An Introduction to Smooth Manifolds
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The Exterior Derivative 5
Lecture 64

Hello and welcome to this lecture. So, let me finish with the proof of existence and uniqueness of the exterior differentiation operator on manifolds.

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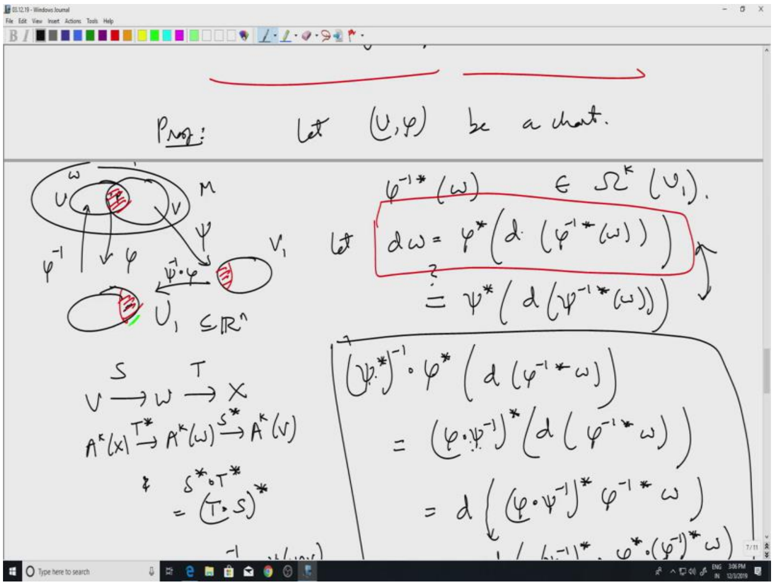

Well, we had stopped at this point. So, first of all, we will define the operator using local coordinates by this formula. Basically, we push forward the, or rather pull back the form to an open set in Rn, apply d there and then pull it back to the manifold. And then we showed, that this is well defined. As well as, this way of defining d satisfies properties 1 to 3. I just showed 3, but then left 1 and 2 as exercises.

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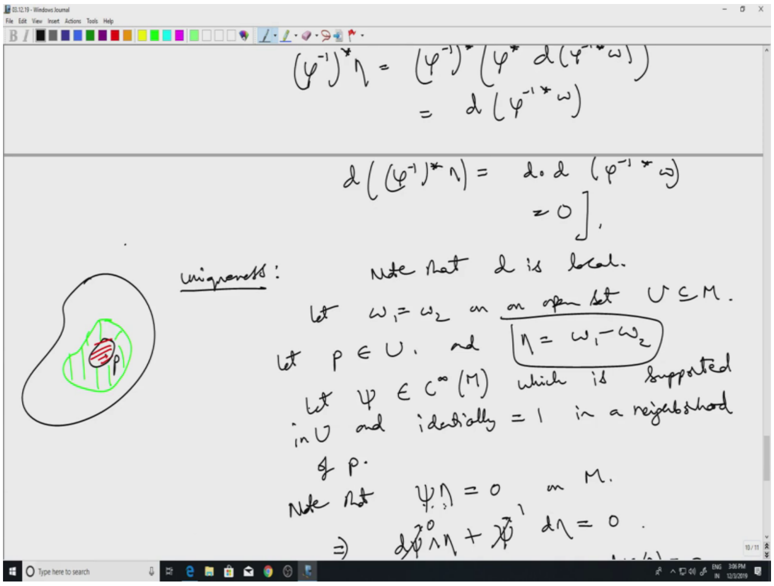

