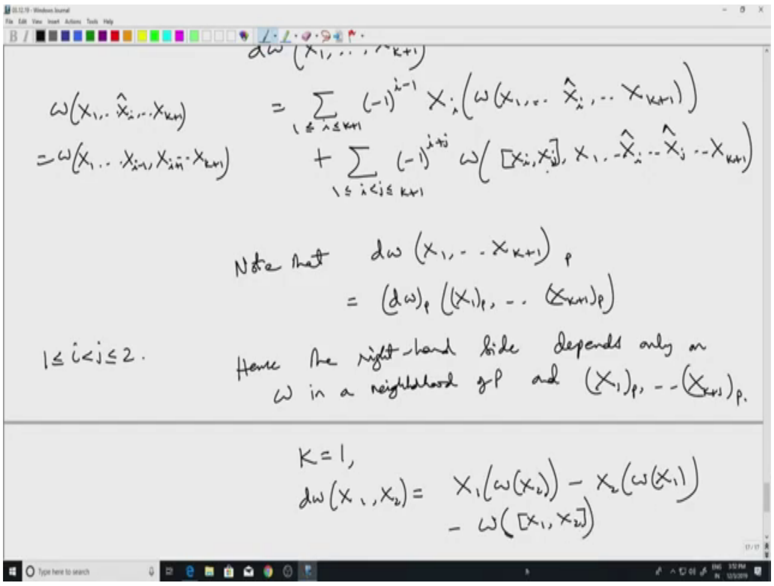

And this of course, the left hand side just depends on X1 at P X k plus 1 at P. But on the other hand, the right hand side because of 2 things, one is the lie bracket term here, even if I evaluate at P, if I were to evaluate this P, the way the lie bracket is defined, it is crucial that these are vector fields, not just tangent vectors otherwise it does not even make sense. So, and moreover, this thing actually depends not just on Xi at P and Xk at P, it depends on how they are defined in neighbourhood of P. So and likewise, this thing the first time as well. I am going to take directional derivative along Xi of this function, it will matter how this X1, X2, X k plus 1 are defined in a neighbourhood of the point.

So, the up short is that the left hand side depends only on the values of X1, X2, X k plus 1 at P, but the right hand side seems to depend on the values of Xi in the neighbourhood of P. But what this formula shows, is that in fact hence, the right hand side depends only on Omega in the neighbourhood of P, and X1 P X k plus 1 P. So, the definition of right hand side, if you are just given X1 P Xk plus 1 P, the right hand side does not even make sense for instance, the lie bracket does not make sense, neither does this the first term. So it is essential that we have vector fields rather than individual tangent vectors.

But even though we start with vector fields on the right hand side, ultimately what we end up with just depends on tangent vectors at a given point, not on the full vector fields. So, which is not obvious, unless, of course one once one has this formula, then it is clear. And this formula is sometimes it is useful. I will give one example where this can be used, but oftentimes, it is more difficult, it is easier to work with local coordinates.

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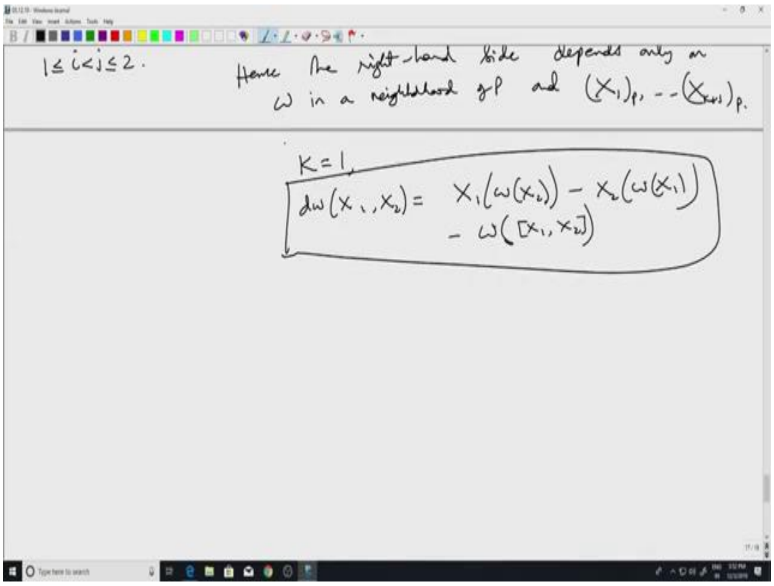

