An Introduction to Smooth Manifolds
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Lecture 55: Alternating Tensors 7

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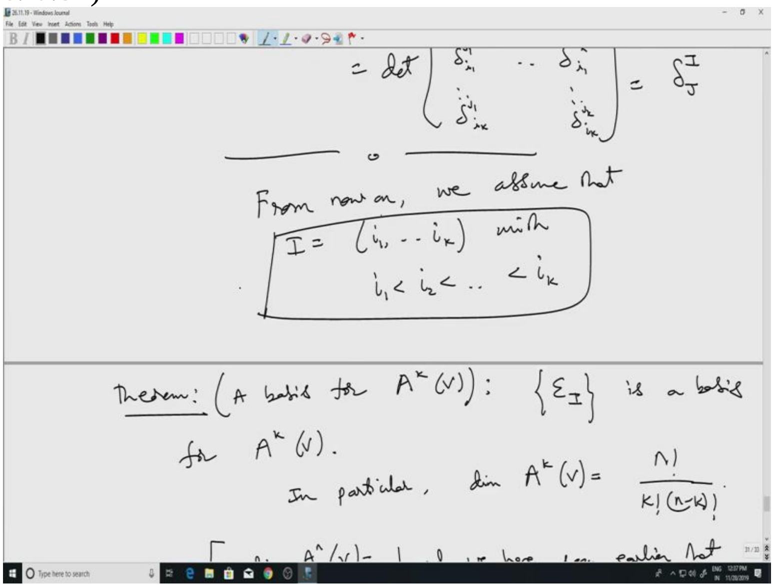

