An Introduction to Smooth Manifolds
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Lecture 57 - Alternating Tensors 9

Hello and welcome to the 57th lecture in this series.

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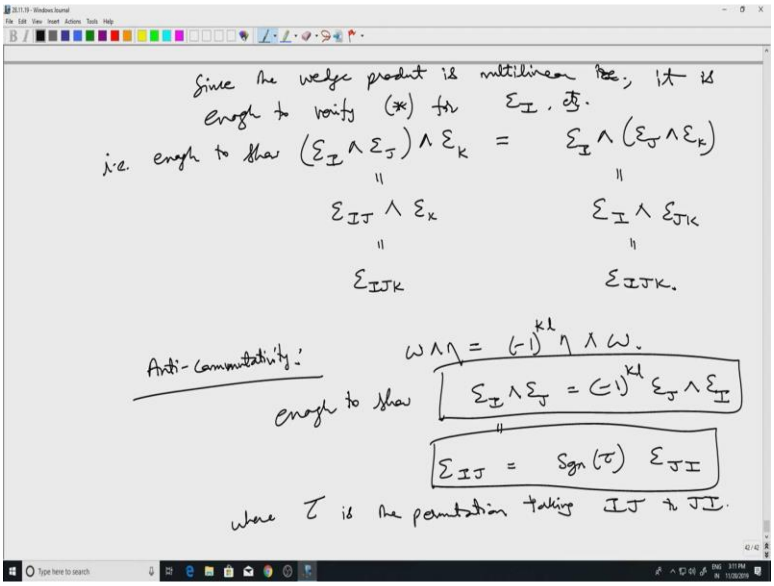

Let me resume from where I had stopped last time, that was showing anti-commutativity of the wedge product operation. So I remarked that it is enough to show that for a basis that we have constructed this the same property holds.

Well, again, let us use this, the main lemma which is about concatenation and wedge product, so this equals epsilon i j while the right hand side is, okay before I move on to the right hand side let me just, this is the same as sin tau multiplied by epsilon j i, where after all i j just consist of concatenating these two indices.

Now the thing is that and j i as sets without any order i j and j i are the same sets. So we can use a permutation to go from i j to j i that is the permutation tau. And we know that when the index set is changed the tau, the epsilon changes in this fashion. This is one of the lemmas that we have proved earlier, where tau is the permutation taking i j to j i. So this is what we get.

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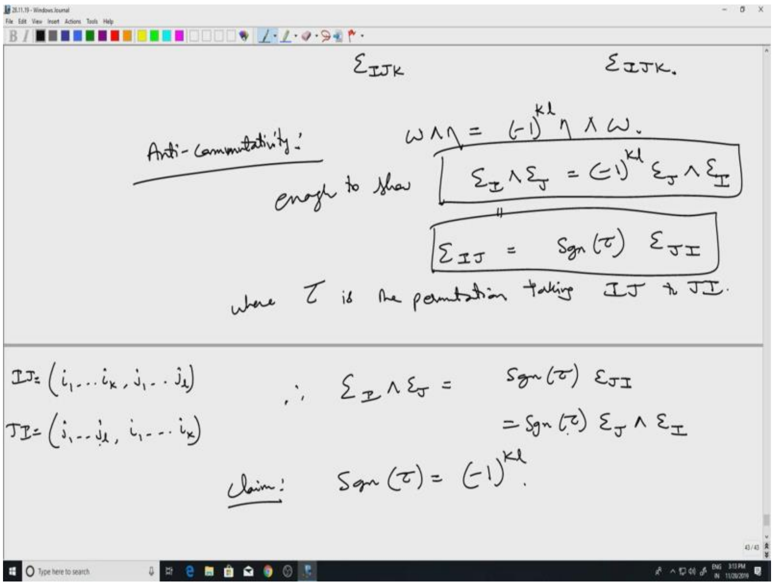

And finally, so therefore epsilon I write epsilon J equals sin tau epsilon J I, which I again split as epsilon J wedge epsilon I. Now all that remains to be shown is that sin tau equals minus 1 raise to k l. So I will not do this, this can be perhaps discussed in the tutorial, it is just a matter of communitoric.

