An Introduction to Smooth Manifolds
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Lecture-61

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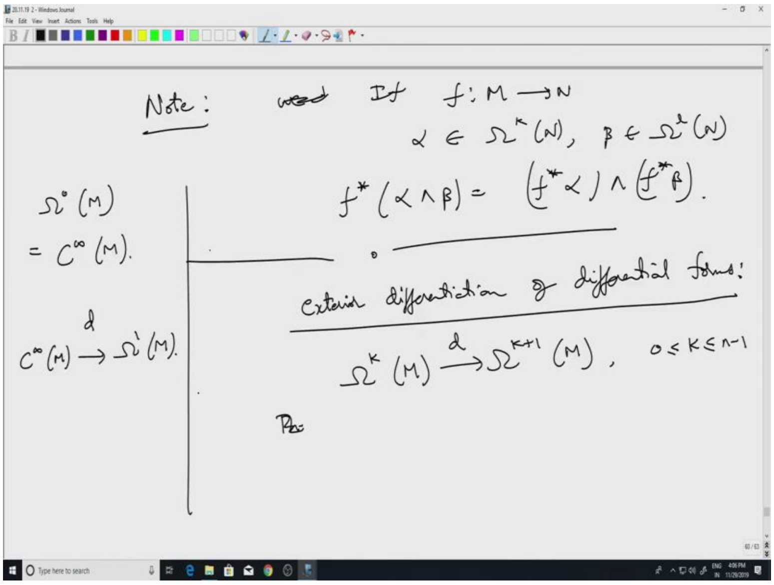

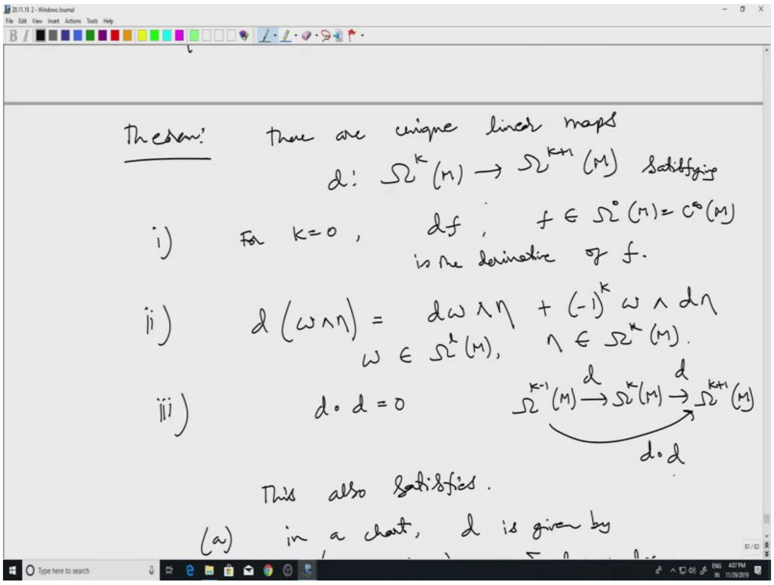

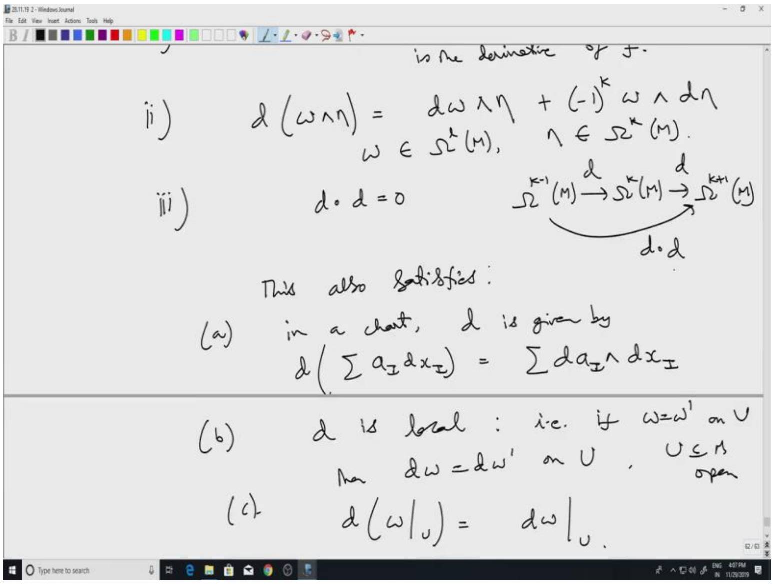

So, welcome to the 61st lecture, the series. Towards the end of the last lecture, I start, I talked a bit about this the important operation called exterior differentiation on differential forms. So, the formal statement is that there are unique linear maps from omega k to okay plus 1 that when k is zero. We defined 0 forms to be infinity functions.

