An Introduction to Smooth Manifolds
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Lecture 54- Alternating Tensors – VI

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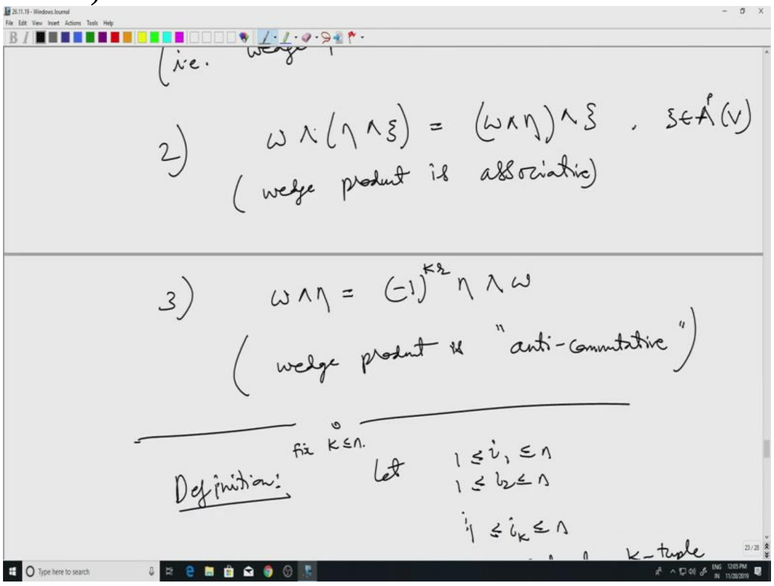

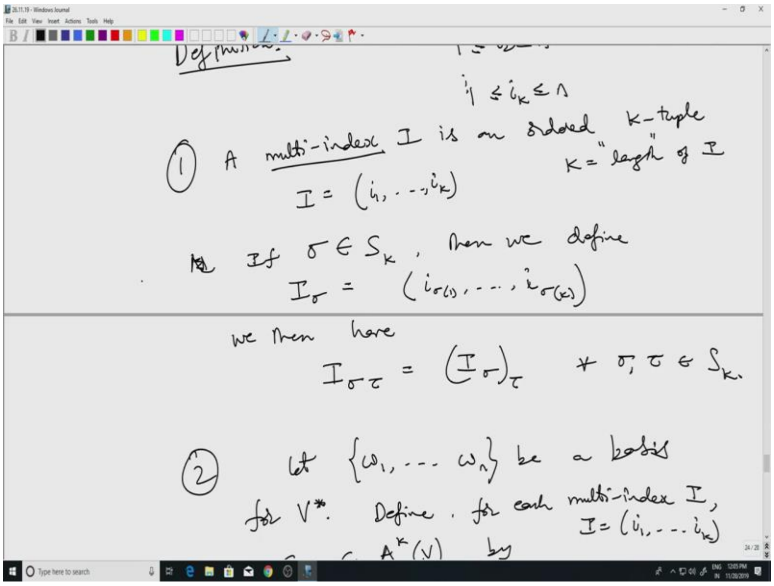

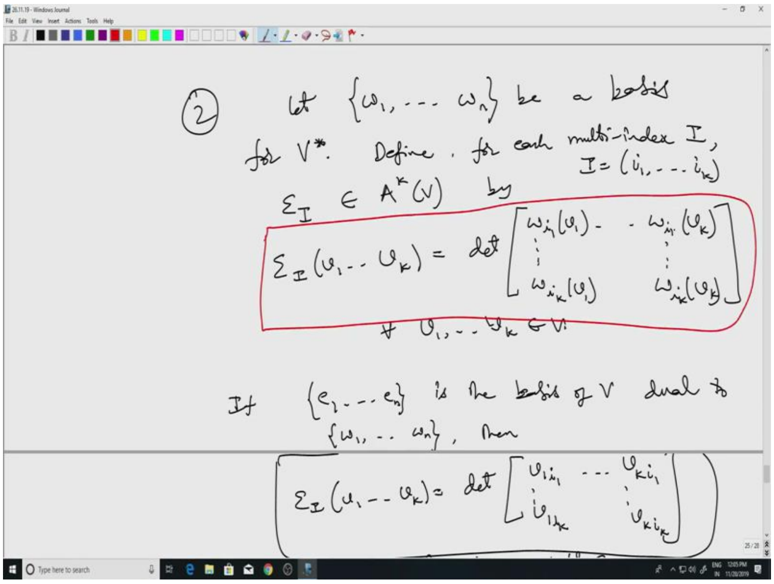

Welcome to the 54th lecture in this series. Last time I was in the process of proving some basic properties of the wedge product or the exterior product, namely, the associativity of the wedge product and the anti-commutativity. To do this, I introduced the notion of a multi-index, then corresponding to each multi-index, we obtained a, and yeah, well, we have to fix a basis of the dual space, we fix a basis of the dual space, then corresponding to each multi-index, we obtain a k-form in this fashion. So, this is the definition we are working with.

