An Introduction to Smooth Manifolds
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Lecture-06
Constant Rank Theorem

Hello, so we will begin the fourth lecture in this series. I will begin by talking a bit about the implications of the inverse function theorem and then briefly discuss infinity functions with compact support and then we will see some constructions involving these. Alright, so let us start.

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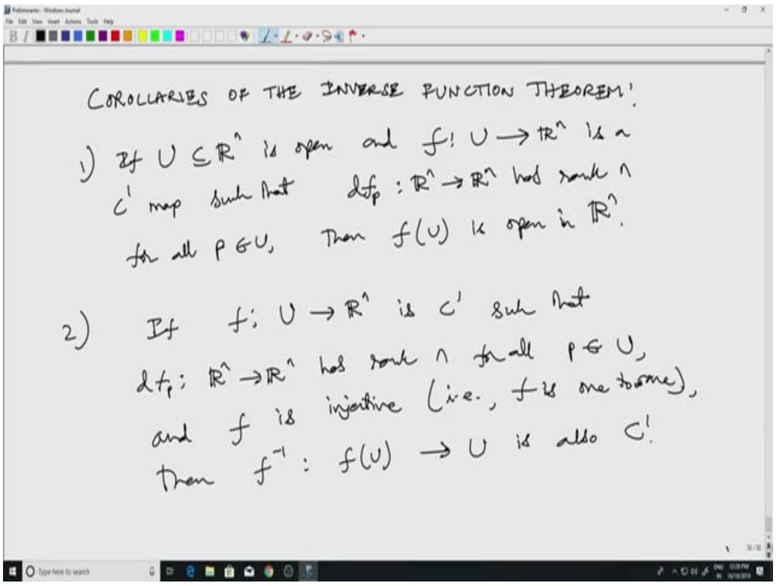

So last time, I menti1d the following consequence of… So, last time I mentioned this, first 1 is that if U Rn is open and f is a C1 map, such that the derivative of f which is a linear map from Rn to Rn has rank n at all points P for all P in U then inverse function theorem will imply that the image of U is open in Rn. So, today I will start by mentioning another immediate corollary of the inverse function theorem. So, again, I have a smooth map, if f from U is again an open subset of Rn, like the first 1. And if you have a C1 map with such that d fP from Rn to Rn, has rank n for all P in U.

Now we make an additional assumption that f is injective, in other words, i.e. f is 1 to 1. The conclusion is, then f inverse which is a map from f of U back to U is also C1, also C1. Well I will just remark that this when to even say that some map is C1, 1 needs a domain to be an open subset of Rn. Now, the first corollary guarantees that f of U is open in Rn. So, it does make sense to say that f inverse is C1. But the point is this assumption that the derivative has rank n for all Ps essential for this result to hold, as the following simple example shows.

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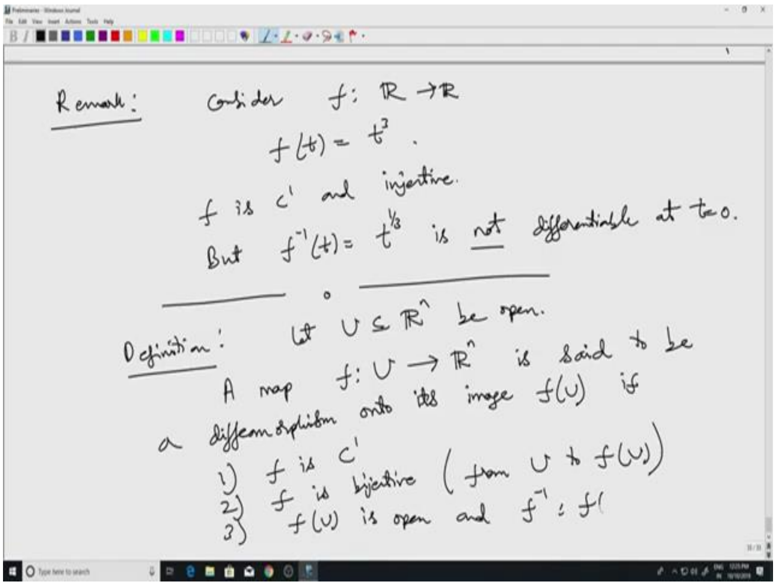

If we drop this assumption, remark, consider f from R to R, f of t equals t cube. Obviously, this is C1, f is C1 and injective, but f inverse of t which is given by t to the power 1 over 3 is not differentiable at t equals 0. So even though the inverse may exist, even though the function may be C1 and its inverse may exist, the inverse may not be differentiable. We, what the previous corollary asserts that, if the function is differentiable and the derivative is an invertible linear map, then of course, inverse function theorem will tell us that the inverse is also differentiable.