So, this d f, this d that we are defining is the same as the usual derivative. And then there is sort of Leibniz rule here. Then this mysterious property that d compose with the 0 and then one has 3 additional properties which that in a chart, this d is given by this formula here like this, and these local and restriction.

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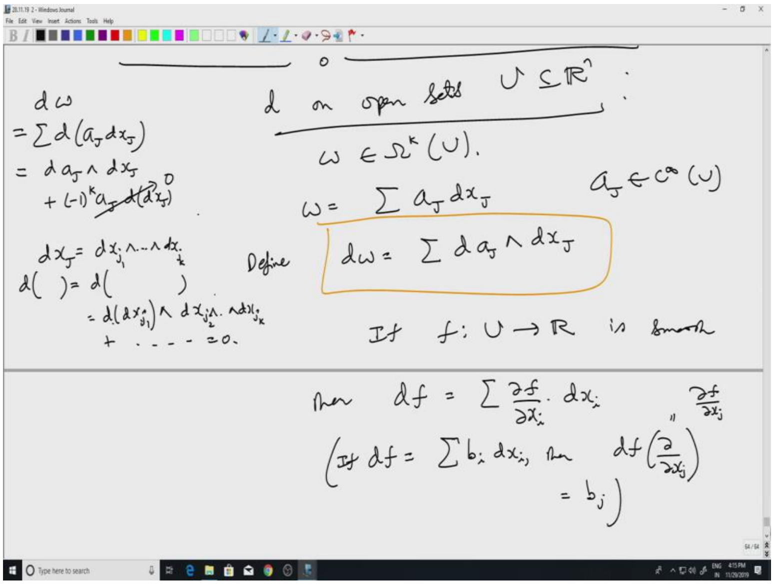

$$= b_j \Big)$$

$$\therefore \quad \boxed{d\omega = \sum_{i, J} \frac{\partial a_J}{\partial x_i} \, dx_i \wedge dx_J}$$

example: If ω is a 1-form, then

$$\omega = \sum_j a_j \, dx_j$$

$$d\omega = \sum_{i,j} \frac{\partial a_j}{\partial x_i} \, dx_i \wedge dx_j$$

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$$= \sum_{i<j} \frac{\partial a_j}{\partial x_i} \, dx_i \wedge dx_j + \underbrace{\sum_{i>j} \frac{\partial a_j}{\partial x_i} \, dx_i \wedge dx_j}_{\sum_{i<j} \frac{\partial a_i}{\partial x_j} \, \underbrace{dx_j \wedge dx_i}_{-dx_i \wedge dx_j}}$$

$$= \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) dx_i \wedge dx_j$$

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[It turns out that if ω is a 1-form

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

$\forall \; X, Y \in \mathfrak{X}(M)$]

Now, all this might make d seem a bit abstract, but the key lies in this local expression for d. So, in fact, let us use that first to talk about d in R n. So, d on open sets you and R n. So, let us take a k form. I know that I can write it as aj d xj, where aj is C infinity functions on U and so, let us define d omega equal to d aj wedge d xj.

Now, why do we define it this way? Well, if you want the second property and the third property to hold, this is pretty much forced on us, because if I start with d omega, omega of this form, then d omega and of course, I also need d to be linear. So, d omega would be some of d of these terms, d omega is sum d aj d xj then if I impose the Leibniz rule condition, I would get d aj wedge d xj plus, minus 1 to the power k a j wedge.

Now, this is a function. So, function at a point is just a constant. So, we had just this usual multiplication so aj times d of d xj. And well, this d xj is what we mean by d xj is we was d xj 1 wedge d xj k, this was d xj. So, if I take d of this, and we will be taking d of this right hand side. And I'll be using the Leibniz rule many times.

So, but so I will get lots of I will get exactly k terms if I apply Leibniz rule but in each term, for example, the first term would be d of d xj 1 wedge the remaining stuff would remain, remaining terms would remain the same, d xj k plus dot, dot, dot you will get signs also but and in each term, you will be doing d of some d x, some index.