So one would like to write the permutation tau as a product of transpositions and it turn out that they need k l transpositions to do that. So essentially, i j consist of i 1 up to i k, j 1 up to j l, this is i j and j i will be j 1, j l, i 1 i k. So in order to move from i j to j i, I have to, or rather suppose I start from j i and I moved to i j, I have to move this i 1 past all this j 1, j 2, all the way up to j l. And i 2 so first I do that I move it all the way, so move it meaning that each time I can apply a transposition, I can interchange i 1 and j l.

So j l will come here and then I interchange i 1 and j l minus 1 and so on. So keeping track of the number one gets k times l number of transpositions. Now so that completes our discussion, so let me just mention a couple of things which should not get lost in the whole process.

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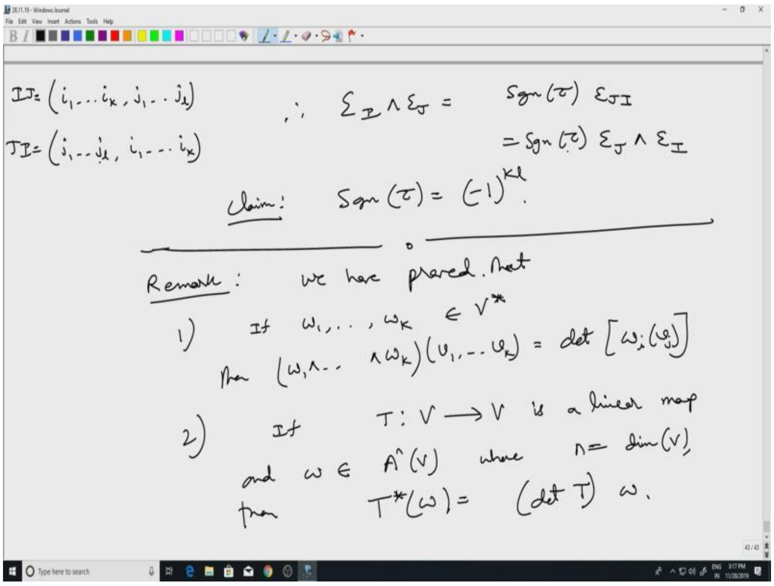

So remark, we have proved, I wrote that we have proved this but I mean we have not explicitly stated it. But this is a thing which will be useful if you start with, I have being using the notation omega 1 and omega k as for the dual basis. But let us say one just starts with any k forms, k 1 forms and I evaluate it on any k vectors then it is equal to that omega i v j that is the what one would like to say.

This is one thing and the second thing is that this is something which has been explicitly proved. If t is a linear map then and omega belongs to A n V, where n equals dimension of v then t star the pullback of omega is determinant of t times omega.

Now I proved this in the case v is r n but exactly the same proof will work for any vector space, so you fix a basis and use the same basis for writing down the, for both the domain v and the target v write down the matrix of t in that and calculate the determinant and one has this. So the same proof as in r n will work for this, now as for the first one that this equal to this, this can also this can be proved by similar methods as what we have been employing so far.

Namely, one wants to check this, the right hand side is also a multi-linear map, multi-linear form, where we have seen this multi-linear map in the context of when omega 1, omega 2, omega k are parts of a dual basis. But the fact that they came from a dual basis is irrelevant for the multi-linearity, so one can define a multi-linear map using any k1 form in this fashion. And this so in order to check that these two maps are the same, it is enough to take v 1, v 2, v k to be basis vectors. Well, even before I come to that, in order to check that, so what we are claiming here?