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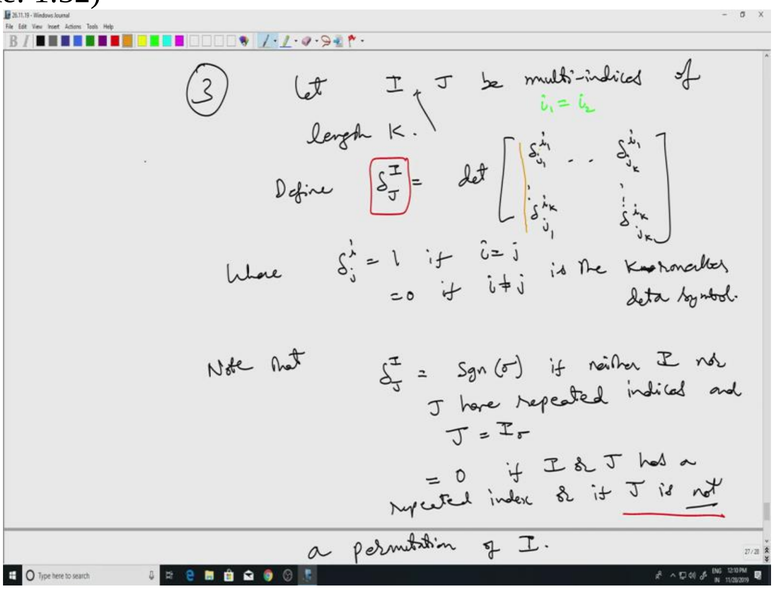

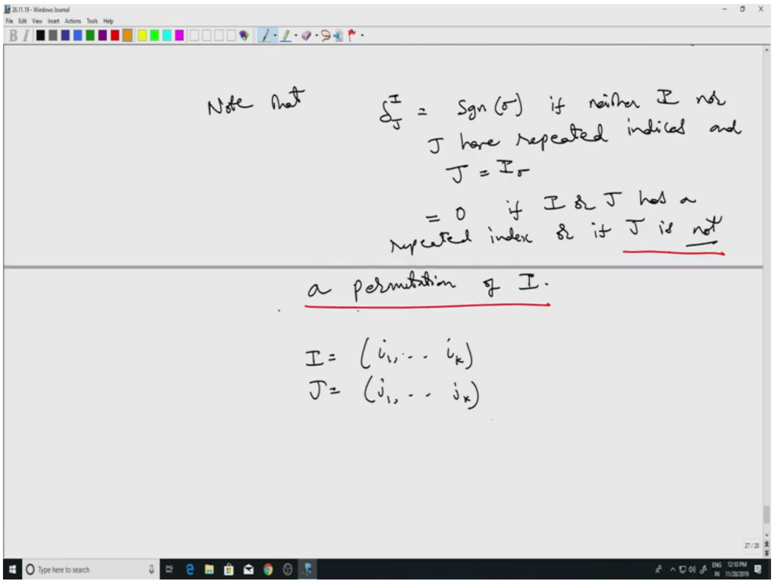

And then we, and then also introduced this new notation, the Kronecker, Kronecker Delta symbol, where for multi-indices. So I have two multi-indices, I and J, then I define this one delta I J.

Then the, I said that this number delta I J is actually just given by the sign of sigma if neither of them, I or J have repeated indices and if J equal to I sigma otherwise it is 0. And let us quickly see why this the case, as I was saying last time this can be seen just by using the definition of the determinant, one just have to expand it. Well, I will come to that, but first let us see what happens if one thing is clear, if I or J, so in the, has a repeated index, so in other words, as you remember that I was i 1, i k; J is j 1, j k where all these entries of this k-tuple, numbers are lying between 1 and n.

Now when I say that I has a repeated index, I mean that one of these entries in I is equal to the other one, for instance, i 1 could be equal to i 2, et cetera. If that happens, going back to the definition of this Kronecker delta, if for example, if i 1 equals i 2, i 1 equals i 2, then the first column here will be the exactly, first row here, would be the same as the second row in this matrix. So, therefore, the determinant would be 0. Similarly, and this I just took i 1 equals i 2 but it same logic works for any two of the entries of i.

We will get two rows equal, similarly, if some of the entries of J are equal, two of the entries then you will get two identical columns in this matrix and therefore, the determinant would be 0 again. So it is clear that, this is equal to 0 if I or J has a repeated index. And it is also clear that if J is not, so let us look at this thing, J is not a permutation of I. If J is not a permutation of I then what it means is that, some entry here in the of, in J does not occur in the, as an entry for I.