Now, the second condition tells us that d of d 0. So, here I have d of d and actually I am using the first condition as well, in the first condition tells us that the d which is occurring here, by the way, is the classical interpretation of derivative of a smooth function as a one form.

This d that I am writing is the new operation d. So, what this old d and new d coincide, so d compose with d is 0, so all these terms will be 0. So, this entire thing goes away. So, I am left with d aj wedge d xj. In other words, if we impose linearity, condition 1, condition 2, condition 3, this is pretty much the only possible definition of d omega.

Now let us look at it, a bit more carefully this what we have here. First let us try to see what is if f is from as from U is as infinite function df we know is a one form. So, but what are its and any one form can be expressed in terms of d x1, d x2, d xn. So, when we do that what are the coefficients? I claim that then d f is equal to summation del f by del xi, d xi.

And this is quite clear because all one has to do is, it is the same old way of getting hold of the coefficients whenever we have an expansion like this. If d f is equal to some let us say bi d xi, then df evaluated on del by del xj is this, bj all the other things will be 0. So, but this is the same as the del f by del xj, right. So, one gets this. So, therefore, one has this expansion. So, all the coefficients such as partial derivatives and one has this.

So, going back to this and plugging instead of f of course, here we have a j. So, therefore, d omega I can write as summation over j summation over i del ai del aj by del xi, d xi, then wedge d xj. So, let me remove this two summations and just put a summation over i and j where i comma J of this thing here. Now, let us see what, what we get when we have a one form.

So, these are examples actually, if omega is a one form, actually in this example, if we smooth, so I, so this is just a simpler way, it is not really an example. But let us now this is an example, example. If omega is a one form, then we know that then I can write omega as aj d xj then d omega would be going by the stuff that I have here. So, this is summation or summation over i comma j del aj by del xi, d xi wedge d xj.

Now, if i equals j I will get d xi wedge d xi which will be 0. Because of the anticommutativity property of one forms. So, I there are only, so I have to consider two cases i less than j del aj by del xi d xi wedge d xj plus i greater than j del aj, del xi, d xi wedge d xj. Now, the second thing I will just interchange the indices and write it as i less than j, del ai by del xj, del xj wedge del xi and notice that this is the same as negative of d xi wedge d xj.

Now, I can combine these two terms the first term and this term. Therefore equal to i less than j, here it is del aj by del xi, here, it's del a i by del x i but with a negative sign. So, del aj by del xi minus del ai by del xj, d xi wedge d xj. So, this is the expression for, a d of a one form. Well, that is just an example. And it turns out interestingly enough that there is a nice way of writing this without using coordinates. It turns out that this is true on a manifold as well.

d omega, if omega is a one form, d omega is a well, it is a two forms, and it is action on any two vector fields is given by x as a derivation acting on omega y, which is a function. In fact, if I take a one form and plug in a vector field, I get a function. So, then I will be differentiating that y omega x minus omega, the lie bracket of X and Y for all X, Y n.

So, this, this is only for one forms and here one does not. One is not using coordinates. Sometimes this formula is useful. It can be shown that at any rate, we have not still defined it on a manifold. So, let us continue with the that d that we have here. Of course, one can check that this formula is valid for open sets on Rn as well. What I have here, but I really do not need this.

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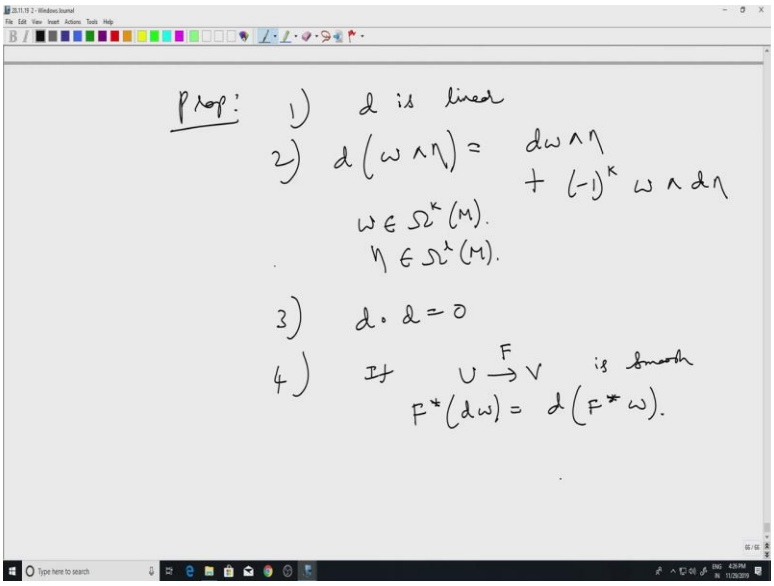