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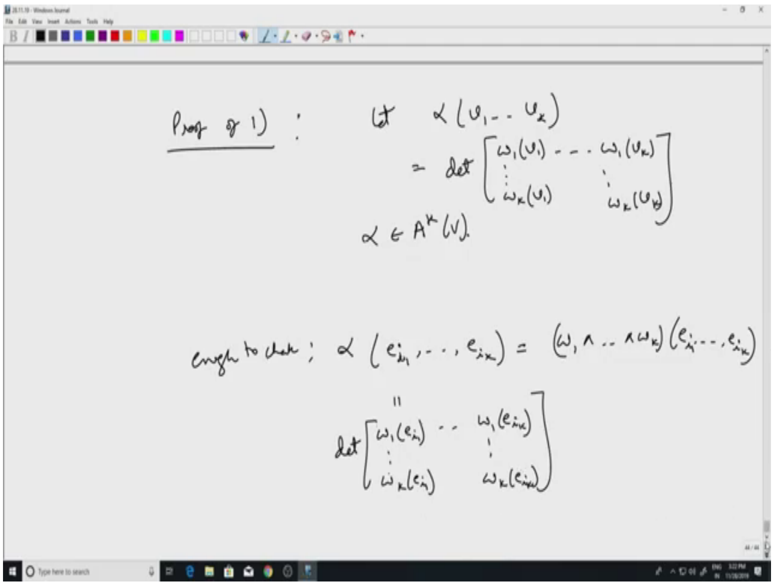

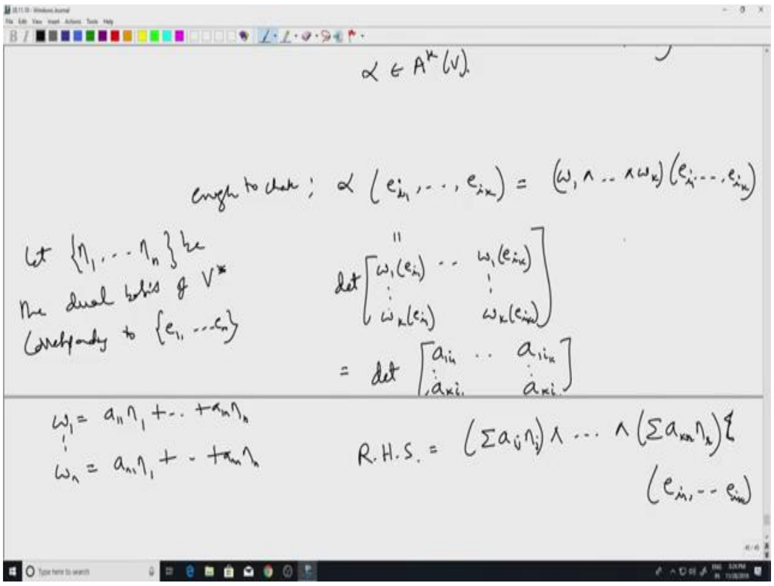

Prove of 1, I will just briefly outline it, so the right hand side is a multi-linear map. Let alpha of v 1….v k be determinant of omega 1 v 1 omega k v 1 omega 1 v k omega k v k. Then alpha is a multi-linear omega A k v.

So one is interested in showing that alpha is equal to this. So what one wants is, let us use the basis what that we have constructed A k v for, and expand alpha in terms of the basis vectors. Actually, is it needed? No, maybe not. Well, the basis at this is stage is not needed. So let just evaluate it and we know that this is alpha is multi-linear, this is multi-linear, so just evaluate it on any e i 1 as usual e i k.

It is enough to check and enough to check this equal to omega 1 omega k e i 1 e i k. Well, when we plug in this basis vectors here, so here, so when we plug in this basis vectors here, then this alpha of this just going by definition will give me determinant of omega 1 e i 1, omega k e i 1, omega 1 e i k, omega k e i k.