So, but 1 must keep this simple minded example of f of t is equal to t cubed in mind. Now I will briefly talk about another major implication of the inverse function theorem. So, to do that, first I have to define the notion of a diffeomorphism. A map from, notice that for this definition I require that the domain is an open subset of Rn and the target is also exactly the same Rn. So a map from here to here is said to be a diffeomorphism onto its image f of U if well first of all, f is C1, second, f is bijective as a map from…

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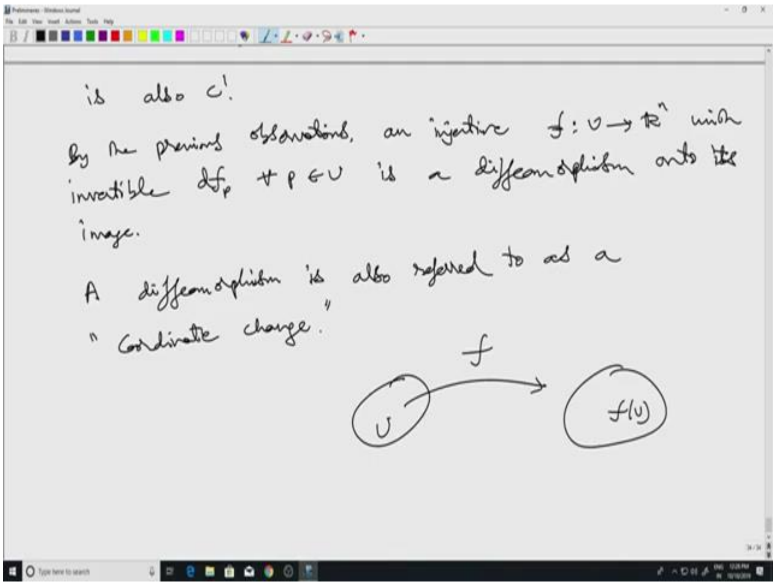

And the third thing is f of U is open and f inverse from f of U to U is also C1. So, notice that by the previous observation, an injective f from U to Rn with invertible derivative for all P in U is a diffeomorphism onto its previous observations, let me make it observations here, observations, diffeomorphism onto its image. Because the previous two observations tell us that the image under the condition that the derivative is invertible the first corollary told us that f of U is open, which was 1 of the requirements for the diffeomorphism.

And second corollary told us that f inverse is also C1. A diffeomorphism is also referred to as a coordinate change. And the reason for that is, well, you have an open set U and f maps it to some other open set f of U. So to begin with, of course, 1 has coordinates, let us say this is an open subset of Rn, so I have, for instance I have Euclidean coordinates for every point in U. But since f is a bijective map from U to f of U corresponding to every point here, I will get a exactly 1 point in f of U. And that new point has, well, some other Euclidean coordinates.

So essentially what you have done is, if you think of a point as being identified by its coordinates, the coordinates have changed under f. And everything happens in a smooth way. That is the point about saying that f is C1 and f inverse is C1.

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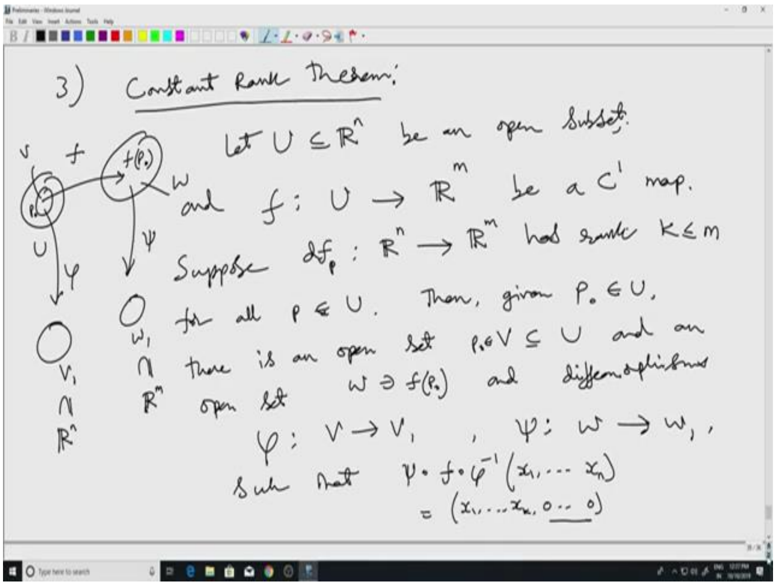