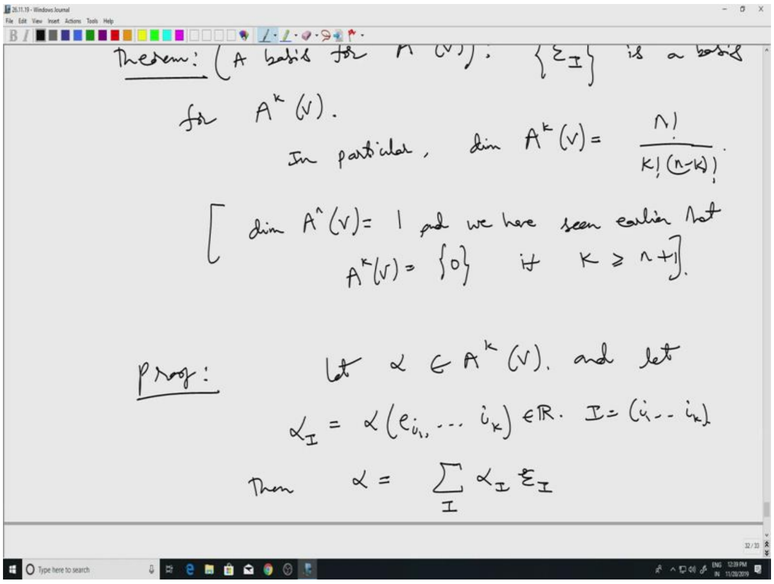

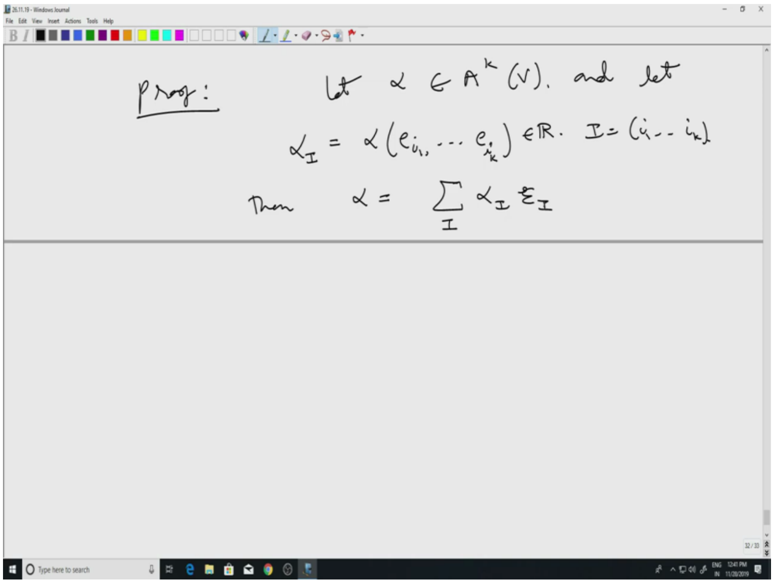

Welcome to the 55th lecture in this series, so let us continue our discussion of the wedge product. Last time, we constructed a special corresponding to each multi-index. We constructed a K-form, multi-index of length K will give rise to a K-form, and of course, as usual we have a fixed basis for 1-forms in the background, with so, with that basis and a multi-index I come up with this epsilon I and the claim is that now, I am going to restrict to multi indices, where all the entries are ordered, they have to be in strictly increasing fashion.

And if I confine myself to such multi-indices, and look at the corresponding K-forms, that forms a basis of A k V, and that immediately gives us the dimension of A k V to be n choose k, n factorial by k factorial times n minus k factorial. So, I was in the process of proving that these special K-forms actually span A k V, so I started with an arbitrary K-form and I came up with these numbers, where I just evaluate, which are obtained by just evaluating the K form on e i 1, e i k for a multi-index I, again multi-index of that type strictly increasing.

With these numbers the claim is that alpha equal to alpha i epsilon i and the reason why this is the case, is well, if you want to see this, that this is true as we have seen many times in order to check the 2 K tensors are equal, we just have to evaluate on, evaluate both tensors on basis vectors by multi-linearity and that is what we will do.

Well, so we just have to check that alpha equal, and alpha and the right hand side are equal, when evaluated on, oops, here I should, there is a small typo, let me correct this, e i k, so I just have to check that these things both sides are the same. Give me the same number when evaluated on basis vectors, so let us do that. In fact, what I claim is that, that is what I said is true for any tensor, K tensor but when you have an alternating tensor, it is enough to evaluate them not just on any basis vectors, on a sequence of, finite sequence of basis vectors where with this property that the indices are increasing.

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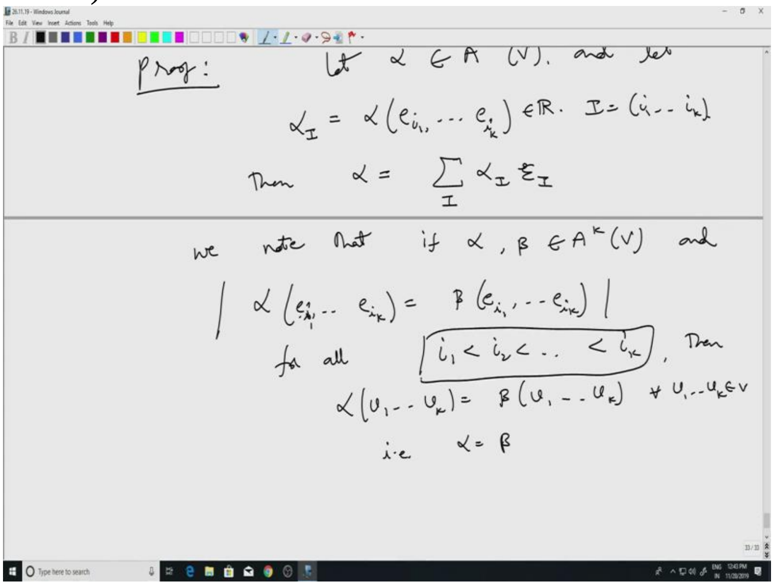