Now I will use the dual basis. Unfortunately, I have already used omega for this arbitrary 1-forms. So the dual basis of e i let eta 1 eta n be the dual basis of vista corresponding to e 1 of v. So this omega 1, I can write as a 1 1 eta 1 plus a 1 n eta n and omega n a n 1 eta 1 plus a n n eta n. When I evaluate omega 1 on e i 1, I will just pick up the a i 1 coefficient. So this is determinant of a 1 i 1….a and then this is a k i 1 and here I will get omega 1 i k…..omega k i k.

Now on the right hand side, I can again plug in. So this is the left hand side. On the right hand side, I can again plug in this expressions for omega 1 omega 2, etcetera. Well, what one gets is first of all, one knows that a k or let us say eta r evaluated on, whole thing evaluated on e i 1…..e i k.

Now this expansion again using multi-linearity, lots of terms will come but one notices that the same index eta 1 for example, if eta 1 occurs twice then this wedge product will be 0 and so basically, with any, the same thing you can index, so essentially all the terms here will have to involve eta i's with different indices.

And when there is a, so the only things which will survive in this sum, so there be one summation I can bring this sum outside, the only terms which will survive are, things with different eta's, eta's with different indices which is, then you will be evaluating on e i 1….e i k. So not only should they be different indices that index set which will survive will be, have to be a permutation of this i 1 up to i k. And then the actual value will be just turn out to be just the sign of that permutation, which we have seen already.

So essentially, one is reducing this the left hand side to this basis forms that we had and then we get this. So I will not go into details of this, but all the ingredients are there.

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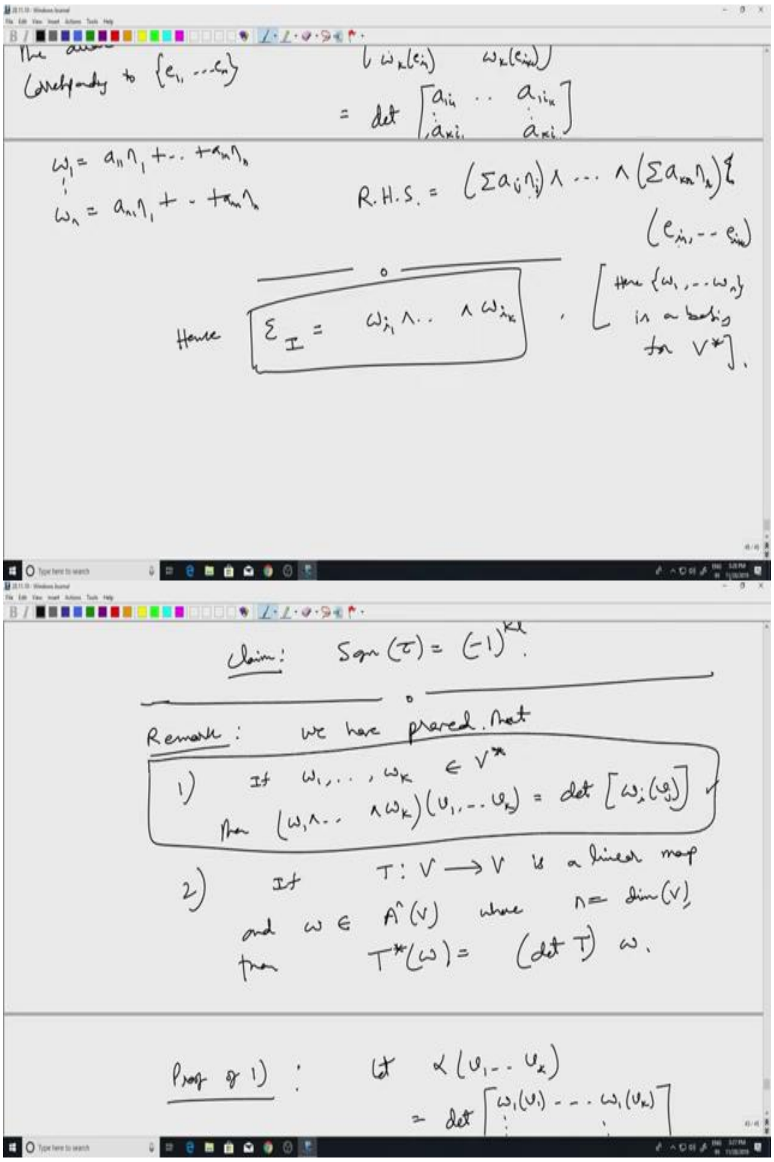