So the case k equal to 1, something that I had mentioned earlier, in that case, d Omega acting on X1, X2 would be, well so here, I would have to start with in the first term, I would have to start with i equals 1 so X1 Omega X2 minus X2 Omega X1 and then here I will get i plus j, both of them. So here, well 1 less than or equal to i less than j less than or equal to 2. So there are only two possibilities i equal to 1, j equals to 2 in this. So essentially, there is only one term in the sum and I will get a negative sign so Omega acting on, and if I omit these two vector fields, there is nothing else remains. So I am just left with the lie bracket of X1, X2, this is the formula that I mentioned earlier. So this is only for one-forms. So I want to talk a bit about special classes of one, special classes of K forms, which are especially relevant in topology.

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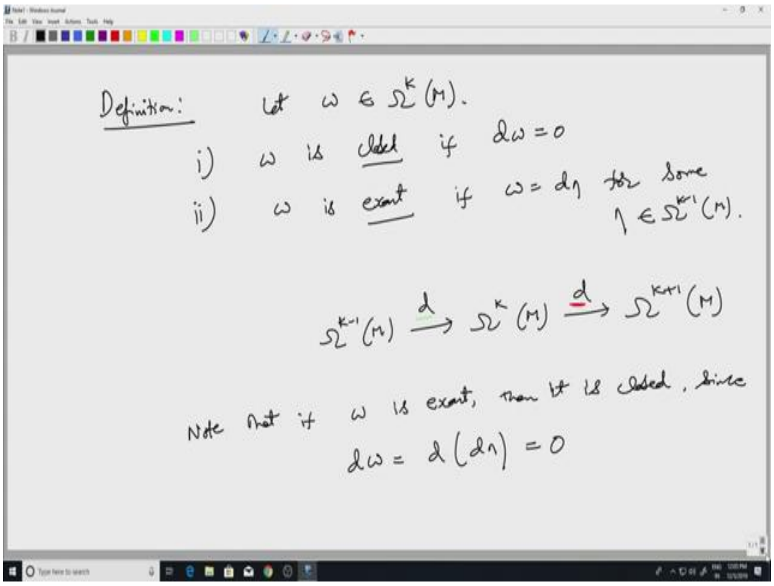

So let us make a definition. Omega is said to be closed if d Omega is 0, here 0 of course refers to the 0 form and as I regarded as a k plus 1 form. And the second thing is Omega is exact if Omega equals d Eta for some Eta in Omega k minus 1 M. So let us recall this, the two maps which are involved here, the d here refers to two different d's, one is from Omega k minus 1 M to Omega k M, and this d goes from Omega k plus 1. So, the first condition is that Omega is closed if d Omega is 0. So in other words, if Omega belongs to the kernel of this d here then Omega is said to be closed. And in the second condition Omega is exact if Omega is d Eta for some Eta in Omega k minus 1.

So, in other words if Omega is in the image of this green underlined d, then Omega is exact. Let us make some observations about these two conditions and then I will give an example. The first thing is that note that if Omega is exact, then it is closed, since, well if Omega is exact Omega is d Eta, therefore when I take d Omega, this is d of d Eta, which is 0 since we know that d squared is 0, d composed with d is 0.