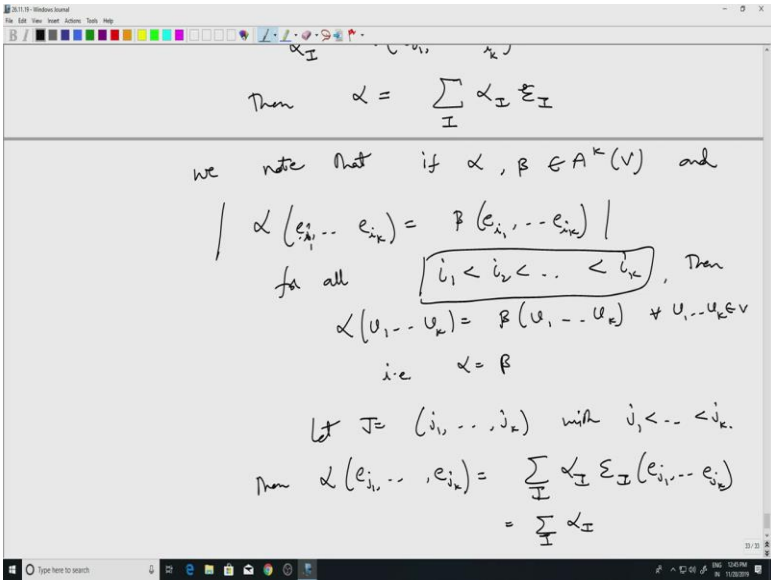

We note that, let me write it as a small fact, we note that if alpha and beta are in A k V, and alpha e i 1, i k equal to beta, oops, here that should be i 1 beta e i1, e i k for all i 1 less than i 2, less than i k then, alpha v 1, v k is equal to beta v 1, v k that is alpha is equal to beta.

So again, this is something which follows immediately from the definition of the, an alternating form. So first of all, one notices that, if any two indices i 1, i P and i R are the same, then both sides will be 0 by the alternating property. So one can assume that i 1 up to i k, they are all distinct and if they are not in an increasing order, you can interchange, keep on interchanging the indices so that they get into an increasing order, and the number of, each time you do an interchange, you will acquire a minus sign but that minus sign will occur for both left hand side here and the right hand side here.

So ultimately, you can cancel the, so you can cancel the negative sign and you keep on doing it till you put them in an increasing order, so that this property we can assume this. So it is enough to check for sequence of basis vectors with this indices ordered like this and when I have such as index sequence, let us, so let us start with something like that, let J equal to j 1, j k, with j 1 less than j k as usual, as will be the case for everything that we work with, so then alpha e j 1, alpha e j k equal to I epsilon I e j 1, e j k and this is the same as alpha I.

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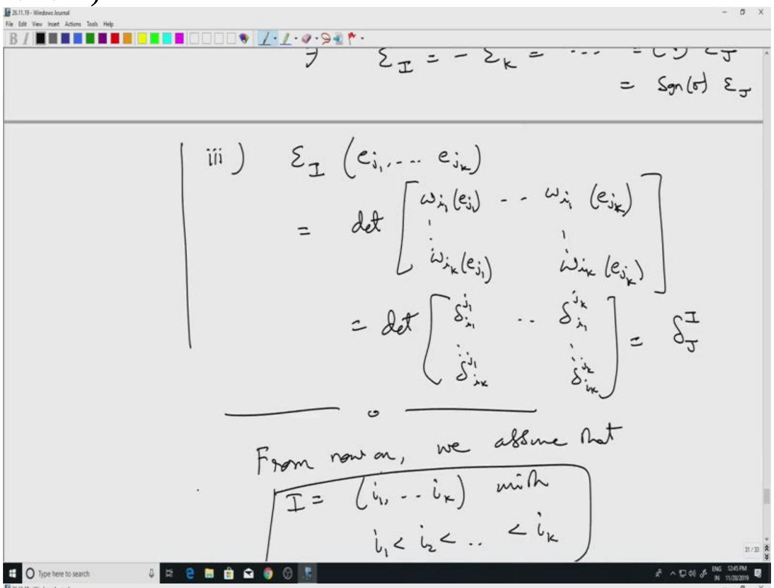

