An Introduction to Smooth Manifolds
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The Exterior Derivative 4
Lecture 63

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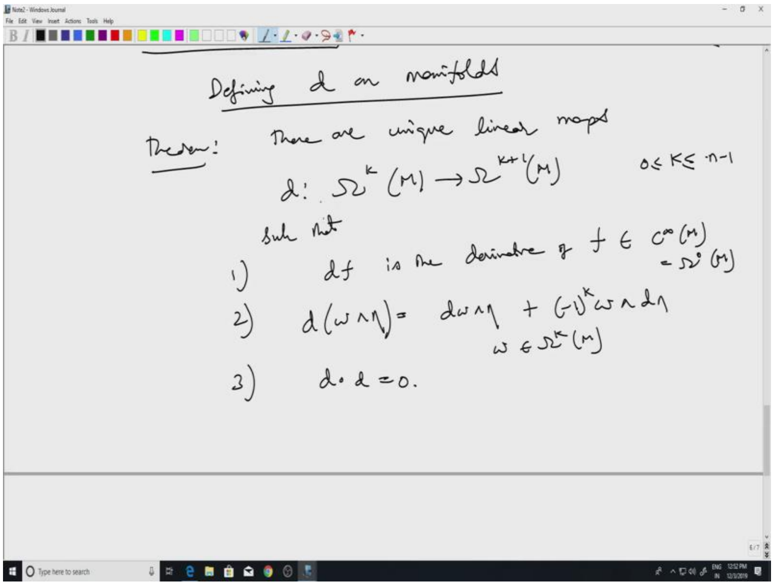

Hello and welcome to today's lecture. So, finally in the, we have got to the point of constructing the d operator on manifolds. So, this was the, where I had stopped last time. So, the statement of the theorem is this. And let me complete the additional properties statement.

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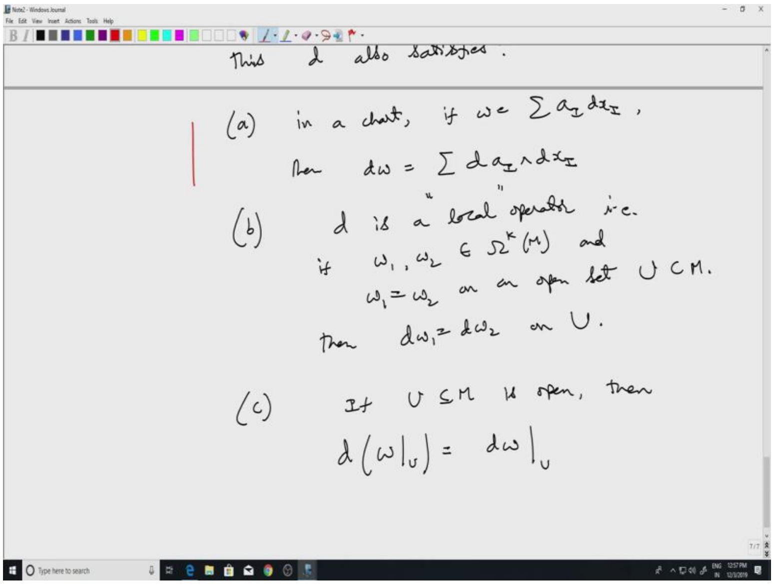

This d also satisfies a, in a chart, if omega is ai, dxi, then d omega is equal to dai wedge dxi, d is a local operator. The word operator in this context is slightly inconsistent with standard usage of this operator, because the vector spaces involved here are not the same. So, normally we say a linear map as an operator, the domain and target vector spaces are the same. But this word operator comes more from differential, from the differential equations point of view, d is a partial differential operator, d is a local operator i.e.

So, what does this mean? If omega 1, omega 2 are 2k forms and omega 1 is equal to omega 2 on an open set U in M, then d omega 1 equal to d omega 2 on U, which is not that surprising, if one thinks of, I mean the same property is true for our usual derivatives of functions. If you have 2 functions on Rn and if they agree on an open set, then the derivatives agree on the open set as well. And the third thing is regard restrictions. If U is open, then d of omega restricted to U, is just d omega restricted to u. So, let me say few words about this additional property as well.