Now with this in mind, let us state another major corollary of the, so I will label it as 3 of the inverse function theorem. This is sometimes referred to as the Constant Rank Theorem, Constant Rank Theorem. So, again the setup is U is an open subset, open subset. Now, 1 major change is I no longer require the target to be Rn again. So, let U be an open subset and f from U to some other Rm be a C1 map.

We demand now that suppose the derivative d fP which is a map from, linear map from Rn to Rm has rank k for all the P in U. The point is that this rank, this k should not depend on the point P, at every point it is, in this the corresponding linear map should have the same rank for all P in U. Then what the theorem says is that we can change coordinates on U and on the target, so that after composing with this new coordinates, f assumes a very simple form, then given P0 in U there is an open set V which contains P 0 and is contained in U.

And an open set W which contains f of P0 and coordinate changes, and diffeomorphisms. Well to be, to illustrate what is going on, so, let me, so here this is my original U and so let me not label, let me not draw an open set here. Let me just start with U here and f. So, I started with some P0 that goes to f of P0. Now I am saying that there is a V, smaller open set V and some W here and diffeomorphisms phi from this V to V1, Psi from the W to W 1. So, let me just change the picture slightly.

So this is V that is getting mapped to V1 by this Phi. And this is C that is getting mapped to W 1. So recall that this V1, by the definition of a diffeomorphism, this V1 has to be an open subset of Rn and this has to be an open subset of Rm. Okay, so this is the picture. And diffeomorphisms like this, such that when I look at the composition the composition is… So I have 3 maps now, one is the original F, then I have a diffeomorphism Phi and a diffeomorphism C.

Well, I can start by looking at phi inverse. So I will go from V1 to V, then I compose with F and then I will compose with C such that, so first is phi inverse then f, then C. This is going to, let us say the claim is that this is a very simple map. So x1… xn and so I just started with some point x1 Cartesian coordinates x1 up to xn inside V1. This has the form x1… x k… 000. So that is it. So well.

So in other words, what this theorem saying is that if a map has constant rank, then it assumes a simplest possible form, which is just a projection onto K coordinates. But there is one small thing that I have to remark here, which is that now the maximum possible rank of this, if when you have a map from Rn to Rm, the maximum possible rank is of course, M, the target dimension. It can very well happen that this k equals m. If k equals m, then there will not be any 0s here. So here k, as k is always less than or equal to m. If k equals m, the 0s that I have put here will not occur, so I will just get x1 up to xn.

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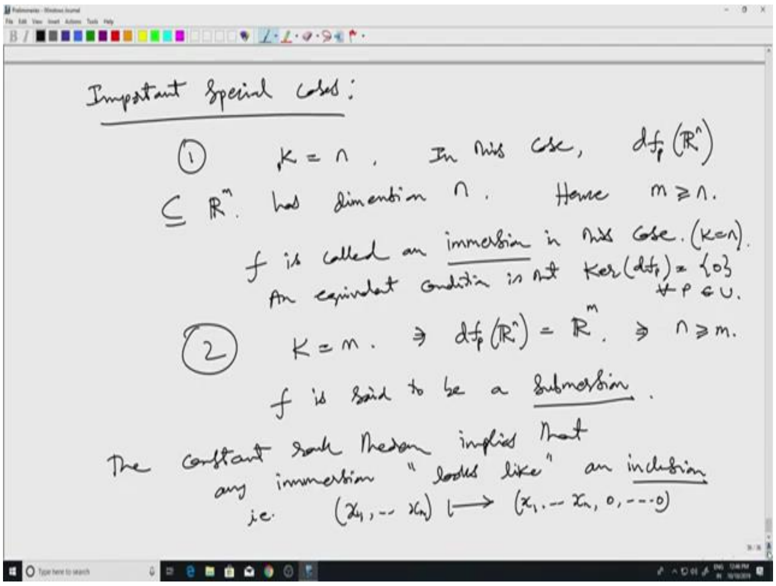