So, proposition, we already, so the point is we now have a definition of d on open sets, namely, this expression inside the orange box is our definition of d. We claim that this d is linear, which is a quite trivial. I will not prove that the d satisfies this, d omega wedge eta plus minus 1 to the power k omega wedge eta, d eta, where omega is in the k form and eta is a l form.

So, let us the other thing is d compose in this and the fourth thing is if f is` a smooth map from U to V smooth. Then, so U and V are open subsets of Euclidean spaces not necessarily of the same dimension they can be of different dimensions as well. F star d omega, this property will be crucial for going back to the manifold for transferring everything back to the manifold.

The first 3 properties any way we want, it was in the theorem. Now, here from what I have written yet, so let us see so. So, if omega is a, for example a function then I would get the f wedge eta plus f wedge d eta, so that is, that is fine 0 form, so right. So, this is correct. So, in the previous theorem, I think I interchanged the role of y should actually want to make a small change here, this is the other way around. So, this is omega as k and this is l.

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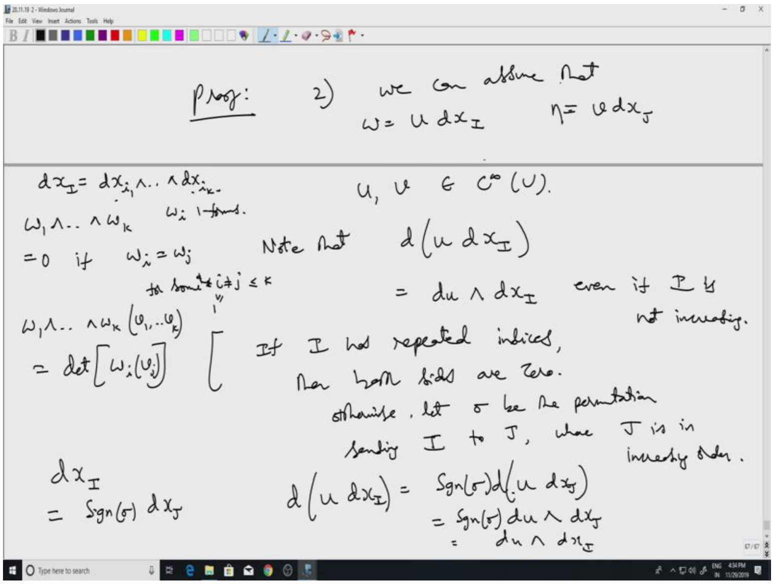

So, let us prove this, proof. Well, 1 is clear so I will not go into 1. Now as for 2, it is we can assume that we can assume that, omega is equal to u d xi. eta equal to v d xj where u and v are smooth functions. u, v are smooth functions on U and the reason we can do that is we can just expand, both omega and eta in terms of their, the basis of the space of k forms and l forms and we can do that on this left hand side and right hand side.

The coefficients that we get will be the same, well, not quite, the coefficients will not be the same, but we can just expand them in terms of their basis. And then use multi-linearity of this wedge product operation and then so on. So, I will not go into details maybe it is a quite straightforward and can be a topic for the tutorial. So, we have this. So, let us work with these 2 forms of this type.

First note that d of u d xi is du wedge d xi even if I is not increasing. The way we define d, everything here this, the index J here was an increasing index. So, with for any, when we have an increasing index and then we had this formula. But here what we are claiming is that it is same thing is true even if I do not have an increasing index. I still get this.

And it is quite simple to see this because as usual, if something is not strictly increasing index, if I has repeated indices then both sides are 0. So, that is a trivial case. Again. So, let me make a comment here. If I has repeated indices then both sides are 0. So, why is this the case? Well, more general comment is after all what is d xi? This is by definition d x, i1, d xi k.