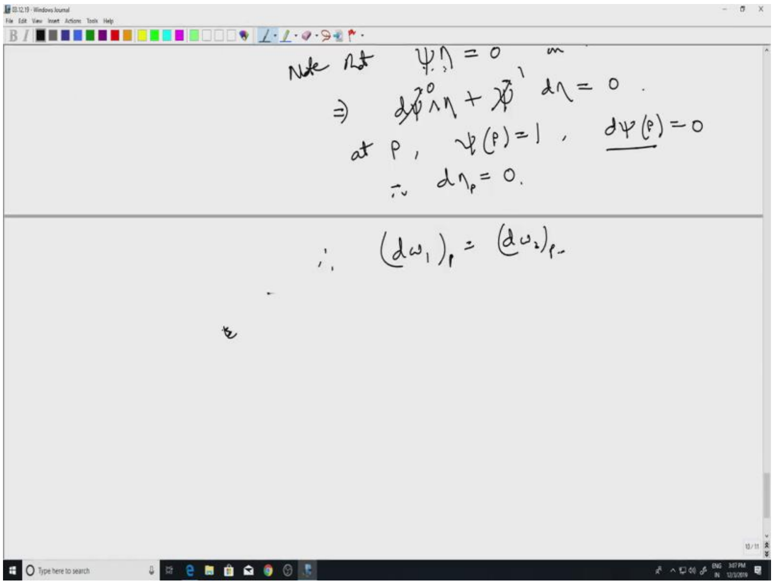

Now, note that yeah, so, the next observation is to get to uniqueness of this. Note that d is local. So, in other words, if we have 2 forms on manifold, which agree on an open set, then d omega 1 will be equal to d omega 2, on the same open set. So, with this in hand, now we can prove uniqueness as follows.

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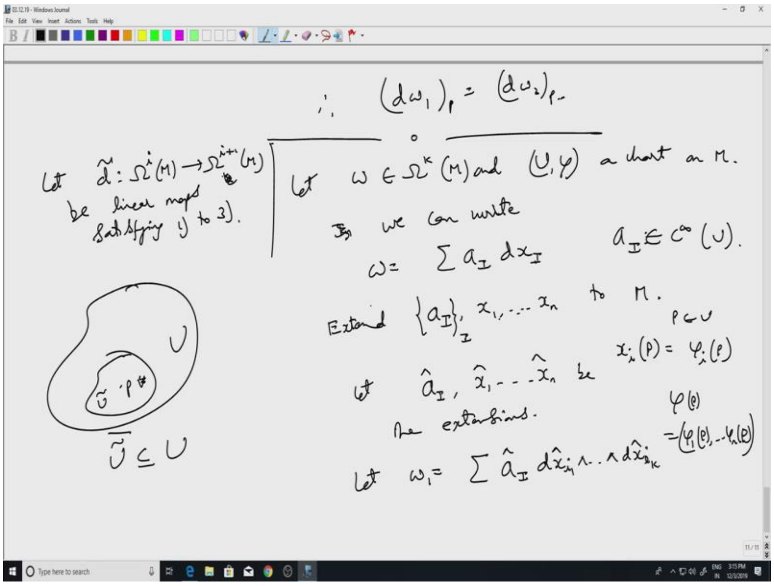