So in which case, for instance, let us say j 1 does not occur, is not equal to any of the entries of I. If that happens, then the entire first column is 0 here in this matrix. Let me use this orange color, so all these entries would be 0 if j 1 does not occur as an entry in I. So then again the determinant would be 0. So the only way it could be non-zero is if J is just a permutation of the entries of I. In that case, one has to check that the value one obtains it might, is actually equal to sign sigma and that as I said can be seen just by looking at the determinant expansion for example, so let us see that.

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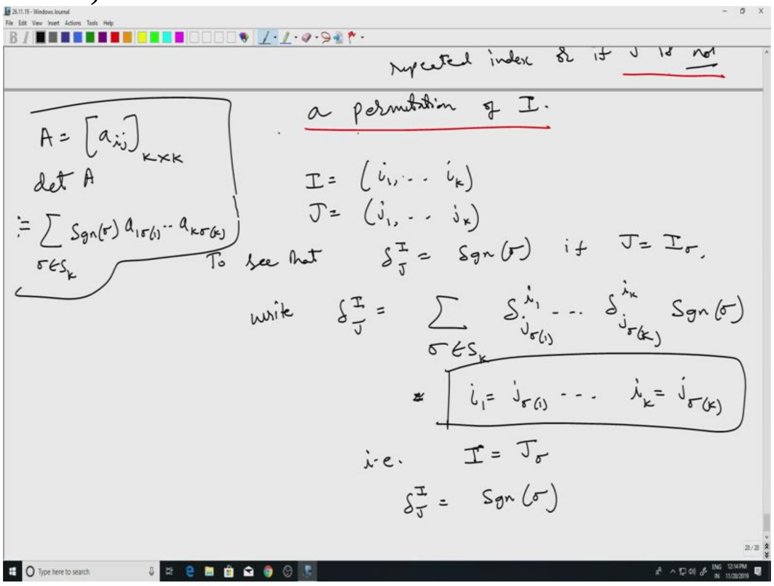

To see that delta I J equal to sign sigma if J equal to I sigma, write delta I J equal to, well, it is a determinant, so let us recall that the, if I have a matrix A equal to a i j, determinant of A by definition is, let us say this is k cross k matrix just like what we have here. Sign sigma, sigma in S k, a 1 sigma 1, a k sigma k. This is in fact just the definition of the determinant in for any k. So this is the definition of that A.

So if we apply this, we will get, this is sigma in S k delta i 1 j sigma i 1, delta i k j sigma i k, then of course I have this sign, sign sigma. And then, the only way any term, it has k factorial terms in this summation, only way any term will be non-zero is all of these deltas should be 1. That will happen only if i 1 equal to j sigma 1, i k equal to j sigma, oops, this is not i k, this is just k. So this is essentially saying i.e. the I equals J sigma.

So I have just started with index set of, index set given multi-index J and then acted on by sigma, so the only term which will survive is the one given by I equal to J sigma. And besides, there is only one permutation which essentially does this job, this can happen only for one permutation. And so therefore, what I will get is, that term will be 1 and corresponding to that, I will be, I will get sign sigma. So delta I J equal to sign sigma, where sigma is given by I equal to J sigma.

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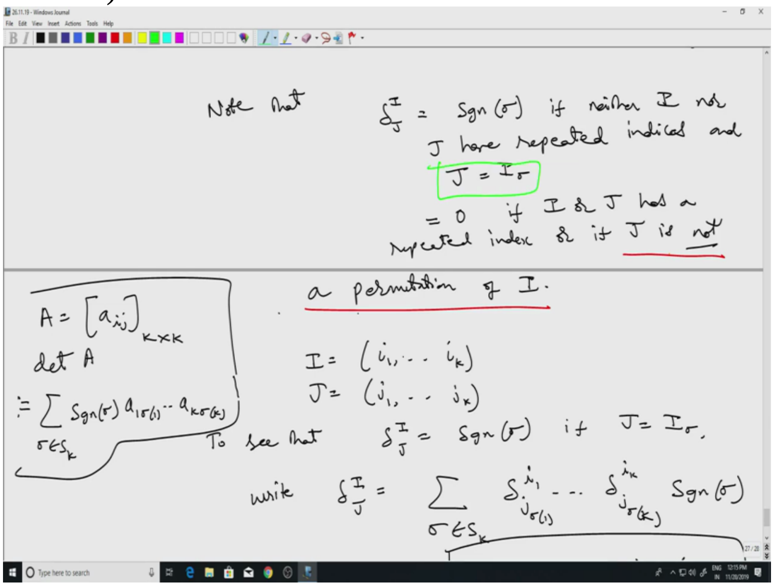