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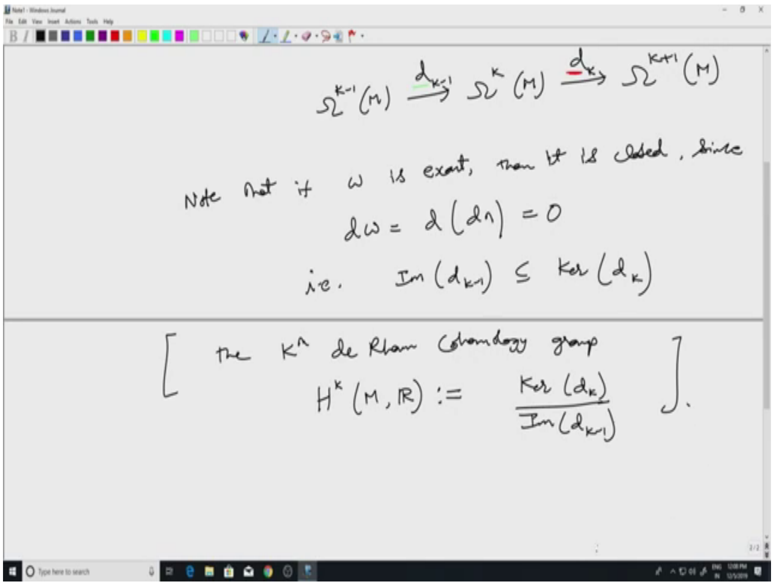

In fact, let us give this for my next point, I will give this some index. So d k and d k minus 1. So this i e, so what I have written here, Omega is exact, then it is closed amounts to saying that this image of this map d k minus 1 is contained in the kernel of d k, so if 1 vector space contained in another, then I can look at the quotient vector space, and that is called the Kth de Rham cohomology group. So I will just put it in brackets, since I am not going to pursue this any further, let me just say that the Kth de Rham cohomology group. In this case, it is actually a vector space and this is H k M, R, this is defined to be the kernel of d k modulo of this quotient vector space.

And the interesting thing is that even though these first of all this Omega k is itself infinite dimensional. And you can check that even this kernel and the major infinite dimensional subspaces of Omega k. But it turns out that if the manifold is compact for instance, then this quotient space will be finite dimensional.

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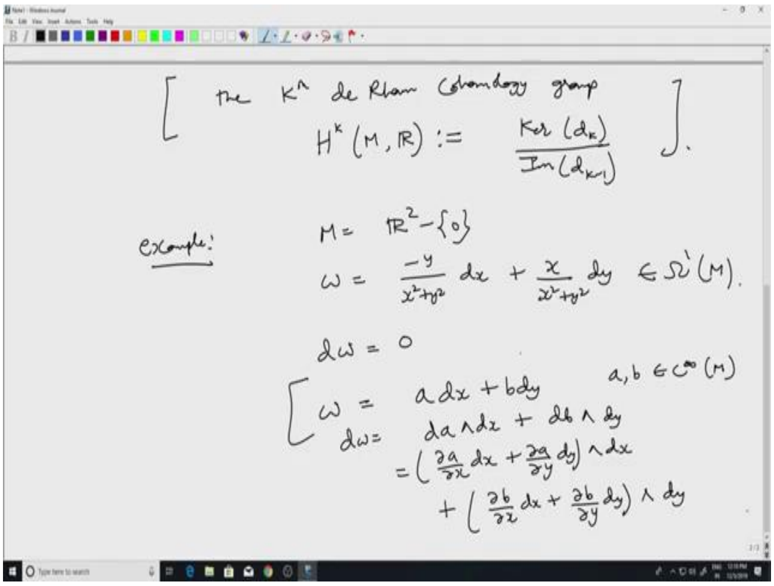

But so let me give an example. Let us take our underlying manifold to be R 2 minus 0, the origin and I will take the specific one form 1 by x square plus rather Instead of writing it as 1, I will write it as minus y this plus x by x square plus y square d y. So this is a one form on the manifold. Now, one can check that d Omega in this case is equal to 0. And how does one do that? Well, it is just a matter of dealing with each term, there are two terms here, each one separately. So for instance, the first term if I write Omega as adx plus bdy, where a and b are functions on M then d Omega is by definition da wedge dx plus db wedge dy and that da we know what exactly it is, it is Del a by Del x dx plus Del a Del y dy wedge dx plus and db is Del b by Del x dx plus Del b by Del y dy wedge dy.