But I will remark one thing that these forms, hence these forms alpha is that I defined earlier in terms of the determinant. Because of this statement, because of this the forms alpha is I defined earlier just will be omega i 1 omega i k. This is the main thing for us. So these things form a basis for what we have shown and what we have used repeatedly in all the proofs, is that this alpha i's. However, we did not know that this alpha i is actually equal to this wedge product.

So here, oh sorry not alpha i, this is what I have been calling epsilon i, it is epsilon i. Omega 1….omega n is a basis. So the remark I made here is for any 1-forms this equation holds but what is more familiar and relevant is that, if we start with basis for 1-forms. Here we just need k of this, but if we start up with a full basis omega 1 up to omega n v star and in that setting, we had come up with this epsilon i k-forms, which we call epsilon i corresponding to any index set i.

And with this remark we see that this epsilon i is same as omega i 1 omega i k. So this is something that I will repeatedly use later on. And that pretty much concludes my discussion of constructions on finite dimensional vector spaces involving multi-linear forms. Now let us return back to the setting of manifolds and let us try to define the objects there.

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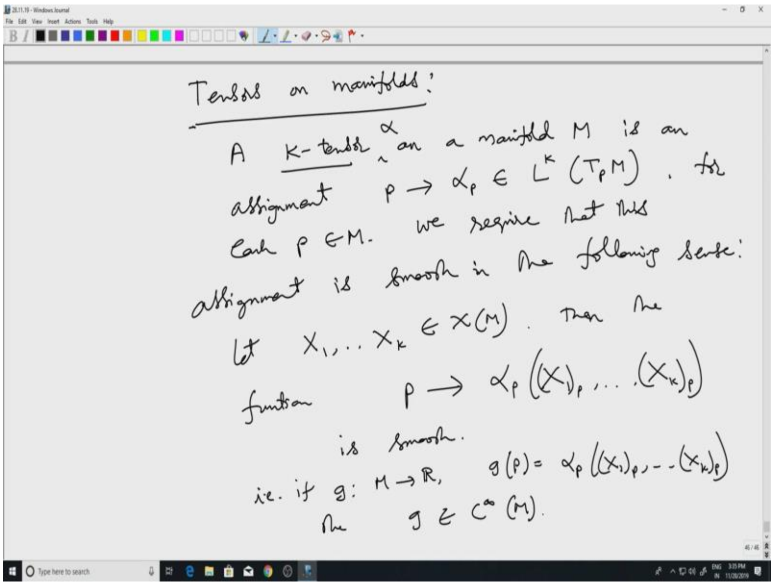

So tensors and, let me just say tensors on manifolds. So what we need is on each tangents given a manifold at each point we have the tangent space. And we know that once we have a finite, once we do not even need finite dimensionality, once we have a vector space, we can talk about multi-linear forms of that which we called tensors. So when we given a manifold at each point, we have a, if we have a multi-linear map, we call it a tensor on the manifold.

And as usual, we would like to demand some kind of smoothness that as you change from point to point, this multi-linear map should change in a smooth way. And to quantify that smoothness, we use a trick which we had used when talking about smooth vector fields. We will see its action on something which is already smooth. And then, we want to, the net result should be smooth again. So let us say a k-tensor, on a manifold k-tensor let us say T on a manifold, no T is not the good symbol, let us take it as alpha on manifold M, is an assignment.