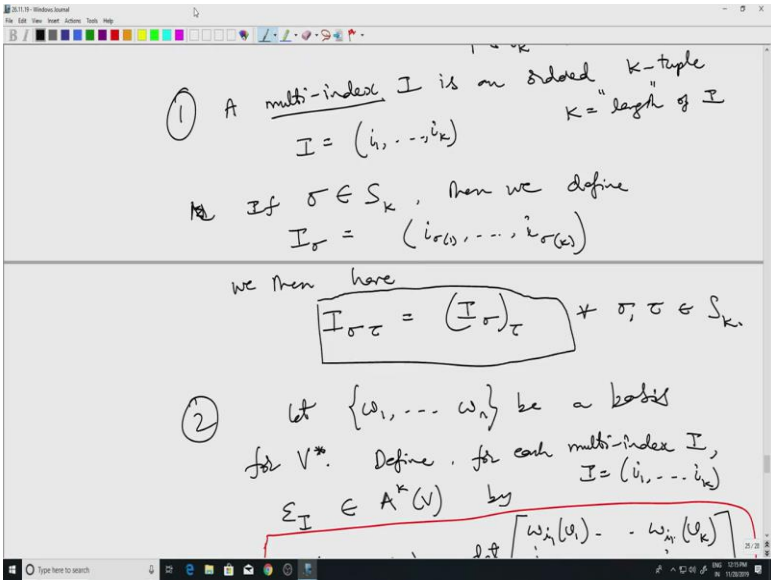

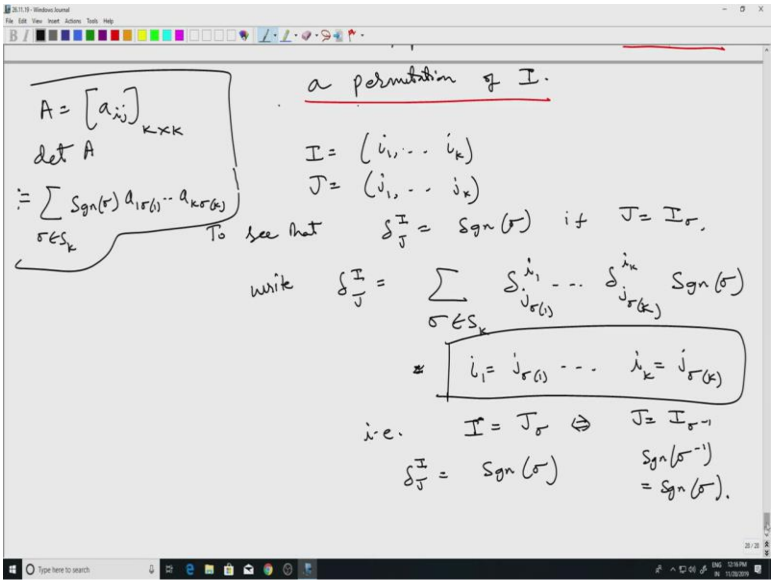

It looks slightly different from what I had here, because here I had said, it is sign sigma if J equal to I sigma rather than I equal to J sigma. But it is the same thing because, this is the same thing as saying J equal to I sigma inverse and that is because of this property that what I had here that, this property.

This will because of this, to say that I equal to J sigma is the same as saying that J equal to I sigma inverse. That property and the fact that the identity does not do anything, identity permutation does not change anything. So, J equal to I sigma inverse and sign of sigma inverse is the same as sign of sigma. And the sign of sigma inverse being the same as sign of sigma in turn reduces to saying that if sigma is written as a product of transpositions, sigma inverse will be, we can obtain a transposition decomposition of sigma inverse using a transposition decomposition of sigma and noting that the inverse of a transposition is itself.

So all these facts can be discussed in the tutorial, I will not have time to go into details at, through all the steps, so I try to cover as much as I can. So, I get this, okay, write this.