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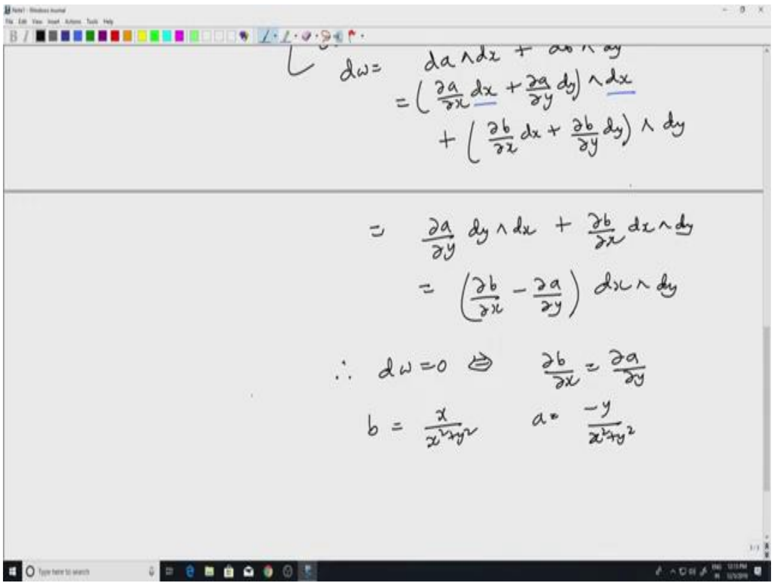

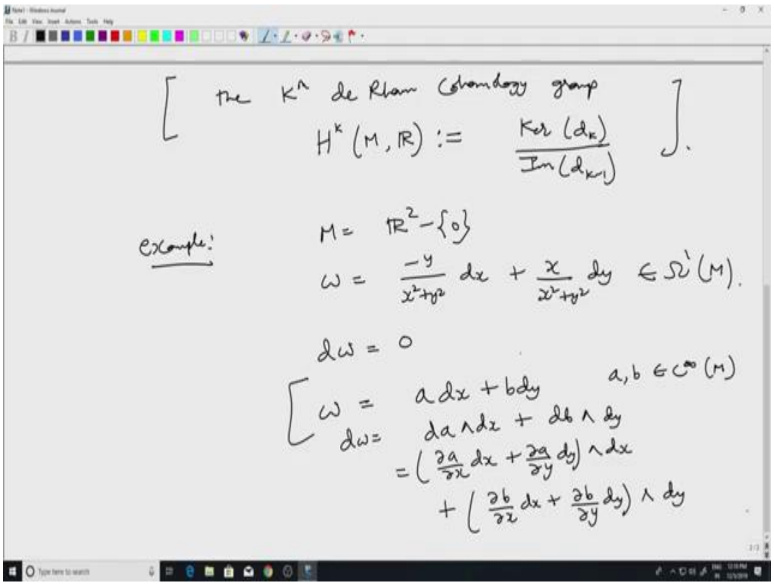

And so, in the first expression here this dx, I have a dx here and dx wedge dx will be 0. So I will just get one term dy corresponding to dy wedge dx. Similarly, here I will get dx wedge dy so let us write those two terms Del a by Del y dy wedge dx plus Del b by Del x dx wedge dy. And I can swap this dy wedge dx is minus dx wedge dy. Though I can, after doing that and combining, I will get the Del b by Del x minus Del a by Del y dx wedge dy. This is a general formula valid on any manifold if I take d of one form, of course on a manifold like I do not have a dx and dy defined on the whole manifold, in a chart I can do this computation.

And to say that d Omega is 0, therefore d Omega 0 if and only if Del b by Del x equal to Del a by Del y. So in this case, so going back to this example that we have here, so, all I have to do is I have to compute, so this is b and the first thing is a. So I have to compute the partial derivative of this with respect to x and this with respect to y and check that they are equal. And in fact, so they will, so in our case, b is the function. So let me just do this one example, so b is the function x by x square plus y squared, a is the function minus y by x square plus y square.