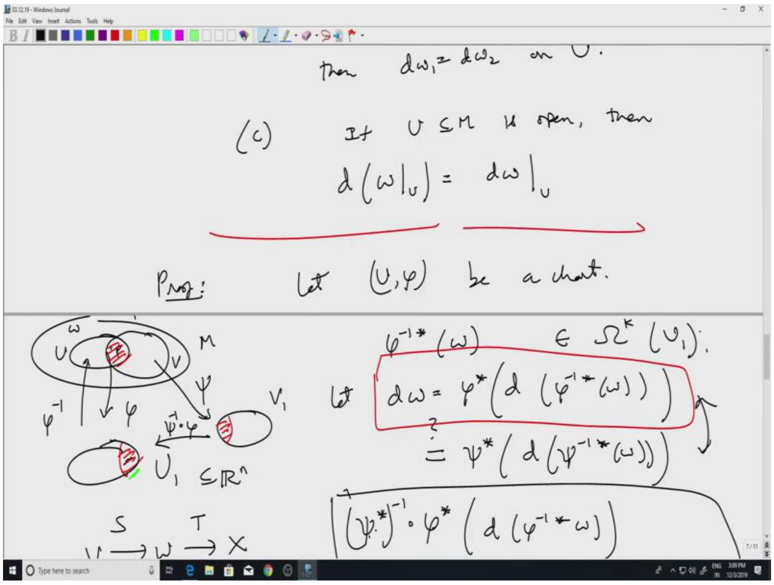

So, let us fix a chart, any chart on M and then we know that in, before that, I should remark that let d tilde be another set of omega i M omega i plus 1 be linear maps satisfying 1 to 3. So, suppose there is other bunch of possible candidates for exterior differentiation, which I called d tilde. I would like to show that this d tilde is actually equal to the d, that we just defined. Now, the moment you have these properties 1 to 3, this first, the local property just used properties 1 and 2, in the proof. So, the local property will continue to hold for d tilde. Whatever argument we just gave, gave in the previous lecture, we will hold for any operator, which satisfies 1 to 3 in the theorem.

So, with that in mind, let us trail it omega be any form and you take a chart on M. We can write omega equals ai dxi, where ai are smooth functions C infinity U. Now, what I should remark is that, we can write this, yeah. So, the definition of d that we gave, namely pulling it to Rn, applying d there, this definition, the one in the red box. This definition is amounts to saying, will imply extent ai, x1, x2, xn. This is not good notation. So, the bracket should come after this, this ai functions, as well as this coordinate functions x1, x2, xn to M. To begin with, ai was defined just on U and similarly this x1, x2, xn.

This x1, x2, xn of course are what one means is, by this is that xi at a point P and U is the chart map phi i of P, where phi of P is phi 1 P or phi n P. Coordinates just refer to the components on this phi 1, of phi. And, so one has this and these are defined, again, the xi's are defined only on U. So, we can extend them to all of M. Well, actually, if one wants to be very precise, then one has to take a smaller open set. So, this is U and you pick a point P and then take a smaller open set, smaller chart U tilde, such that the closure of U tilde is contained in U and work with and extend it to, extend this ai, x1, x2, xn to all of M.

In other words, extend it outside U tilde, rather than U. The reason being that, at the boundary of U, for instance, this xi P, it can go to infinity in Rn. So, in order to avoid technical difficulties like that, one normally takes a smaller chart and works with that. So, let us assume that we have done that. And the point is that, let be the extensions. Now, look at this form. Let omega 1 equal to ai hat, then again dx 1, dx n hat. Sorry, xi 1, oops, this is not quite correct. d hat xi 1, dx hat ik.

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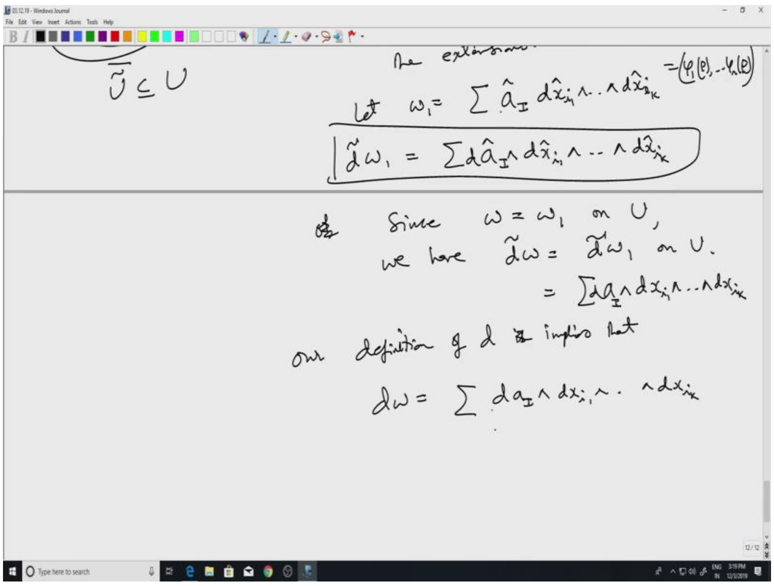


Since $\omega = \omega_1$ on U,

we have $\tilde{d}\omega = \tilde{d}\omega_1$ on U.