As for the first one, a, this is a statement, I mean we cannot, as of now we cannot use this as the definition of d, because we know that the usual problem of a point, I mean a point being in different coordinate charts arises. So, if we use a specific coordinate chart, omega can be written like this, then one can do d, then one can take d omega and it turns out to be equal to this. But d omega, the expression on the right-hand side, dai wedge dxi very much depends on which chart one is on. So, unless one proves that for different charts, when one gets the same right hand side, one cannot directly use this as a definition of d. So, it is a property.

Then the other thing is this third property c, what we are saying is that if you want to calculate what is d omega on an open set, if we want to see what d omega is on an open set, then we do not have to worry about omega on the rest of the manifold. We can just restrict omega to U and then take d of that, that will give you d omega. Which is of course something we expect from our knowledge of derivatives as well. If you want to take the derivative of a function at a point, all that matters is, the behavior of the function in any neighborhood of that point. What the function does elsewhere in the manifold or even simpler, even Euclidean space hardly matters.

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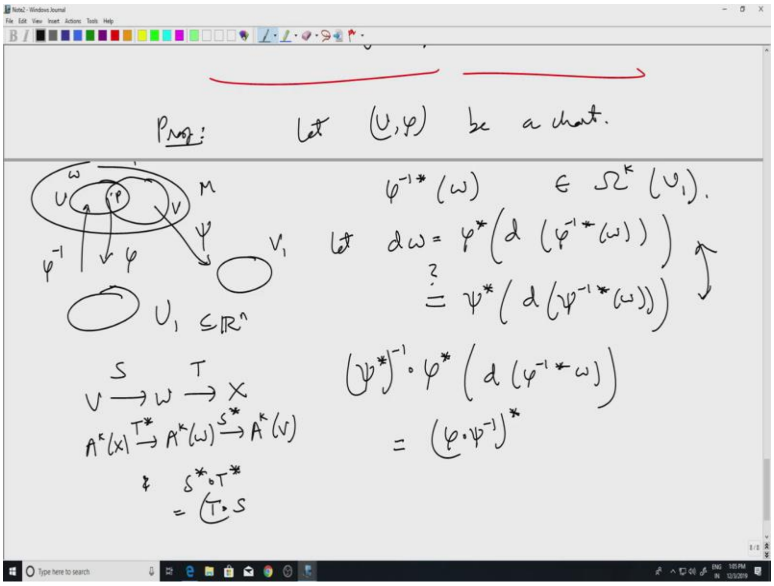

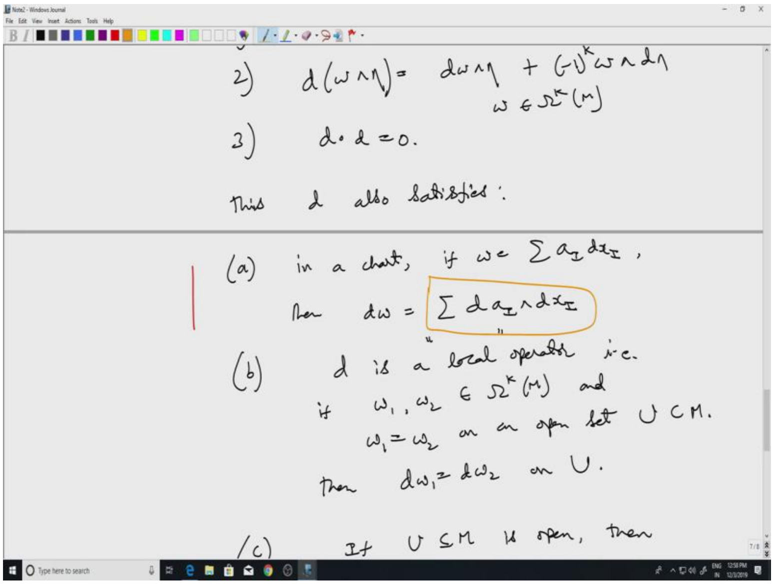

So now let us prove this. So, let U phi, even though I said that one cannot use this as the definition in the proof that is precisely what I end up doing. But of course I have to show that the right-hand side is independent of which, rather this expression here, this thing. If I use a different chart, I should get a (simi), write down a similar expression. That expression should be actually equal to this. Then this would be independent of the chart and we have a well-defined d. So, what I do here is, well, let us start with, be a chart.