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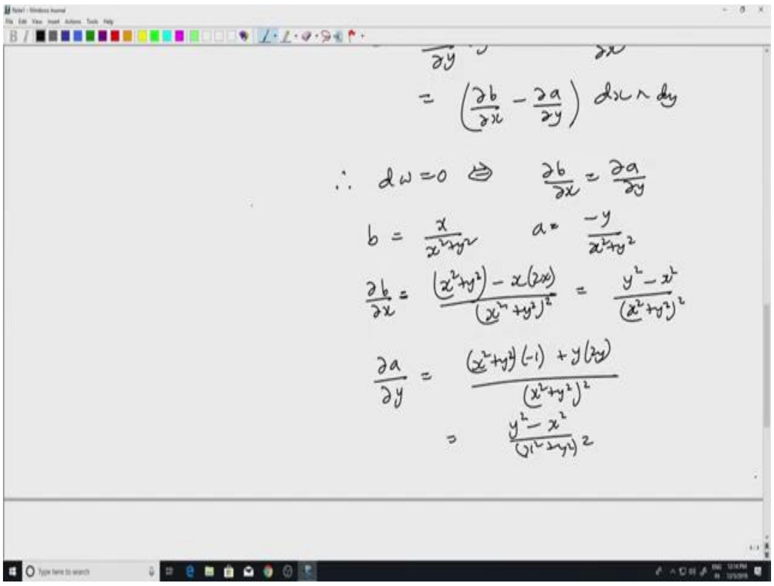

And if I take Del b by Del x, I will end up getting well to do a bit of calculation here. So this is use the quotient rule. So x square plus y square times partial derivative of this minus x times the partial derivative to x square plus y squared whole thing squared. Well that is what I, so y squared minus x squared divided by x square plus y square squared. And if I do it for this, then Del a by Del y will be similar so x square plus y squared, I will get a minus 1, so plus it becomes plus 2y. And then x square plus y squared whole thing square. So 2y squared minus again I will get the same thing, y square minus x square divided by x square plus y square. So this form is closed.

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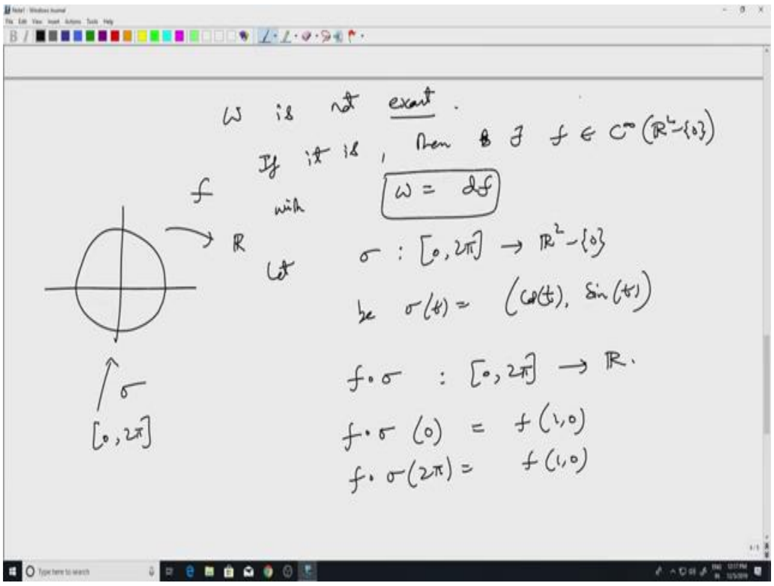

I claim that it is not exact Omega is not exact. If it is, then there exists a C infinity function on R2 minus 0 such that with Omega equal to df. And let us see why this gives us quickly see why this gives a contradiction. So, let I look at the circle the unit circle let sigma from 0 to Pi to R2 minus 0 be sigma t equals cos t sin t. Now I will compose this f and sigma, f composed with sigma. So f is a map on this and sigma is a map from here to here. So f is to R and this is from 0, 2 Pi, so 0, 2 Pi to R. So I will get a map and what we need about this map is like two things; one is f composed with sigma 0, sigma 0 is f of 1, 0. And f composed with sigma 2 Pi is the same thing f of 1, 0. But on the other hand, so this is not yet used this condition like I could do d f. So let us do that now.