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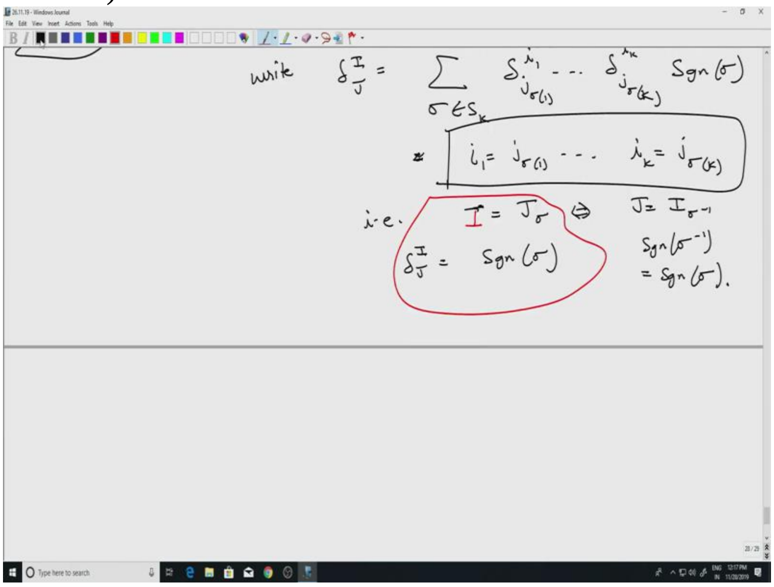

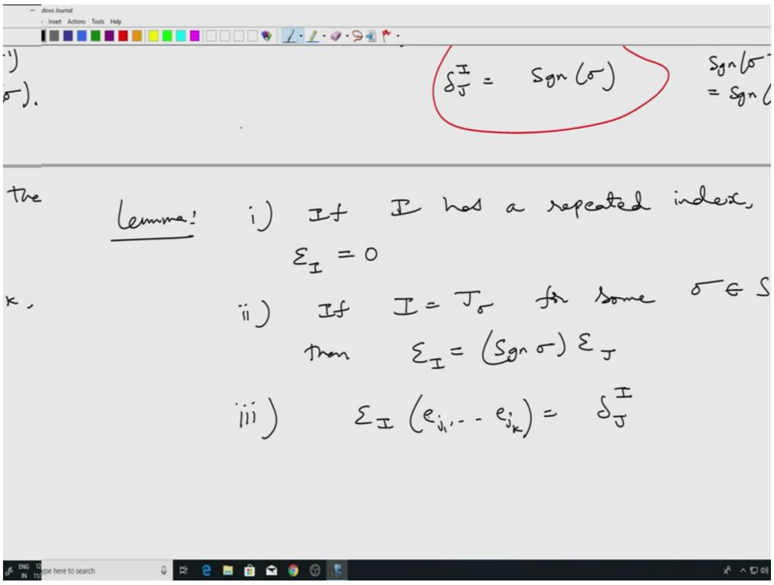

And then the main point here is this one. Not quite this. The main point is this. Whether you write I equal to J sigma or J equal to I sigma inverse, anyway you will get the same thing. So it is this. Now with these preliminaries in hand, let us move on to differential forms, well, yeah. Now we will move on to, back to so far we had been just talking about multi-indices in this part 3.

So now, let us take it as a lemma. If I has a repeated index, in other words, 2 entries of I are the same, then the corresponding k-form is just the 0-form, that is one claim. The other thing is if I equal to J sigma for some J in, no sorry, sigma in, oops, S k, then the k-forms associated to I and J are related by this equation, epsilon I equals sign sigma epsilon J.

And the third thing is epsilon I acting on e i, e j 1, et cetera, oops. And also this line should be erased here, okay. So fine, here I will write e j 1, e j k is this was the point of introducing this Kronecker delta for multi-indices. So epsilon I acting on e j1, e j k is delta I J where of course, J is the same j 1, j 2, j k which occur inside these brackets here.

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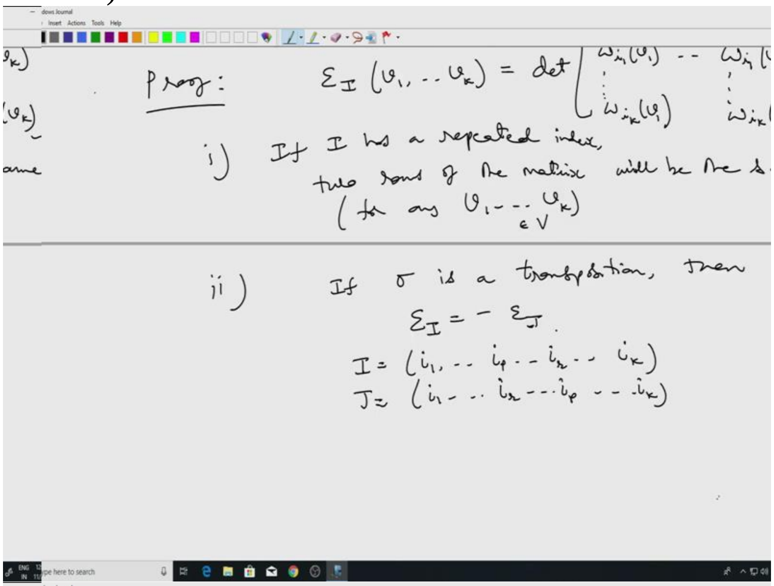