Alright, so there are two special cases of this… Important we will revisit these notions when I talk about Smooth Manifolds. But for now, let me just mention the following, which is that, the first case is that rank the k equals n, the dimension of the domain. Of course, in this case, linear algebra tells us that. So the image of this in this case, when I look at the derivative, at any point and look at d fP, the image of the entire vector space Rn the image of that inside

Rm. So d fP Rn, this vector space which has to be a subspace of Rm has dimension n by definition of rank. Of course, this rank is by definition the dimension of the image.

So, the image has dimension n, but this is sitting inside Rm. So, necessarily m greater than or equal to n. If this happens, we say that, if k the rank is equal to n for all this thing, then f is called an immersion in this case, so in this case of course, I mean that k equals n. The 2nd extreme case is k equals m. So in other words, d fp Rn, then, d fp Rm is a subspace of Rm, but we are told the assumption is that this subspace has dimension m again. So if you have an m dimensional subspace of Rm, it has to be Rm itself. So, this is an onto map, this is onto an f is set to be a submersion.

Here of course, yeah, regarding dimensions if d fp Rm is equal to Rm then we know that this the rank nullity theorem will imply that this implies actually that n greater than or equal to m. I should also mention that in the previous case the emotion case is the condition that k equals n is an equivalent condition, is that d fp, kernel of this d fp is that, kernel of d fp is the trivial subspace for all P in U. This again follows from the rank nullity theorem. If the kernel is trivial then the image of d fp has the same dimension as the domain and conversely if the image has same dimension, then the kernel is trivial.

Both of this statement as well as the surjectivity in the second statement, follow from the rank nullity theorem. So, here also the equivalent condition, that is to say that k equals m is the same thing as saying d fp is surjective, which I have written here. All right. So, these are two important cases. I give more, I will give some examples when we talk about this in the context of manifolds. For now, let us just keep these definitions in mind.

And notice that the Constant Rank Theorem, Constant Rank Theorem implies that any immersion, I will put it in quotation marks, looks like an inclusion i.e. x1 all the way up to xn. Remember that in the case of immersion, the target has more dimension than the image in the domain. So x1, x2, xn, getting mapped to the x1 up to xn remaining coordinates 0. So this map is called, is the inclusion map of Rn inside Rm and it is the simplest possible immersion.

What the Constant Rank Theorem tells us is that, given any immersion, it looks like this, of course here looks like means that after changing coordinates in the domain as well as the range, like I specified in the last slide, f composed with these two coordinates changes has this form. This is regarding and this is Yeah, okay.

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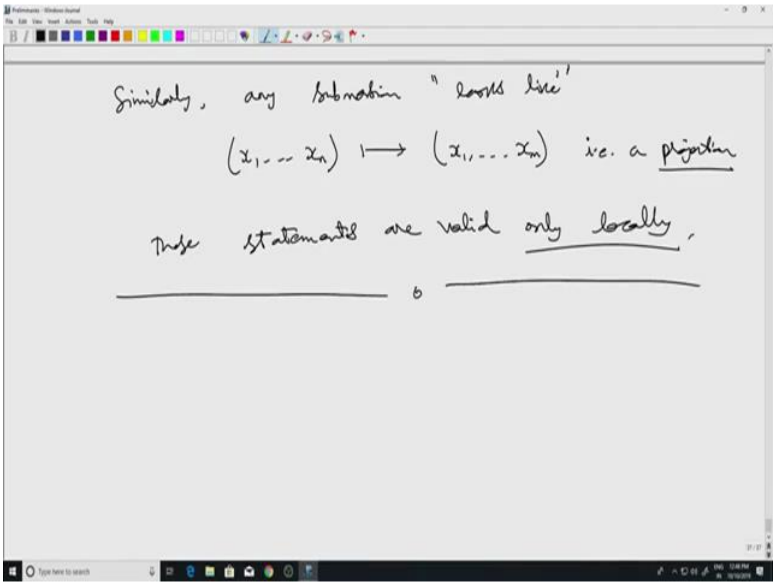

And then similarly any submersion looks like x1… xn will get mapped to, well, in the case of a submersion, so I just pick out the first m coordinates. This is a what is called as a projection of Rn into Rm i.e. a projection. Of course, these statements are valid only locally, this should be kept in mind that the Constant Rank Theorem does not tell us that, nowhere do we say that on entire u I can change coordinates on the whole of u, so that I get some nice form like this that is not the case.