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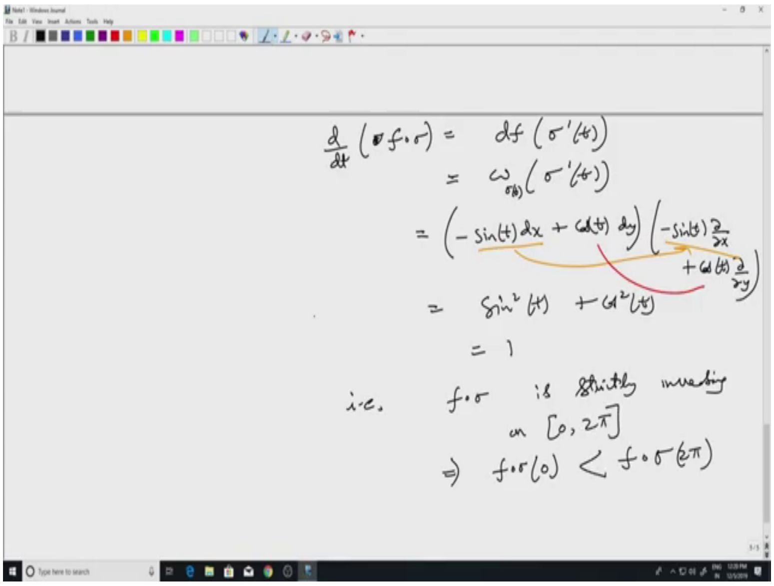

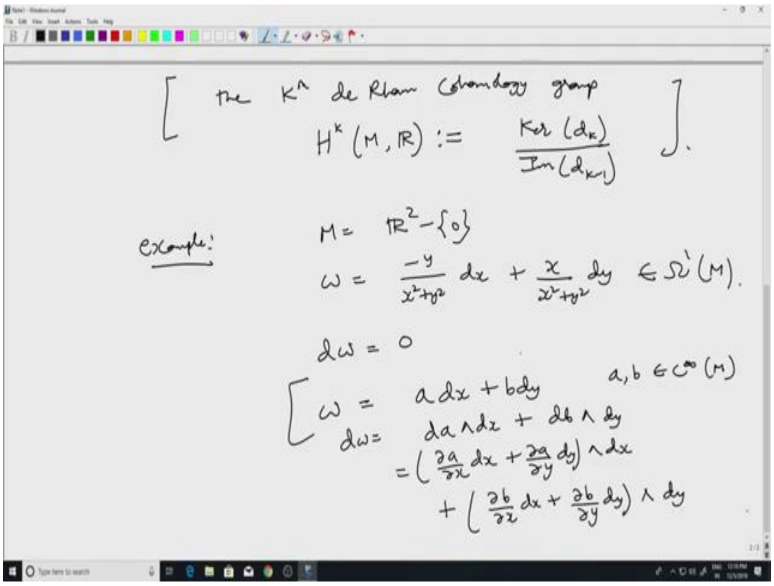

So let us compute the derivative of f composed with sigma d by dt of f composed with sigma is by the chain rule, df of sigma prime d, where here of course, d f is the, I am using the derivative interpretation of df, which is the same thing as the one form df. And this is Omega, since I have assumed d f is Omega. So this is Omega of sigma prime t. So Omega at sigma t at the point sigma t. Now, sigma t is a point on the circle, so if I look at the formula for Omega, this x square plus y squares becomes 1 on the circle. So I will just get minus y dx plus x dy, which is the y coordinate is minus sin t dx plus cos t dy and this thing evaluated on sigma prime, which is, well it is sigma equal, so minus sin t cos t.

Yeah, it is better that I write it as a tangent, use the tangent vector notation, then we will see it is even more clear. So, minus sin t Del by Del x plus cos t Del by Del y. Now, so there are four terms and but the point is these are dual one forms, so dx acting on Del by Del y will be 0, etcetera. So I will just end up getting this, and this will combine to give me sin squared and this and this will combine to be cos squared, so sin square t plus cos square t, and therefore, which is 1. i.e. f composed with sigma is strictly increasing on 0, 1 which will imply that f composed with sigma 0 is strictly less than f composed with sigma 1 which is a contradiction, because we saw that these, oh sorry not 1 it is 2 Pi, so 2 Pi. We saw that these two values are the same, so this equal to this so therefore, this form is not exact. So let us stop here and resume with the discussion of orientation and their connection with differential forms and manifolds. Thank you.