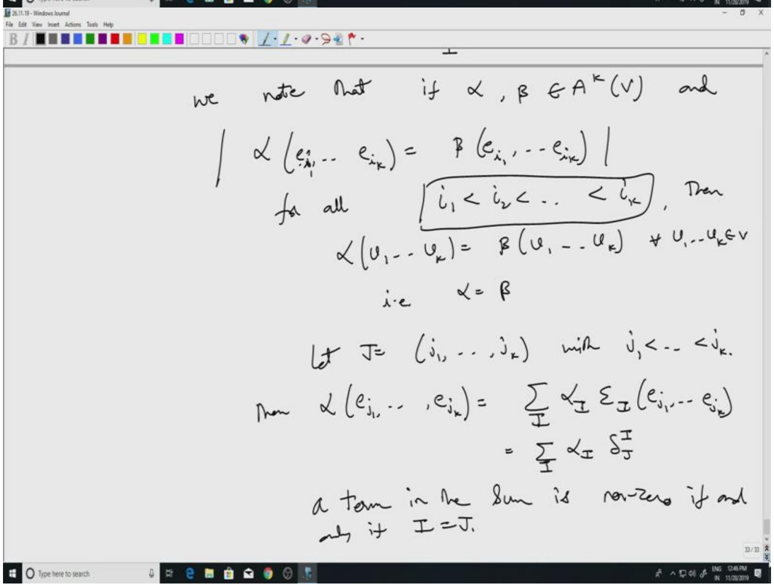

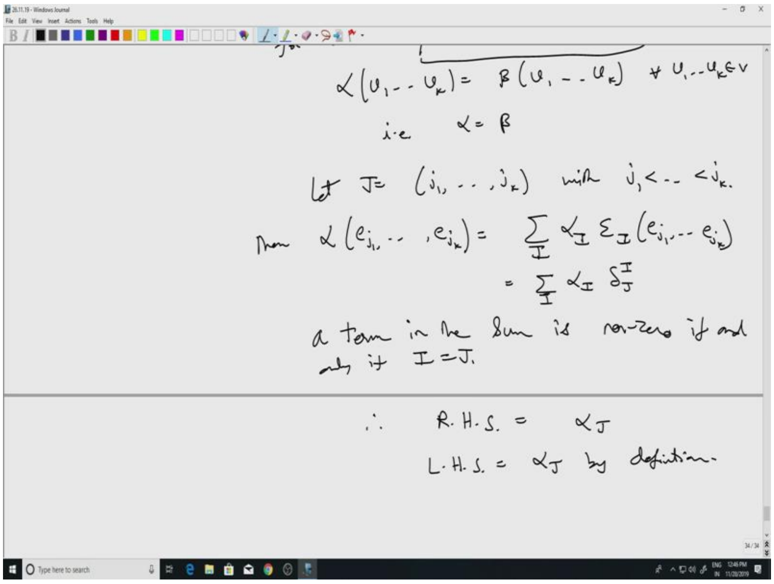

And we have seen already small lemma that we had, here, tells us that this is the same as delta, the multi-index Kronecker symbol I J. Now this delta I J will be 0 unless I is a permutation of J or the other way around, but since they are increasing the I and J are both increasing indices, to say that one is a permutation of the other the only possible permutation which will keep the increasing property is the identity. So, a term in the sum is non-zero if and only if I equal to J. So, the permutation involved will have to be just the identity permutation.

So in which case, therefore, the right hand side RHS equal to alpha J and the left hand side equal to alpha J by definition, so both sides are equal and this is true for any J, of course J with this increasing property. So we are done, so the two forms, so this forms will span the space of all k linear forms.

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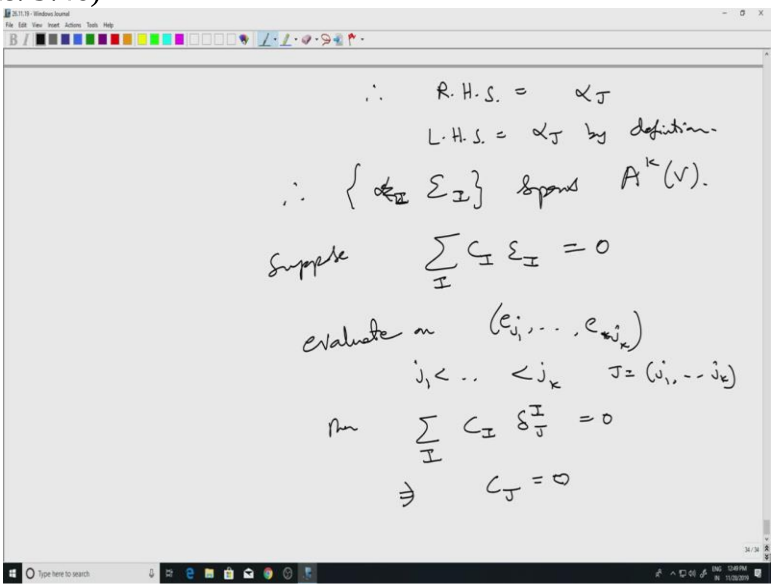

Now, as per linear independence it is pretty much the same argument, so, what we have shown is that alpha I, no, epsilon I spans A k V. As per linear independence, suppose we start with a linear combination of this epsilon I. Now there will be some coefficients involved here which I will call it again by multi-index C I, summation over all I, this equal to 0. I would like to prove that all the C I are 0, we do the same thing that we did here, evaluate on e j 1, e j k, j 1 is less than j k, let us call this as multi-index J.