So I am defining it exactly the way I defined a vector field. In a vector field, to each point I associated a tangent vector. Now I am going to associate to each point in the manifold an element of L k T P M. For each P in M, I will pick out that particular tangent space, a multi-linear map. And we require that this assignment is smooth in the following sense: Well, as I said the idea is to act, so this is after all a map, a multi-linear map and so it is going to act on vectors and give me a number, but I can also since it is there is a multi-linear map at each point on the manifold, I can act it on k vector fields, smooth vector fields.

Let X 1, X k be smooth vector fields, then the function P going to alpha P evaluated on X 1 at P, X k at P is smooth. So this is a function on the manifold. So if I define g from M to R, g of P is alpha P X 1 P, X k P, i.e., if g is this then g belongs to C infinity M. This is the requirement and this should hold for any choice of vector fields X 1 up to X k. Then we say that it is smooth.

Then we say that this K-tensor alpha is smooth. Well and remember that in the case of vector fields, we had a more workable definition of smoothness in terms of local coordinates or charts. Here too, one can write this in terms of charts and I will come to that.

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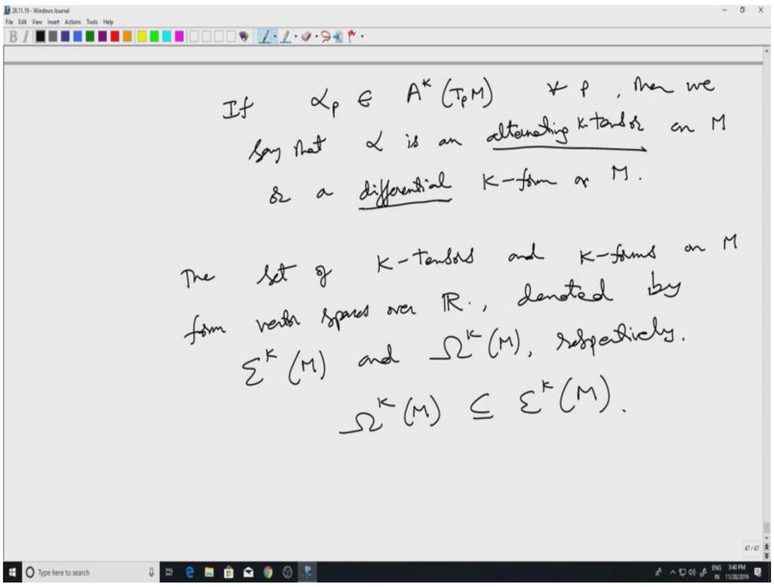

Before I do that, let me, if alpha P belongs to A P, no A k T P M for all P, then we say that alpha is in alternating tensor on M or alternating K-tensor on M or differential K-form on M.

The unlike the vector space context, so now we adding this adjective or just I mean it is not even an adjective, it is just a, for historical reasons one says differential forms when talking about. What would be just called an alternating K-tensor and alternating K-form on a vector space is now called a differential K-form on a manifold. As far as a tensor is concerned, there you do not say differential, it is just say K-tensor on the manifold.

So, now that these objects are defined, I will come to the coordinate description shortly. But before that this let us give it some names, these things. The set of K-tensors and K-forms I will not say differential, it is understood when I say K-form, it is understood that it is a differential K-form; in other words, it is an alternating K-tensor. Both these sets, the set of K-tensors and K-forms on M form vector spaces over R.

So we can add two K-tensors just by adding them at every point, and likewise for K-forms, denoted by let us say Epsilon K M and omega K M respectively. The notation for the space of K-tensors is not standard, some text use this but this omega capita omega K M for the space of forms is pretty standard. So, I will use this and of course, omega K M is a subset of epsilon K M. Alternating K-tensors are just special kind of tensors.

Alright. So we will stop here. Next I want to talk about, these are just the objects then we have these operations, the tensor product and the wedge product, both of them carry over to manifolds without any difficulties or certainties. The additional element which was not there in a vector space will be this exterior differentiation operator on forms. So, I will talk about that in my subsequent lectures. Thank you.