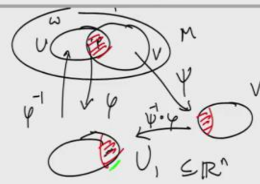
$$= \sum da_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

our definition of d implies that

$$d\omega = \sum da_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$\therefore \quad \tilde{d}\omega = d\omega \text{ on } U.$$

Proof: Let (U, φ) be a chart.



$$\varphi^{-1*}(\omega) \in \Omega^k(U_1).$$

Let $d\omega = \varphi^*\left(d\left(\varphi^{-1*}(\omega)\right)\right)$

$$\overset{?}{=} \psi^*\left(d\left(\psi^{-1*}(\omega)\right)\right)$$

$$V \xrightarrow{S} W \xrightarrow{T} X$$

$$A^k(X) \xrightarrow{T^*} A^k(W) \xrightarrow{S^*} A^k(V)$$

& $S^* \circ T^* = (T \circ S)^*$

$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$

$$(\psi^*)^{-1} \circ \varphi^*\left(d\left(\varphi^{-1*}\omega\right)\right)$$

$$= (\varphi \circ \psi^{-1})^*\left(d\left(\varphi^{-1*}\omega\right)\right)$$

$$= d\left((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega\right)$$

$$= d\left((\psi^{-1})^* \circ \varphi^* \circ (\varphi^{-1})^*\omega\right)$$

$$= d\left(\psi^{-1*}\omega\right)$$

This d also satisfies:

(a) in a chart, if $\omega = \sum a_I dx_I$,

then $d\omega = \sum da_I \wedge dx_I$

(b) d is a "local operator" i.e.

if $\omega_1, \omega_2 \in \Omega^k(M)$ and $\omega_1 = \omega_2$ on an open set $U \subset M$.

Then $d\omega_1 = d\omega_2$ on U.

(c) If $U \subseteq M$ is open, then

$$d(\omega|_U) = d\omega|_U$$

So, now apply d tilde to this, d tilde omega 1 equals. Well, d tilde satisfies properties 1 to 3 in the theorem. So, therefore, one has Leibnitz rule. And moreover, again, d squared will be 0. So, by the same argument that we have been using, we get this and this equation is valid on all of M. And of course, one is interested in the original form omega, not this omega 1. So, but observe that since omega equal to omega 1 on U. The reason being these are extensions of the ai and xi. So, that means they agree on U. And that is what I have written. So, therefore the corresponding forms, omega 1 and omega will also agree on U.

So, since this happens, since omega equal to omega 1 on U, we have d hat, no, d tilde omega equal to d tilde omega 1, on U. Well, so, in other words, this is d tilde, but d tilde omega 1, we have an expression for this on all of M. When again when I restrict to U, this a hat i will become ai and so on. So, this is just ai dx i 1, dx ik. And moreover, yeah, this is what one gets for d tilde. But our definition of d, as what I had written earlier, this definition in the red box is equivalent to saying that, that is another way of writing, this implies that d omega is given by, I think I have forgot something.

So, this dai wedge, here to dai wedge, dai wedge. So, our definition of d implies that d omega is exactly the same thing, dai wedge dxi 1. And the definition in the red box, that this is, that is equivalent to this definition. I leave it as a exercise. But it is quite straight forward. And so, therefore, we conclude that therefore, d tilde omega equal to d omega on U. So, for any chart, we have proved that d tilde, on in any chart, d tilde omega and a equal to d omega. And that concludes the proof of uniqueness.

So, well, now the another crucial property of d, that we have not formally, that was not sort of mentioned in the theorem is, and also, there was the third in a theorem, we said we had three main properties. And I said that, the moment we have these three properties, a, b and c also follow. So, we have already seen. We just saw that, the chart expression is forced to be this. And we have already seen the local property. And let us see this restriction, if U is in open, then d omega, that is also quite straight forward. It follows from the chart property.