So, as usual, this is U, this is phi and this is what I have been calling U1. This is an Rn, this is the manifold. Well, so phi inverse goes in this direction. So let us we can, what we can, omega is here on the manifold. So, I can pull back omega by phi inverse star to U1. So, I look at phi inverse star of omega. This will be a k form on U1, k omega k U1. And then, I can do the d operation here, since we already know, we have already studied the exterior differentiation on open sets and Rn, I can do d here. Whatever, and I will get a k plus on form, then use phi to pull it back.

So, let d omega be equal to phi star d of phi inverse star of omega. So, when I write this, in this form, what I mean is, if I start with any point in the manifold, I first put it in a chart U phi, then I define d omega at P, by this expression. So, the main question is, if I put the same P, so, this is the point, it can be very well contained in another chart V. So, let us use, this is U. So, what I should get here on the right-hand side should be the same, even if I start with V. So, let us check that. So, V comes with its own diffeomorphism psi and here is V1. So, well, the idea is to, as usual to use the smoothness of the transition function.

So, let me write down first the expression for, in the V psi coordinate chart. So, this is what, what I want to show is, is this equal to, so, I will put a question mark. So, is this expression equal to this. And the answer is yeah. Well, so let us do this, what I first do is I just take, so, this would be equal to, these two would be equal. Here, let us remember that both these phi star, phi upper star and psi upper star are isomorphisms, because, well phi is a diffeomorphism, phi and psi are diffeomorphisms. And the induced map on, the derivatives are linear isomorphisms. Therefore, the induced map on this space of alternating tensors are also isomorphism.

So, this would be equal to this, if and only, if I can multiply both sides by psi star inverse. And I would end up considering psi star inverse, then compose with phi star, d of phi inverse star omega, which I can write as, now these two I can combine and write as phi composed with this, because we know that if we have three vector spaces, S and T, the induced map on the alternating tensor space, so, AkX to AkW, AkV, so, this is, from here to here it is T star, from here to here, this is S star. And we have seen that this star composed with T star is the same as T composed with S star.

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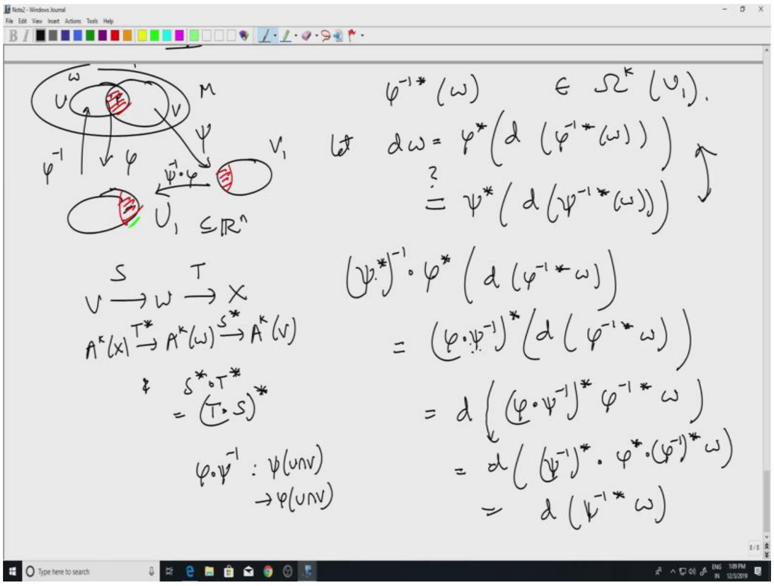

So, the order gets reversed, when you are dealing with pull backs of differential forms, pull backs alternating, pull backs of k tensors, alternating is irrelevant for this. Well, yeah, so, I can write, combining these two, I can write it like this. And then I still have d of phi inverse star omega. And we know that, we have already seen that, now this phi composed with psi inverse, the point is that now we are entirely in the realm of open sets of Euclidean space, because phi composed with, psi inverse composed with phi. This is a map from, as usual this intersection.