If I evaluate on this, then the same logic gives us this is delta I J equal 0 which implies the only term which will survive is when I equal to J, so I get C J equal to 0. So in other words, for any multi-index with j 1 less than j k, I get the corresponding coefficient to be 0, so therefore all these CI's are all 0 and we have proved that they are linearly independent.

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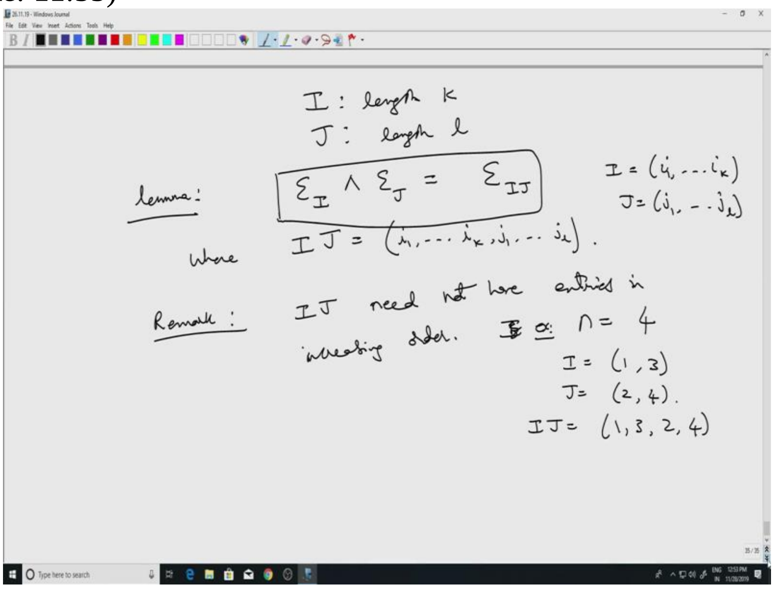

Now let us move on to somewhat slightly more involved lemma, the statement is simple enough, and this will be the crucial ingredient which will help us prove the associativity property. So, the lemma states that, so let us start with K and L, so I length K, J is a multi-index of length l, so I want to claim that, so this is, let lemma below this epsilon I wedge epsilon K, epsilon J equals epsilon I J, where I J is the multi-index which is obtained by concatenating I and J.

In other words, you just write it as, so if I equal to i 1 all the way up to i k and J equals j 1 up to j l, so you just write it side by side so i 1, i k, j 1, j l. Now here I have to mention that the property, see even if I had been written in an increasing order and likewise J, obviously that property might be destroyed when I do the concatenation, so that is one thing to observe. The other thing is that, this what we are trying to prove here, why do we even need a proof?

It is just that the way we have defined this epsilon, is in terms of determinants not wedge products, the whole point is that, this crucially links the wedge product with this the determinant operation, and therefore it does require some proof. Now, so as I was saying this remark I J need not have entries in increasing order. So for example, if I take, let us take n equals 4, and if I take I equals 1, 3 and J equals 2, 4 then I J would be 1, 3, 2, 4. So one has to keep this in mind.

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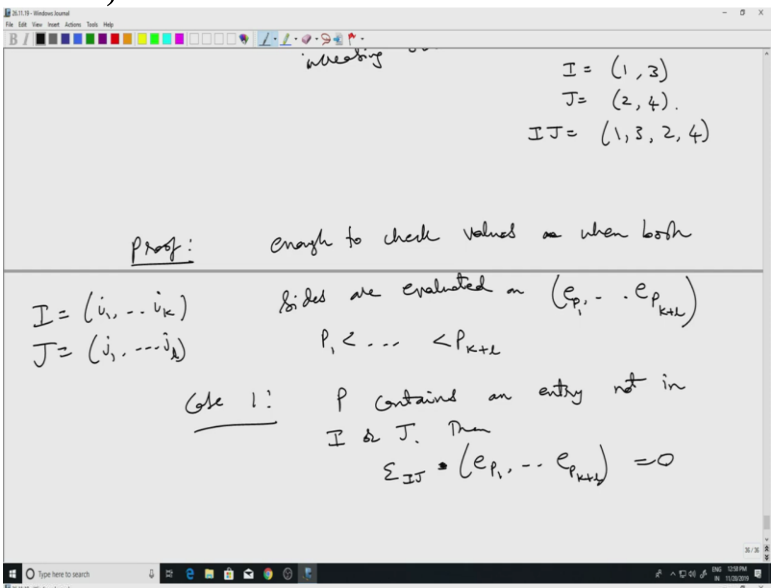