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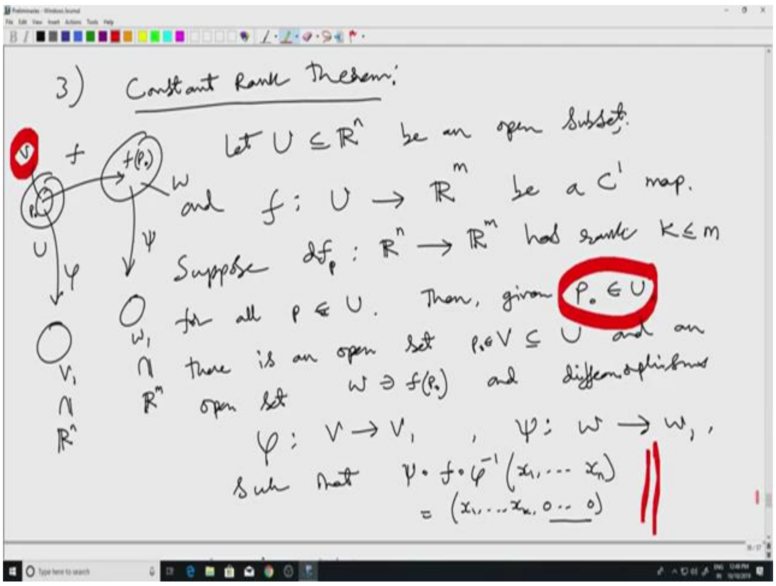

But, so this form that I have here is valid only when we take a specific point P0 inside U, then I can find this coordinate another open set V here and this, I get this nice form which I have put here. So that is what I mean by the statement that this is valid only locally, valid only local. So that concludes my discussion of the inverse function theorem and its corollaries. For our purposes, that is all that we need right now.

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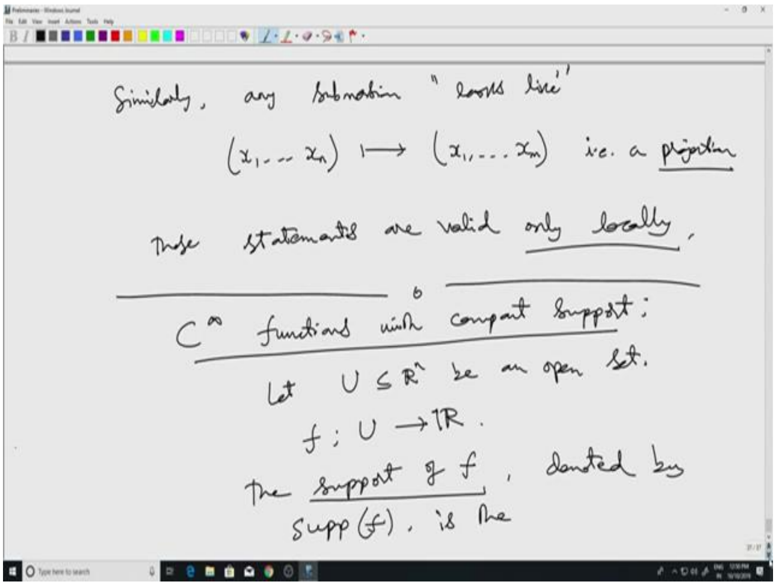

So, the last topic in my set of preliminary topics is C infinity functions with compact support. So what does the support of a function mean? So let, as usual let U be an open set. Let us take a function from U to R, a real valued function. The support of f denoted by supp f is the closure of, closure in U of the set of all x in U such that f of x is not equal to 0.