So, this is psi inverse composed with phi. There is a map from, oops, I wrote it wrong here. This is phi composed with psi inverse, not the other way around. So, phi composed with psi inverse, there is a map from psi of U intersection V to phi of U intersection V. And both of these, domain and target are open sets now, the red shaded regions. So, and this phi inverse star of omega is also a form, is also a form on this open set, phi inverse star. Omega is a form on this part here.

And so, now we can apply our knowledge of how the exterior (diff) of exterior differentiation on forms defined on open sets in Rn. And we know that pull backs and exterior differentiation commute, so what I can write this as d of, I can take this inside, psi inverse star phi inverse star omega. Well, again, on unravels this and one gets psi inverse star, composed with phi star, composed with phi inverse star of omega. This phi star and phi inverse star cancel to give me identity. So, I am left with d of psi inverse star omega. And we are done.

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$$\varphi^*\left(d\left(\varphi^{-1*}\omega\right)\right)$$
$$= \psi^*\left(d\left(\psi^{-1*}\omega\right)\right)$$

This proves the existence of d satisfying 1) to 3).

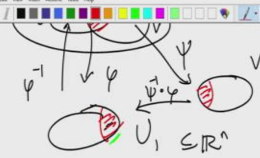
[For 3), $d \circ d(\omega)$
$$= d\big(\underbrace{\varphi^* d(\varphi^{-1*}\omega)}_{\eta}\big)$$
$$= \varphi^*\left(d\left(\varphi^{-1*}\eta\right)\right)$$
$$(\varphi^{-1})^*\eta = (\varphi^{-1})^*\left(\varphi^* d(\varphi^{-1*}\omega)\right)$$
$$= d\left(\varphi^{-1*}\omega\right)$$

Theorem:
$$d : \Omega^k(M) \to \Omega^{k+1}(M) \qquad 0 \le k \le n-1$$
such that
1) df is the derivative of $f \in C^\infty(M) = \Omega^0(M)$
2) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, $\omega \in \Omega^k(M)$
3) $d \circ d = 0$.

This d also satisfies:

(a) in a chart, if $\omega = \sum a_I\, dx_I$, then $d\omega = \boxed{\sum da_I \wedge dx_I}$

(b) d is a "local operator" i.e. if $\omega_1, \omega_2 \in \Omega^k(M)$ and ...



let $\boxed{d\omega = \varphi^*\left(d\left(\varphi^{-1*}(\omega)\right)\right)}$
$$\stackrel{?}{=} \psi^*\left(d\left(\psi^{-1*}(\omega)\right)\right)$$

$$V \xrightarrow{S} W \xrightarrow{T} X$$
$$A^k(X) \xrightarrow{T^*} A^k(W) \xrightarrow{S^*} A^k(V)$$
$$\& \quad S^* \circ T^* = (T \circ S)^*$$
$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

$$(\psi^*)^{-1} \circ \varphi^*\left(d\left(\varphi^{-1*}\omega\right)\right)$$
$$= (\varphi \circ \psi^{-1})^*\left(d\left(\varphi^{-1*}\omega\right)\right)$$
$$= d\left((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega\right)$$
$$= d\left((\psi^{-1})^* \circ \varphi^* \circ (\varphi^{-1})^*\omega\right)$$
$$= d\left(\psi^{-1*}\omega\right)$$

$$\varphi^*\left(d\left(\varphi^{-1*}\omega\right)\right)$$

So, the final step would be, I just transfer this psi star inverse to the right-hand side. So, I will end up getting the left-hand side as, all that remains will be this, starting with phi upper star, inverse star omega. And this is now on this side. So, I will get psi upper star psi inverse upper star of omega, which is precisely, oops, I forgot something here. So, d of this, this d here, which is exactly what we wanted. So, this, the right-hand side is independent of coordinates. So, this proves the existence of d, satisfying 1 to, these properties which are have listed in the theorem, 1 to 3.

Well, why does it satisfy 1 to 3. So, the way we have defined d is, so, this was our definition of d. This, the way we have defined this, essentially, we are just doing d on Rn or other open sets of Rn and then transferring it back to the manifold via these isomorphisms, phi star and phi inverse star. So, from this, it is quite straight forward to check that all these 3 properties hold, because they hold for this d inside. For instance, suppose one wants to check d composed with d0. Then I would write it as, so, I choose a point, choose a chart.