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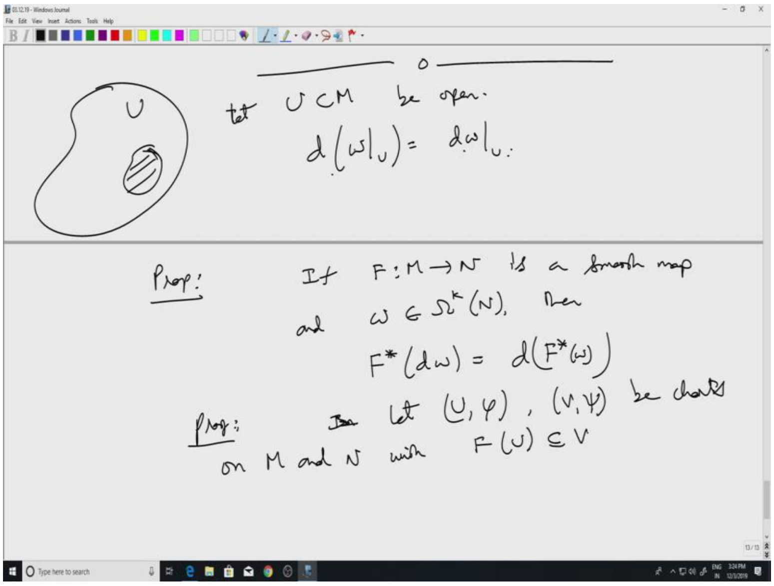

Let U be open. So, we want to check if d of omega restricted to U, equal to d omega restricted to U. So, actually so, this again, if you can, there are couple of ways of doing it. One straight forward way of doing it is just again, just cover, you take an open set and put a chart inside the open set. And then work with the left-hand and then just check directly that in the chart, that this property holds. So, after all, inside the chart, I mean this expression, the moment you express in local coordinates, this expression will, omega restricted to U will have some form, the usual form, I think of this type. And then you will end up getting d of this, which we know is d ai wedge dx i and so on.

And as for the d omega restricted to U, well, we already know that on a chart, d omega is precisely given by this anyway, this expression here. That is the way we define d omega. So, there is nothing much to prove here, actually it more or less follows from our definition of d omega. And so, now the other thing is, which does require a couple of lines of proof is the pullback property. So, if F from M to N is a smooth map and omega belongs to capital omega k N, then I can do F star of d omega. That will be the same as d of F star omega. Again, one can use chart and so on. But that is actually, yeah, so, one can use charts to prove this. In a chart, so, let U phi, V psi be charts on M and N with F of U contained in V.

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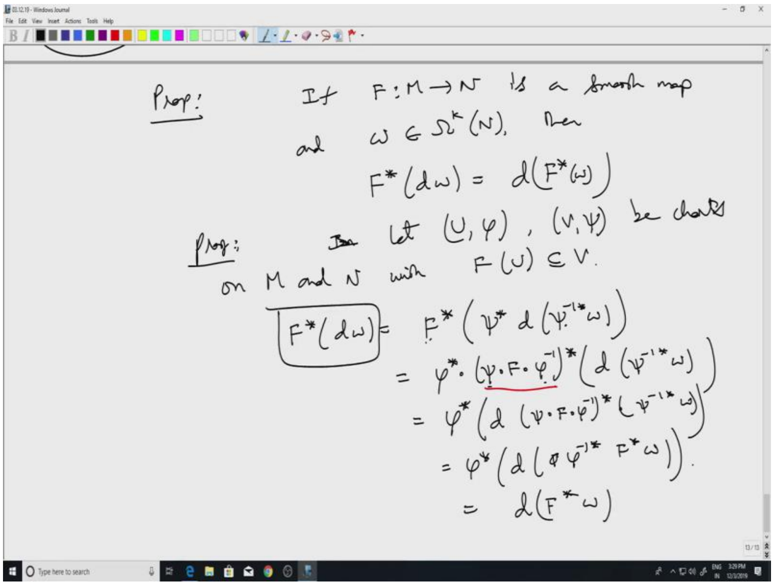

Then F star d omega will be F star. So, here for d omega, I will use the red box definition of d in a chart. So, this will be psi star d, psi inverse star omega. And this will be, well, what I can do is, I can write this as phi star composed with psi composed with F composed with phi inverse star, acting on d of phi inverse star omega. And the reason I can do that is, well, I can just take star of this triple composition, the whole, the order gets reversed. So, I will get, first I will get phi inverse star, which will cancel with this trigonometry identity. Then I will get F star, then psi star, which is what I have here.