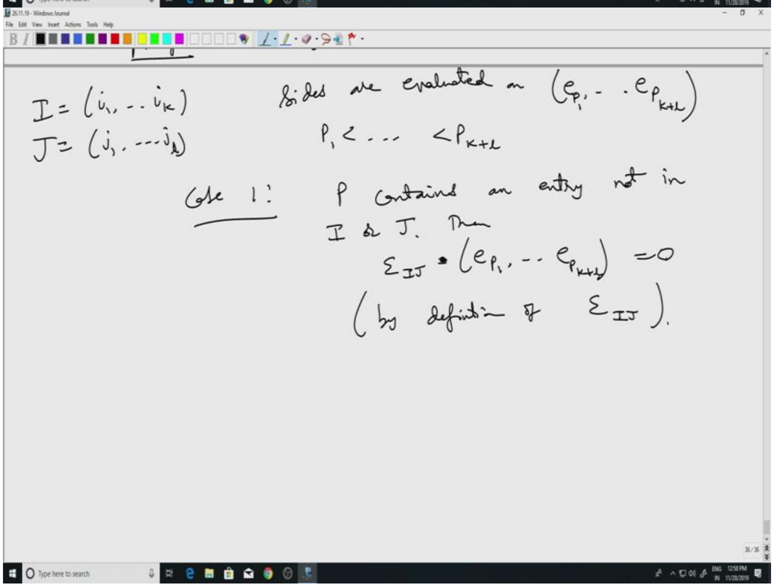

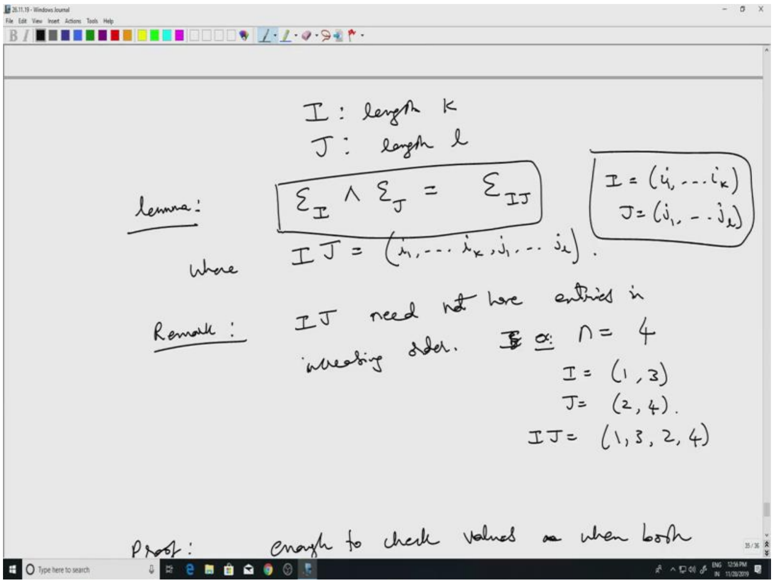

Now, so let us prove this, and we look at, so we want to, as usual, see this is a k plus, both sides in this lemma are k plus l-forms, it is enough to evaluate this on k plus l basis vectors, see that we get the same thing on both sides. So, enough to check values on when both sides are evaluated on e 1 all the way up to, rather not e 1, e p 1, e p k plus 1, but of course I can assume that these p 1 all the way up to p k plus l they are in an increasing order.

Since that is the comment that we made last time in the previous statement, that if we have any two k forms and if you want to check that they are the same, it is enough to evaluate them on k basis vectors where the index, indices can be assumed to be increasing. So, same thing here as well so, I am checking that these two forms are the same. So I can assume that the p 1 less than p 2 etcetera, so p 1 less than p k plus 1.