Then in that chart, d omega, I use this expression, phi star, d phi inverse star omega and use the fact that, so, I want to take d of this. So, again, so this would be my new form. Let us call it as eta. Now going by the same local definition, the coordinate definition, d of eta would be, I am using the same chart, phi upper star of d phi inverse star of eta. Now, phi inverse star eta, eta is phi inverse star phi upper star, d of phi inverse star omega. So, and what, this phi inverse star and phi star cancel out and I am left with d of phi inverse upper star of omega. So, then I am taking d of that.

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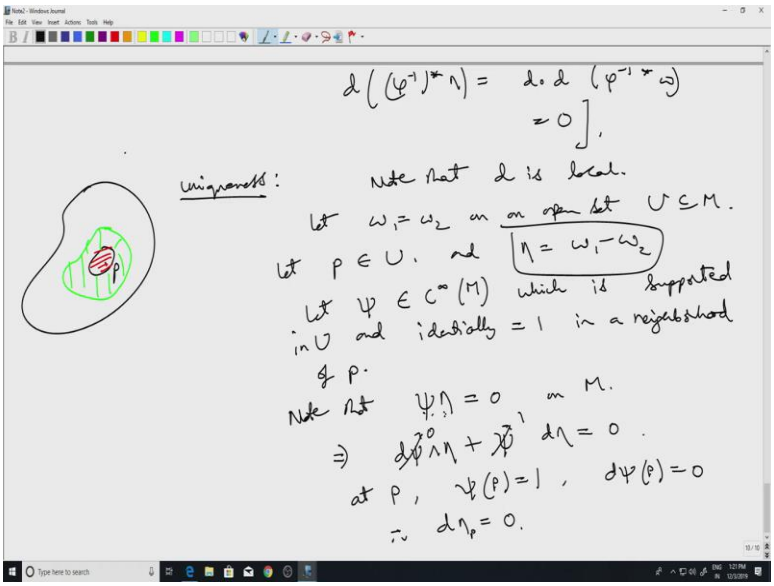