So, more generally, let us look at omega 1 wedge omega k, where this omega is our 1 forms. The claim is that this expression is 0, if omega i equal to omega j for some i not equal to j, between k and 1. So, if these indices, 2 of them coincide, so then which product will be 0 and one can see this in a couple of ways. One is we already had a nice expression for what this is, when we are talking about the vector space setting, we know that this is that of omega i vj.

And if omega i equal to omega j, then 2 columns of this matrix so, no 2 rows of this matrix will be the same. So, therefore, the determinant will be 0 no matter what v1, v2, vk are. So, one gets this, or one can also see it using the anti commutativity property of the wedge product. So, here coming back to this if i has repeated indices, then some dx some index will be equal to dx with some other index. So, therefore, product, wedge product would be 0.

The other possibilities that I have otherwise. All indices in i are distinct but they may not be in increasing order. Let sigma be the permutation, sending I to J, where J is in increasing order. In other words, rearrange the indices, so, that they become, they are in strictly increasing order and call that rearranged in permutation sigma. So, then what will happen? So, we are interested in d u xI is the same as. So, once we do that d xI, which is this will be as we have seen before sign sigma d xJ.

So, therefore this will become sign sigma, u. So, instead of d xI just put the xJ, so the d will still remain d xJ and well sign sigma. Now the point is, this J is in increasing order so d of this, I know by definition will be du wedge d xJ. Again combine the sign sigma with d xJ to get d xu, d xI that is it. So, this shows that this formula continues to hold even in the, now the we will need that because, now let us start calculating this.

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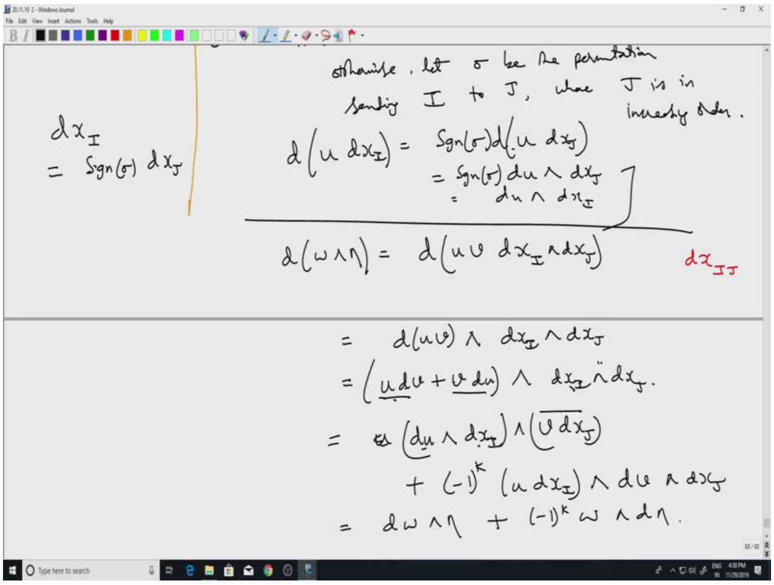

So, you take omega and eta and let us do calculate this d. Now omega I assumed as u d xI and this is v d xJ. So, I get uv d xI wedge d xJ. Now, immediately I will use this the previous remark that the formula, that this formula, this formula holds whatever the index is. So, this d xI wedge d xJ we know is the same as d x IJ, where IJ is the concatenation of I and J. So, we can use the formula to conclude that this is d uv wedge d xI wedge d xJ.

I might have written, I might have written it as this d x IJ. But I will keep it like this. Now, again, use the Leibniz formula here to get udv plus vdu wedge d xI wedge d xJ. So, let me quickly complete this and then, so I can combine this and write it as u. So, this u d v. So, this term, v d u will give rise to du wedge d xI.

So, I am combining these 2 wedge vdxJ. So, this v I am taking to here, plus here is where is the negative sign is coming from, minus 1 raise to k. u d x I. Well, u, right, so I am starting with this and then the d xI am moving past the dv and dv wedge d xJ. d xJ remains wherever it was. So, because I interchange d x I and d v, I got this minus sign, minus 1 raise to k.

And this is the same thing as d omega wedge eta plus minus 1, to the k. After all this is eta, this thing here is eta. This is d omega. And so this is omega wedge d eta. This is d eta. So, that proves the second property. So, we will stop here. In my next lecture, I will deal with the other two properties. Then we will see how we can transfer all this to the manifold setting. Okay, thank you.