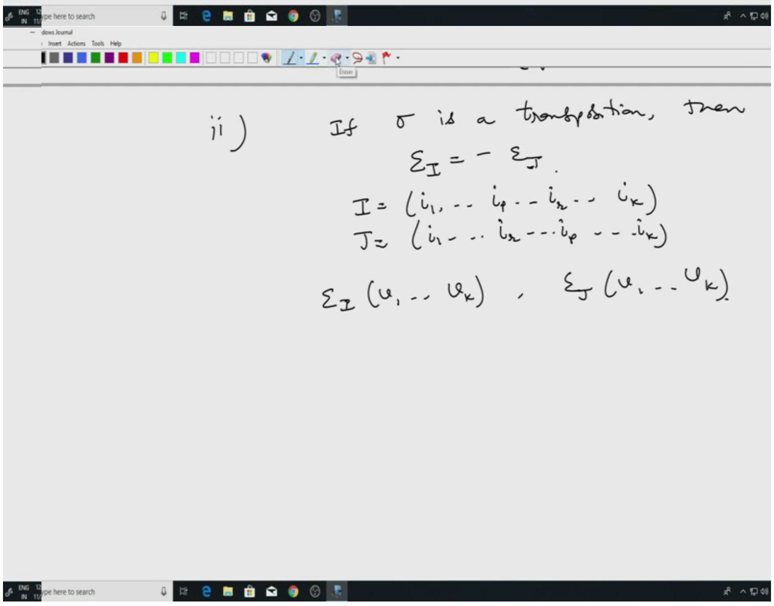

So let us see why this is the case. Proof. To see the, to see this, epsilon I is 0, if I has a repeated index, just go by the definition of epsilon I. This is determinant of omega i 1 v 1, omega i k v k, etcetera, omega i 1 v k, oops, this is not v k here, this is still v 1, omega i k v k. Now, if I has a repeated index, then as we saw earlier what happens is that two rows, if I has a repeated index, two rows of the matrix, this matrix of course, if I have two rows of the matrix, will be the same.

So, for instance, if i 1 equal to i 2, then the first and second row will be the same, so then the determinant will be 0 and this is independent of what the v i's are. For any v 1, v 2, v k, this is true. For any v 1, v 2, v k in V. So therefore, that proves, that takes care of 1.

Now the second one, if I equal to J sigma for some sigma, so what we can do is let us say, if sigma is a transposition, then let us observe that epsilon I equal to minus epsilon J and why is that? Well, if sigma is a transposition, then essentially it is just interchanging two letters in between 1 and k, two numbers between 1 and k and leaving the other numbers fixed. So in other words, the index at I will be the same, will differ from the index at J only in two slots.

So for example, I can have something like this, i 1, i p, i r and then i k. Then J will be something like i 1, i r, i p, i k. And so, if we go back to the net definition action on any k vectors, on the, so if I look at these things here, so the corresponding matrix for epsilon I and epsilon J will be the same except that in one of them the, they will differ by two columns being interchanged. So when I look at the, take the determinant, I obtain a minus sign between, so and that proves this and that is true for any v 1, v 2, v k. So that is essentially the gist of it.

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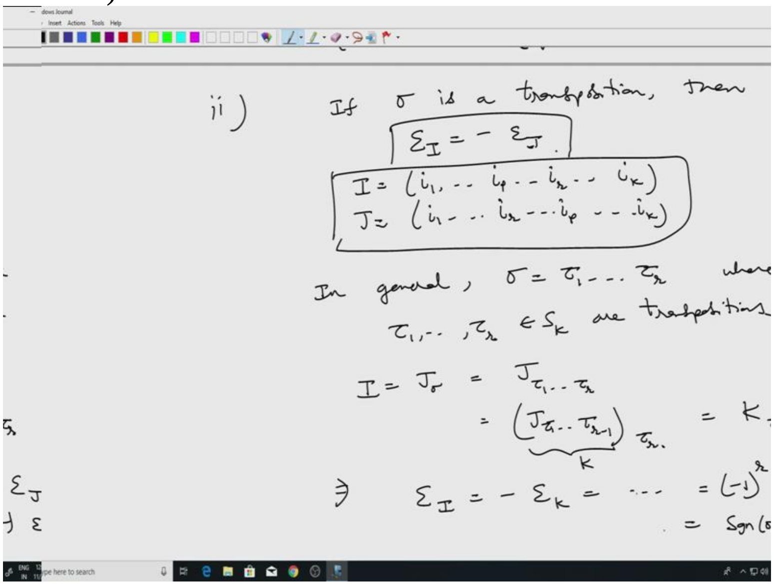

Since I did not write it down, let me erase this but the main point is this thing here, to say that sigma is a transposition will imply that I and J are related this form. So now, what about an arbitrary permutation? So, in general, we can write tau 1, tau r, where tau 1, etcetera, tau r in S k are transpositions and I know that I equals J sigma which is J tau 1 tau r. Well, which I can, basically I do it step by step, tau r minus 1 of tau r. So, what, so if I call this, yeah, let me now give it a new name.