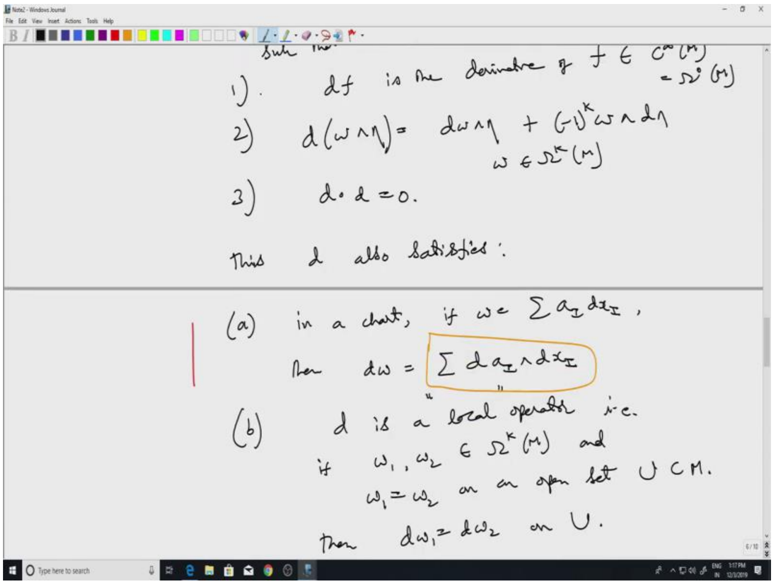

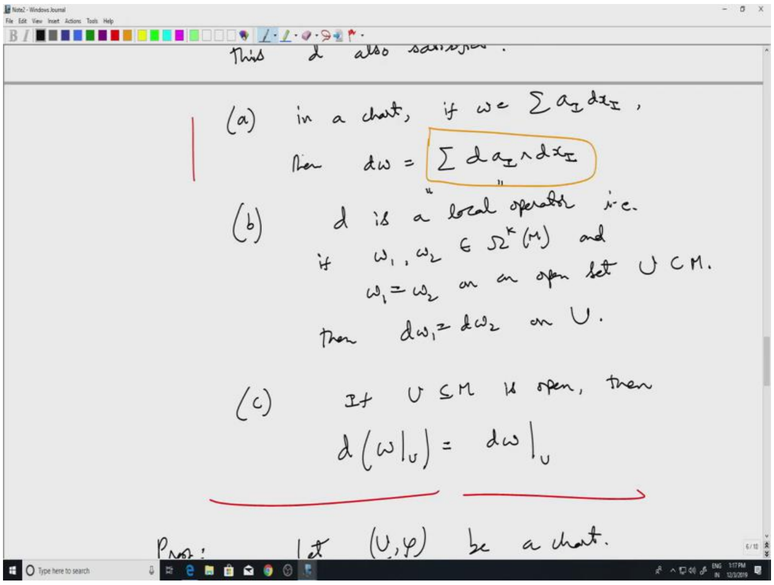

Now, the thing is that, this phi inverse star, d of phi inverse star omega for that matter phi inverse star omega is on Rn. It is a form on Rn. And so, when I do d composed with d, on that I will get 0 just by what we have seen earlier, d composed with d of phi inverse star omega 0. So, it just follows from the fact that we have this on Rn. And the Leibnitz rule is also even more straight forward. And similarly, the first property can also be checked immediately. So, all this, so, I just proved 3, 1 and 2 are quite straight forward. So, this proves that this d defined in terms of charts does satisfy this, what we want. Now, let us prove uniqueness.

In order to prove uniqueness, we have to, actually, in the last class I said, uniqueness follows from these 3 properties, but which is true. But actually, we would rather use this expression in these consequences of these properties and then prove uniqueness. So, note that d is local, in the sense that we have defined earlier. Let omega 1 equal to omega 2 on an open set U in M. We would like to show that this is, d omega 1 equal to d omega 2 on U. So, let us take a point P and U. And let us look at the difference between omega 1 and omega 2. Let psi be a C infinity function on M, which is supported in U and identically equal to 1, in a neighborhood of P.

So, this is the usual cut-off function, that we have been looking at. So, it is identically 1 in this red neighborhood and its support is inside U as well. So, outside this green neighborhood, it is identically 0. So, in this green neighborhood, it varies between 1 and 0. From the way we have constructed it, we can assume it varies between 1 and 0. Though I will not need that for this proof. So, what I do have is note that, psi multiplied by this form eta is equal to 0 on the whole manifold. And the reason is we know that omega 1 minus omega 2 is identically 0 on all of U.

So, if the point is inside U, then this form itself is 0, eta is 0. But if the point is outside U, we do not know about the form, but we know that the function psi is 0, because psi is supported in, so, we have this. And then, I can take d on both sides, d psi wedge eta, plus psi wedge d eta equal to 0. So, at P, well, psi of P is 1. Moreover, we assume that psi is identically 1 in the neighborhood of P. Therefore, its differential is 0 at P, this is 0. Therefore, this goes away. This is 1, I do not have to put the wedge product here, it is just a function times form. So, I get d eta equal to 0.

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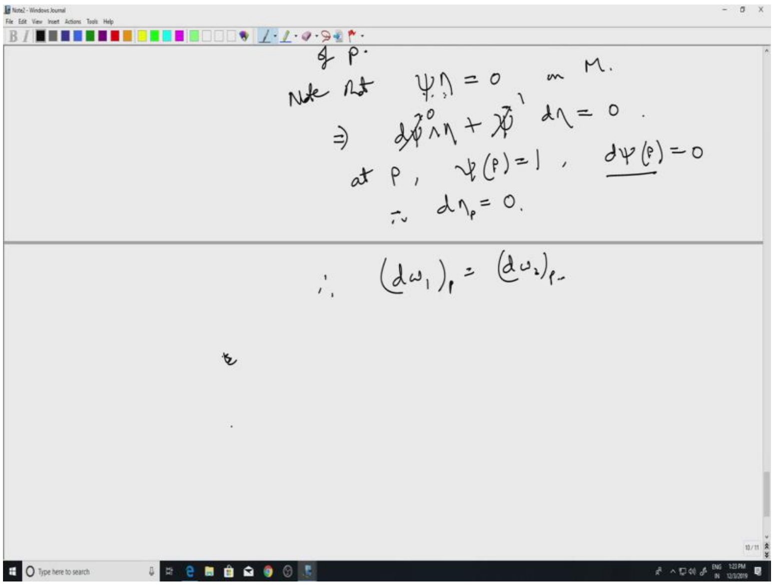