So now, let us look at some the possibilities. First of all, P contains an index so, the first possibility is P contains an index, an entry or an index not in I or J, so remember that I and certain entry so, we are saying that this one of these. So, let me write that again, I equals i 1, i k and J equals j 1 and j l, so one of these P R let us say, will not be, will not occur either in I or J. If that happens then, this side epsilon I J will be, then the right hand side will be 0.

Just by definition, then epsilon I J evaluated on e p 1, e p plus k plus l equal to 0, because well, this is the determinant of the 1-forms coming from I and 1-forms coming from J evaluated on these factors and since we are working with dual basis, one gets this immediately. So, this is by definition of epsilon I J, and that is the right hand side. Now what about the left hand side? Well, left hand the left hand side is a wedge product.

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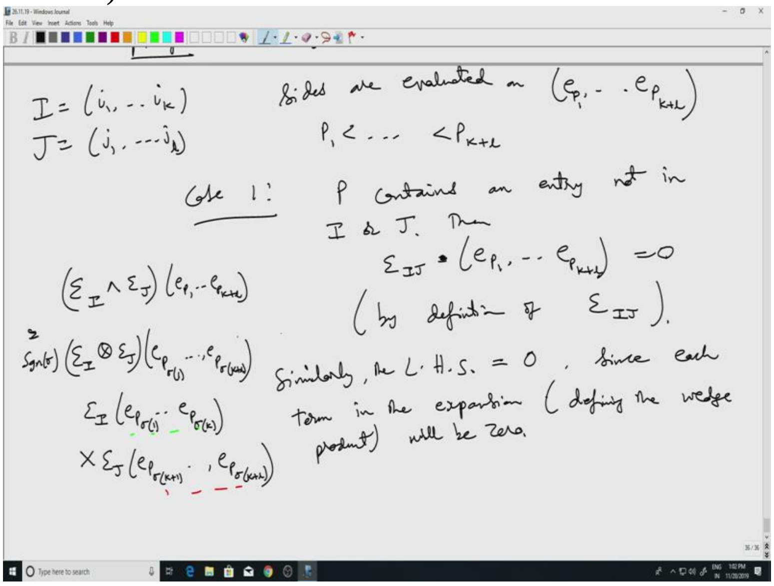

Similarly, LHS is equal to 0, since each term in the expansion defining the wedge product will be 0, so I am looking at this e p 1 etcetera, e p k plus l. This will be, this will have the wedge product is by definition, you will be looking at things like this, epsilon I tensor epsilon J evaluated on e sigma p 1, e sigma p k plus l. Actually, no, I should write it rather like this, so not what one is permuting, is e p sigma 1, e p sigma k plus l, this will be a typical and then there is of course sin sigma.

So the expansion will consist of, will have terms of this form and the tensor product by definition this will be epsilon I acting on e p sigma 1 all the way up to e sigma p k multiplied by epsilon J sigma k plus 1, p sigma k plus l.

Now, the point is that if these, if this index set, if the entry, if the numbers p 1 up to p k plus l had something which was not in i 1 up to i k or j 1 up to j l, even after you do permute them in anyway, the new index set will also continue to have some number which is does not occur as part of I or J. Therefore, that extra number which was in the index, which was an entry for P will occur either amongst these or amongst these.

So whenever that occurs, then the corresponding epsilon I evaluated on that will be 0. Similarly, this epsilon I or epsilon J will be 0, one of these two terms will be 0 and therefore the whole sum is 0. So if this happens, then one can assume that there are no X terms which are not already in I or J.