So, what we have just observed is that this multi-index here, okay, let me give it a name, K. So this I is related to K by the action of a transposition. So this would be K tau r. So from what we just saw here, epsilon I would be minus epsilon K and then you repeat the process. K itself is a, involves several, so you keep on doing this. Finally, you will get minus 1 raised to r epsilon J, which is by definition sign sigma epsilon J, so that proves this as well. The second one.

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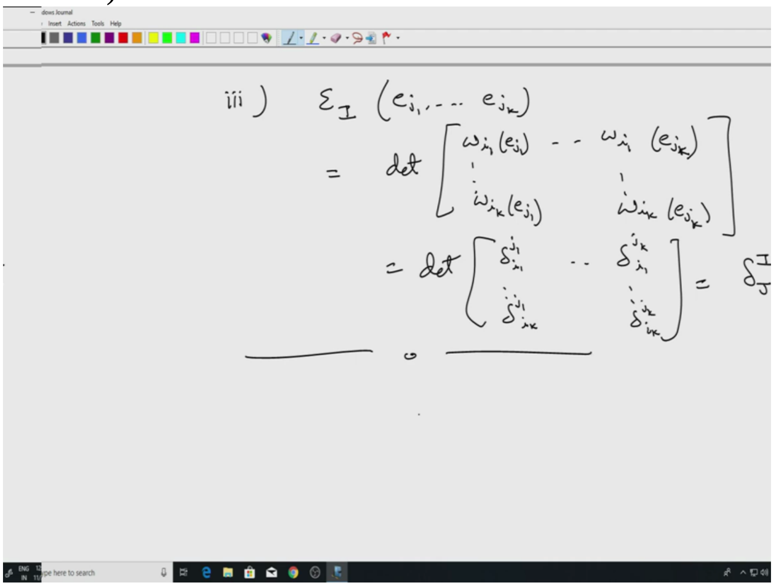

As for three, we already proved that three is epsilon I. No, it is not question of proving, it is just the definition actually. Three is e j k is by definition determinant of, well, it is omega i 1 acting on e j 1, omega i k acting on e j 1, omega i k acting on, no, omega i 1 still, e j k, omega i k e j k. And since these are dual basis, this is 1. If for example, the first entry is 1 if i 1 equal to j 1, etcetera, so we can summarize it by just writing it like this.

So, j 1 i 1, delta j 1 i k, delta j k i 1, delta j k i k and this is what we, by definition is the same as delta I J, delta I J. Maybe I should have yeah, it does not matter that delta is anywhere, whether I write which index I write on top and bottom does not matter, so I leave it like this. So, I have this all right. So, that proves this small lemma. So far we had been assuming that these index sets I consisted of numbers lying between 1 and n but no other restriction.

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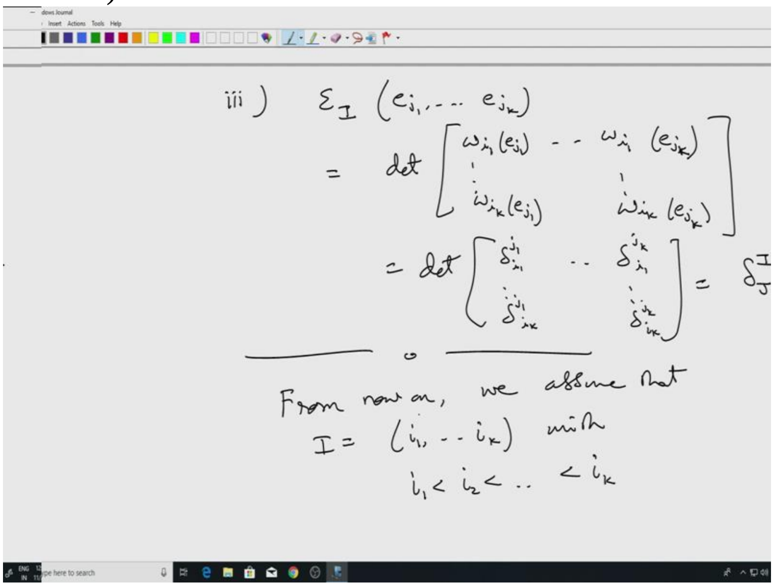

Now I am going to assume, from now on we assume that every multi-index I with i 1, they have been ordered like this, i 1 less than i 2 less than i k, strictly less than. So there are two things being said here, first is that they are written in an increasing order and strictly increasing, so no two indices can be the same. So we work only with multi-indices of this type. That will be enough for our purposes.