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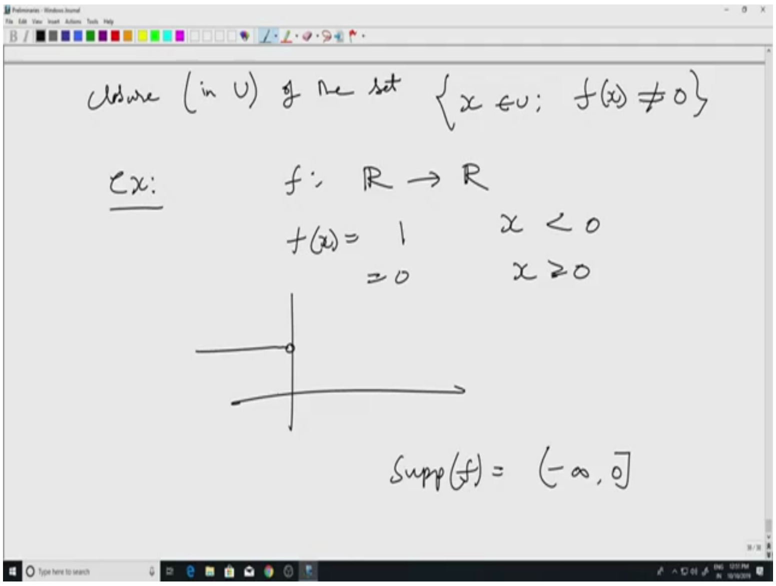

So as a trivial example, let us look at f from R to R, defined by, let us look at the step function. So f of x equals 1 when x is less than 0, is equal to 0 when x is greater than or equal to 0. So in this case, the support of f the graph of f looks like this. Now the set of all x such that f of x is not equal to 0 is precisely the open interval - infinity to 0. When I take the closure I will just get one additional point, so which is 0. So support is - infinity to 0.

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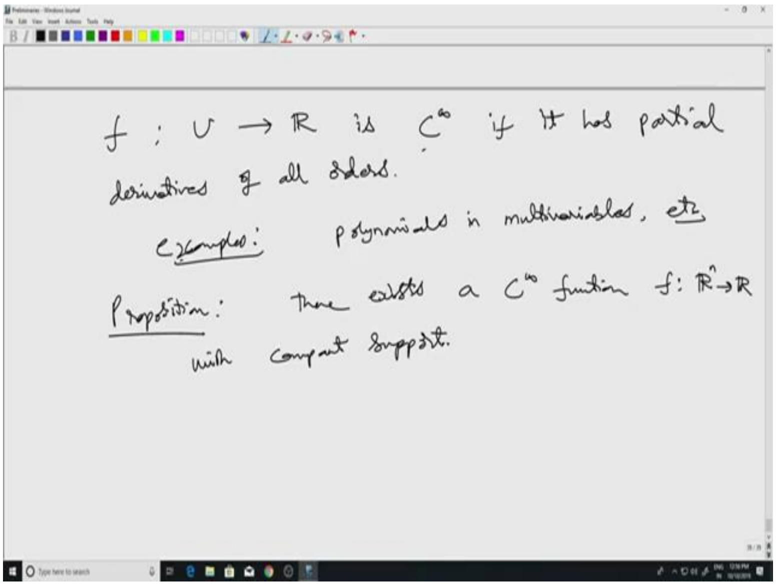

Now, I need one more definition f from U to R is I will say it is C infinity, if it has partial derivatives of all orders. So, of course, one knows of lots of C infinity functions, examples polynomials in multivariables etc. So one can take of course trigonometric functions, polynomials and so on. And then we can compose them do the usual operations. So, there are lots of such things but now what we ask for is something which is not so clear at the outset.

We want to know if there is a given an open set can I find a C infinity function defined on U who support is compact. In other words outside a compact set it is identically 0. Well surprisingly, it turns out that there are such things. So, let me state it as a proposition, proposition, there exists a C infinity function f from Rn to R with compact support. So, support is, the support of this function is going to be a ball, a closed Euclidian ball inside Rn.

So, how does 1 go about constructing such a thing? Of course, one must be I must remark that there is a trivial case, which one must exclude here, namely, if I take the function which is identically 0, then its support would be the empty set and that would be compact. Obviously, we do not want that. So, in fact, this infinity function that we are going to construct will be identically 1 inside a 1 Euclidian ball and identically 0 outside a larger Euclidian ball.

So the way one does this is, so yeah, so let us stop here. So in the next lecture, I am going to show explicitly how to construct this starting from one variable function. And then once I, in other words, I will assume that n equals 1 and do this construction and then it will be easy to do it for any n. Okay, goodbye.