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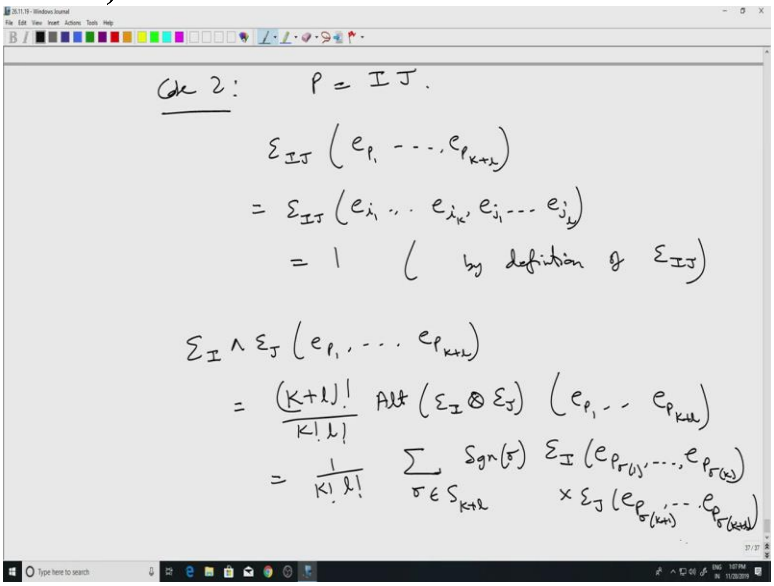

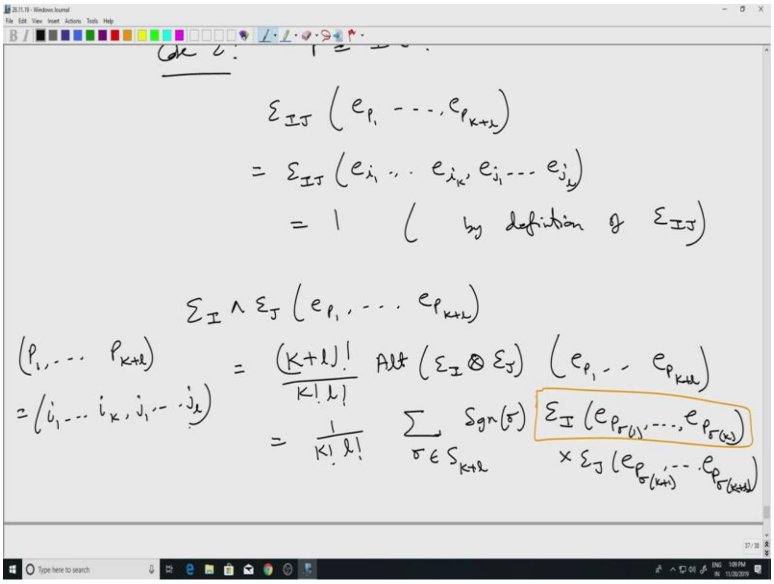

Case 2 is that, well, that still leaves us quite a few possibilities namely, they can actually involve permutations of the entries of I and J but let us assume in fact that a simple case namely, P is actually equal to I J. So, P equals I J, in this case so, one is still interested, so let us evaluate the right hand side. Now, if I evaluate on, so epsilon I J evaluated on e p 1 all the way up to e p k plus l is the same thing as epsilon I J.

So, it is e i 1, since this index that is a concatenation e i k then I will start with e j1, e j l, now this is equal to 1 again by the definition. Go back to the determinant definition of the epsilon forms and then immediately it is clear that this is equal to 1, what about the left hand side? The left hand side is what requires a bit of work, so let us look at the left hand side, this acting on e p 1, e p k plus l is by definition, this is K plus factorial, K factorial, L factorial, Alt of E I tensor E J evaluated on e p 1, e p k plus l.

And Alt already has a 1 over k plus n factorial so, 1 over this, then use the definition of Alt. So sigma (belongs) belonging to S k plus l, sin sigma then here I have epsilon I acting on e p sigma 1, e p sigma k multiplied by, so I have already used the definition of the tensor product when writing this step, multiplied by epsilon J e p sigma k plus 1.

Oops, I forgot the p, p sigma k plus 1 and then I have e the last term k plus l brackets here and here, so again we notice that in this each term, it is essentially a product of two things. Here the only way this is nonzero is that this P sigma 1, P sigma k etcetera belong, is just as I said it should be the same as the entries of I. So in other words, the sigma 1, sigma k should be just permutations of, by the way, so this by definition since, it is a concatenation, let us keep in mind that, the entries of, so I should mention that P1, P k plus l, since P we have assumed that P equals I J, this is the same as i 1, i k, j 1, j l.

So, what I want to say here is that this P sigma 1, etcetera P sigma k, should be nothing but a permutation of i 1 all the way up to i k. So we will stop here, and I will complete the proof of this crucial step next time. Once we have this, everything else is more or less straightforward. We will stop here, bye.