So, it is just rewriting, it is upper star composed with psi upper star in this form. Now this, the point of writing it like this is that, this is now a map between what I have here. This thing is a map between open subsets of Euclidean space. And we know that if open subsets of Euclidean spaces and smooth map between them, we know that the star operation commutes with d. So, I can write it as phi star, oops, again I got the red color. So, phi star and then d, so, this entire map I can underline map, I can take inside the d operation. And I get phi composed with F, composed with phi inverse star. And I still have this thing here, of this.

Well, again, if you take star of this triple composition, the whole thing gets reversed and at the right most side I will get psi upper star. But there is a psi inverse upper star. So, that will cancel to give me identity. And finally, I just end up with d of F. Well, so, this phi inverse star F star omega and this, by definition. So, in the first definition, we used that expression for d in local coordinates on N. Now, we use the expression for d in local coordinates of M. So, this is the same, by definition, this is the same as d F star omega, which is what we wanted. We started with this and we will end with this. So, that proves that. So, by now we have all the basic properties of d, that one usually encounters.

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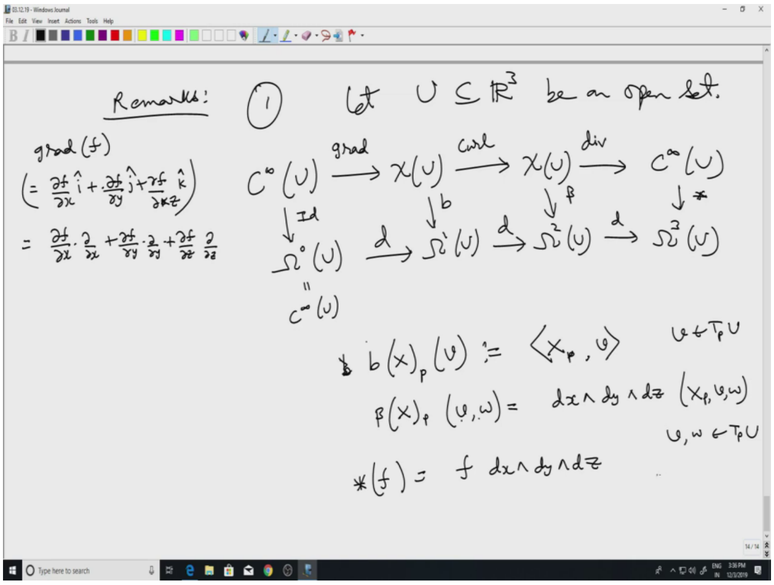

There are some more things involving Lie derivatives and how they, they interplay between Lie derivatives and Lie brackets in particular and d. But since I have not talked about, too much about Lie derivatives, so I will not go into that. However, I would, I will mention shortly relation between d and Lie brackets. So, here are some, the first one is relates d with multi variable calculus, vector calculus as in the classical sense. So, when one studies vector calculus, one has these operations, divergence, curl, gradient.

So, they are very closely related to d. And it is as follows. So, first remark, let U be an open set. Then if I start with a smooth function on U, well, I can take the gradient. And what I end up with is a vector field on U. So, to be precise, the gradient of a function, in classical notation, this is del f by del x i plus del f by del y j plus del f by del k, no, del f by del z k. In our notation, this is, grad f is the vector field, del f by del x del by del x plus del f by del y del by del y del f plus del f by del z del by del z. And once I have a vector field, then I can take the curl of that vector field. So, well, again one can write down the it is the definition of curl that I am working with, the same as the classical one.

And one gets another vector field. And then one has the divergence operator, which gives us back a function. So, you start with a vector field and you get back a function. So, these are the 3 classical operations on functions and vector fields. Now, I have these maps like this. Now, let in the, below this, I will write down d. So, this is C infinity U again. d will take me to one-forms and then d again omega 2 U and then omega 3 U. Well, so, now the claim is that, let us use here, I can, of course both are C infinity U. So, I can identify these 2 spaces, just with the identity map. Now here, I will, it is called sharp isomorphism. So, I will just put b. And then here is a map beta, which I will define. And here is a map star I will define.