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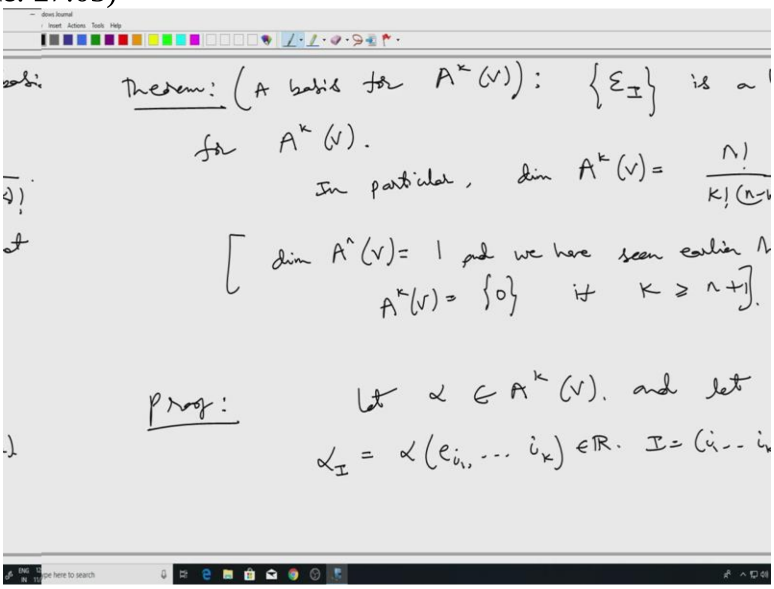

Okay. Now we come to the first main application of all these constructions. So let us write it as a theorem. Basis for A k V: So the claim is that this epsilon I is a basis for A k V. In particular, dimension of A k V equal to, so it is a matter of counting how many, so such k numbers I can choose which are strictly ordered like this. So, one knows exactly how to do this. It is n choose k. So this is n factorial by k factorial n minus k factorial.

So we have already seen that the dimension of A n V, we get it, we separate it fact here but we already proved that this one what I am about to write down, dimension of A n V is when k equals n, then I just get 1 and if k is more than n, well, this formula does not make sense, but we know that if k is more than n then A k V is 0. And we have seen earlier that A k V just consist of the 0-form if k is greater than or equal to n plus 1.

So let us quickly prove this. So let us take let alpha belong to A k V, so we are proving this panning property of this set of k-forms, so we would like to write it as alpha as a linear combination of the epsilon I's. So we have to come up with some coefficients which will enable us to write any alpha, given alpha in terms of the epsilon I. So we define, and let alpha index I be equal to alpha acting on e i 1, e i k where I equal to i 1 up to i k. These are just some real numbers. Then with these coefficients, then the claim is that I can write alpha as summation alpha I and then this epsilon I over all I.

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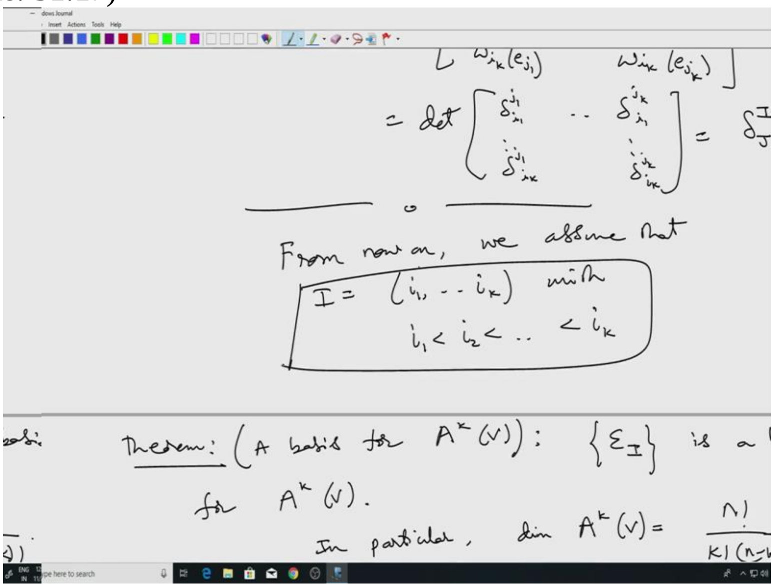

Of course, when I say over all I, remember that we are working with this restriction. So I will stop here. In next lecture, I will complete the proof of this panning property and then also show that they are linearly independent and then move on to the proof of the associativity and anti- commutativity properties of the wedge product. Okay, thank you.