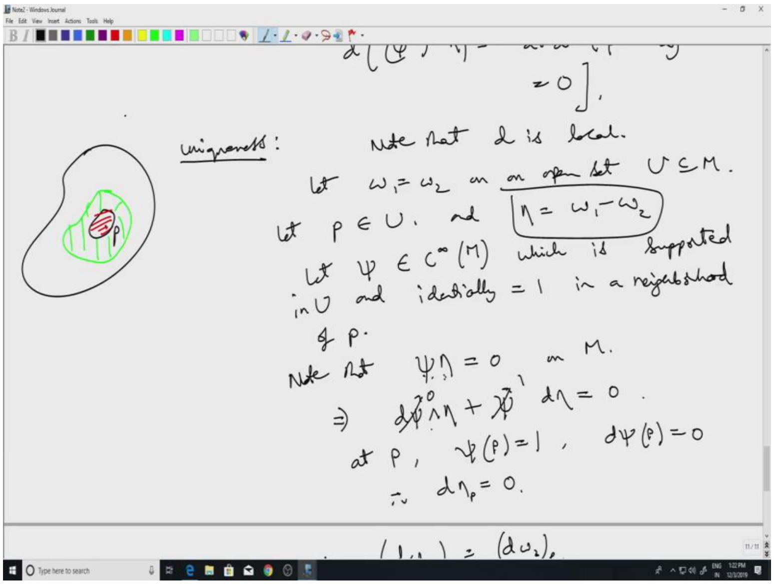

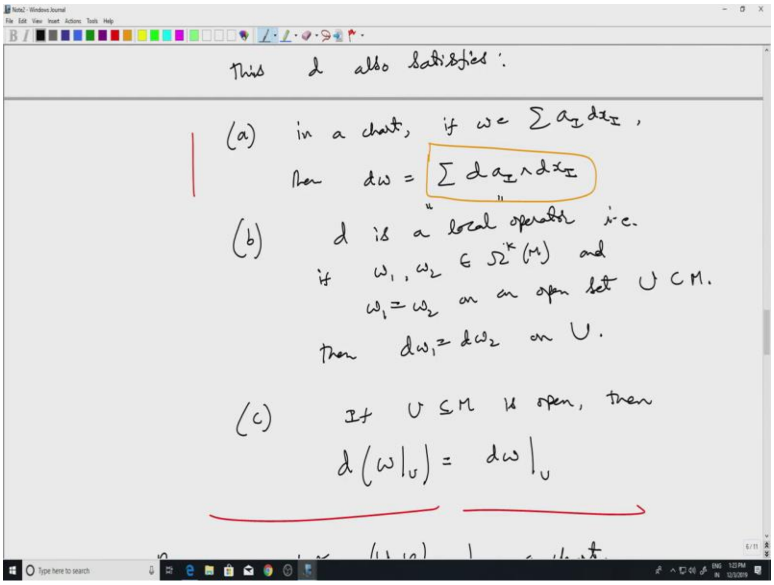

So, d and d eta is d omega 1 at P equal to d omega 2 at P, which is what we wanted. So, we took any point in U and we get this. So, essentially, what did we use about d. The only thing we used, well, we used two things, one is that for a 0 form, d psi coincides with the derivative. That is how we could say that, this is 0. The other thing is Leibnitz rule. So, by using properties 1 and 2, we got this. The other additional property was, well, there are two other additional properties, a, was in any chart, if we write it in this form, d omega equal to d, it has to coincide with this expression here. So, that immediately implies the following, that in the first part, we have shown that it is local. So, now let us check that, this actually implies that d is unique, may be at this point. So, rather than continue with this in this lecture, so, I will just stop here. In the next lecture, I will prove uniqueness and then later on we will discuss some examples. Okay, thank you.