So, this b, the input of b is going to be a vector field and out of that, I am supposed to get a one-form. Well, if I prescribe what that one-form is at any point, as an element of the dual space of the tangent space, then I would be done. So, I define this to be XP in a product v. So, here, v is in T P U, which is this R3. So, this is the way I define, one-form associated to a vector field. And this is the usual way one goes between a vector space and its dual space, in the presence of an inner product.

And as for this, beta, this is something, which does not involve an inner product. So, here, beta, so, again I start with a vector field. I am supposed to get a 2-form. So, I am going to act on 2 tangent vectors. And this is defined to be, I look at this 3-form, dx wedge dy wedge dz on R3. And then evaluate it on the input is, and on left side, there are just 2 vectors v and w but I will use XP as a third vector. And this is for all v, w and TP U. And finally, the star map supposed to start with the function and I should get a 3-form on U. Star of, all I do is I just take the standard 3-form, dx wedge dy wedge dz and I multiply it with f, the function f. So, this gives me (()) (29:38).

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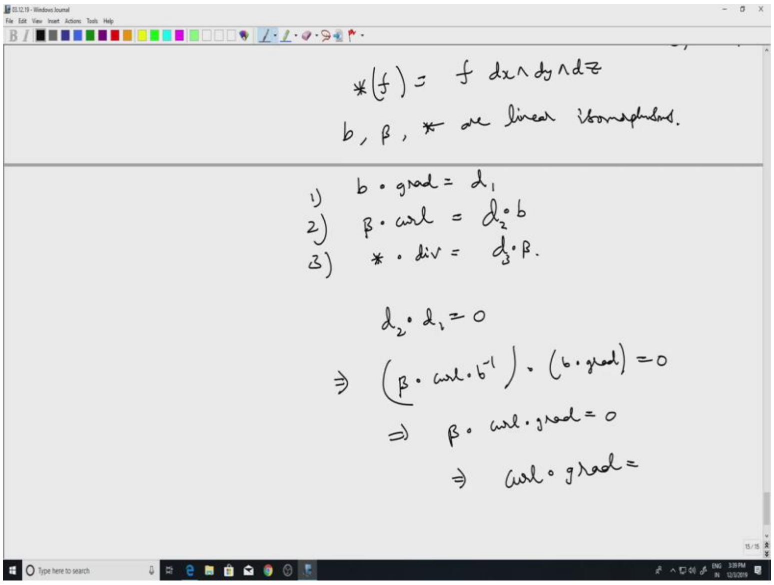

Now one can check that all these are linear isomorphisms between the corresponding spaces. So, sharp beta star are linear isomorphisms. And moreover, the important thing here is that, in a way, keeping these isomorphisms in mind, this classical operations are nothing but d acting on appropriate spaces. In other words, what I mean is, for instance, grad is the same as d. But

I have to keep track of these isomorphisms. In other words, if I start with the space, I apply grad and then come back to this by the sharp, then that is the same thing as d acting on identity. So, for first grad, the sharp isomorphism composed with grad equals d, the first d.

And in the second case, this beta composed with curl of a vector field is the same as d composed with the sharp isomorphism. And in the third case, star composed with divergence is equal to d composed with beta. And from this, one can check that, see, we know that, for instance, that d (comp), let us give it name d1, yeah, d1, d2, d3. The fact that d2 composed with d1 equal to 0, can be implies instead of d2, I can put b inverse of this side, after all b is an isomorphism. b composed with curl, composed with b inverse.

And d1, I can put, use this expression, b composed with grad equal to 0, the b inverse and b give me identity. So, I get this composed with this, it is 0 but this b is an isomorphism. Therefore, I get curl composed with gradient is 0. The classical fact that curl, likewise, you will also get divergence composed with curl is 0, by using these two, the last 2 b's. So, all the classical vector calculus stuff can be captured with in this formalism, or differential forms and exterior differentiation. So, I will stop here and talk a bit more in the next lecture about relation with Lie bracket. Thank you.