An Introduction to Smooth Manifolds
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Lecture 51
Alternating Tensors 3

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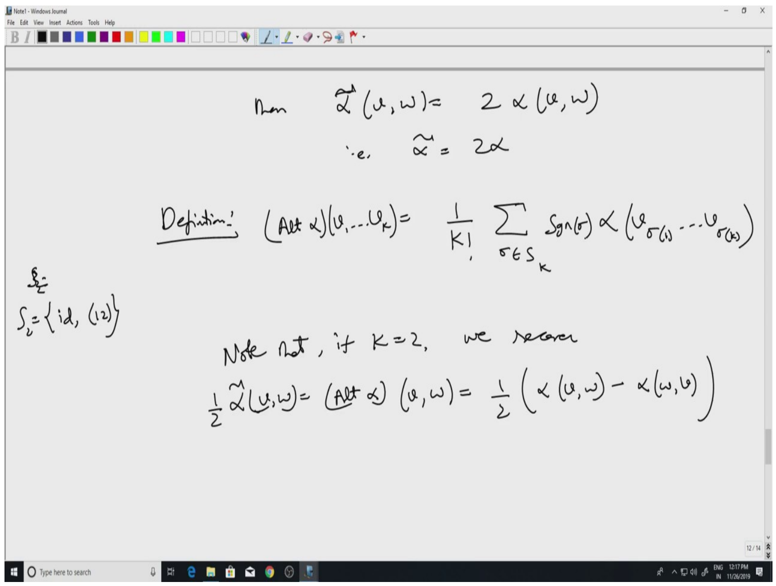

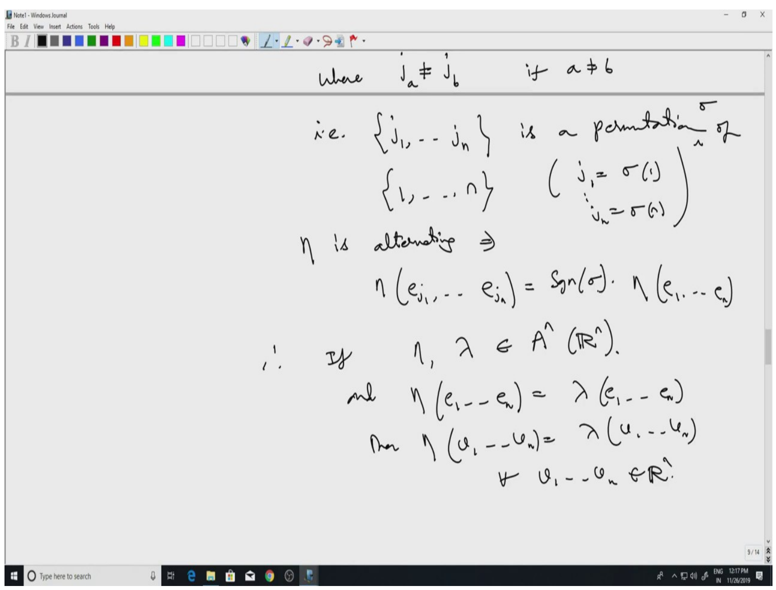

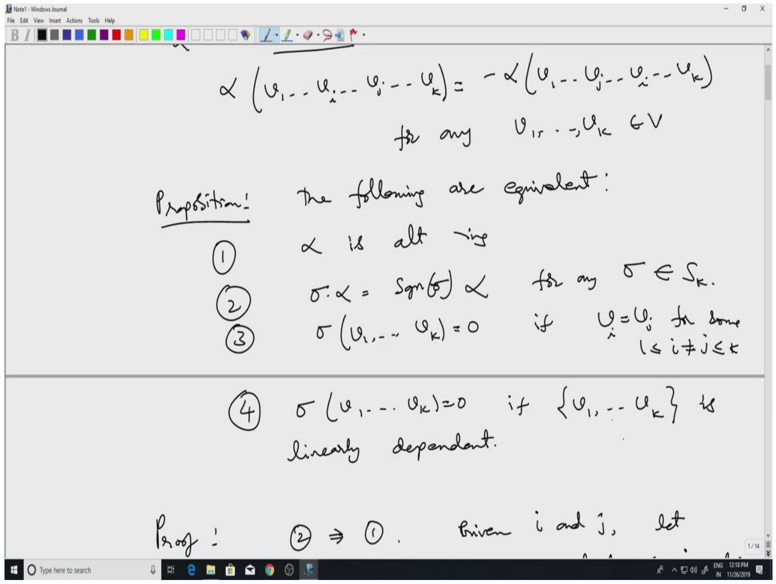

Hello and welcome to the 51st lecture in the series, last time I had introduced this operation of anti symmetrisation, where you start with any multilinear form and you end up getting alternating form. And the definition, so I realized that there is a small correction one has to make, it is more of a typo. All along, I have been writing minus 1 raised to sign sigma, what I meant was just sign sigma, rather than minus 1 raised to sign sigma. So for instance here it should be sign sigma.

And it, unfortunately this, I made this, this correction should be made in various places. Let me just quickly go or, sign sigma. And in fact, even and the proposition, where I say that here, I need to just write sign sigma. So, now coming back to this, the, let us check. So, we are in the process of checking that this definition actually gives an alternating form, that, so I had started writing something. Let me write it in a slightly different way. So, this is what, so I want, I had started with any sigma naught and then I did sigma naught of Alt alpha, then I get this sum.any sigma naught and then I did sigma naught of Alt alpha, then I get the sum.

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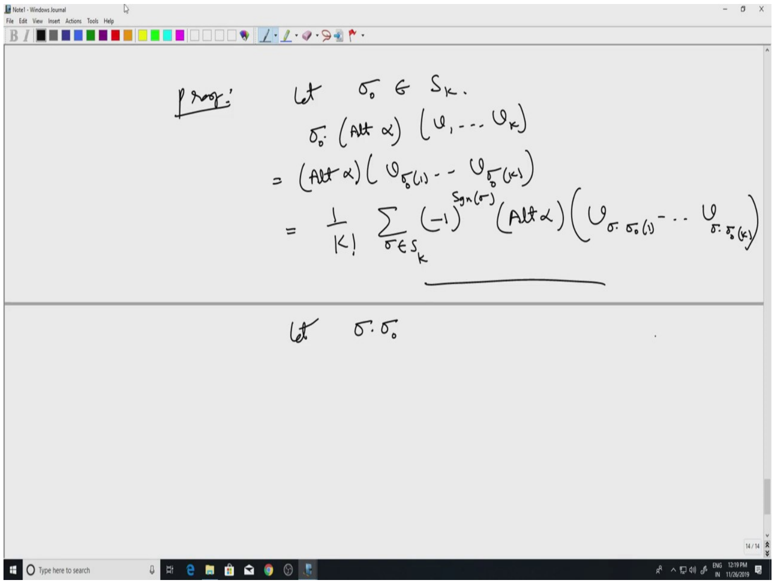

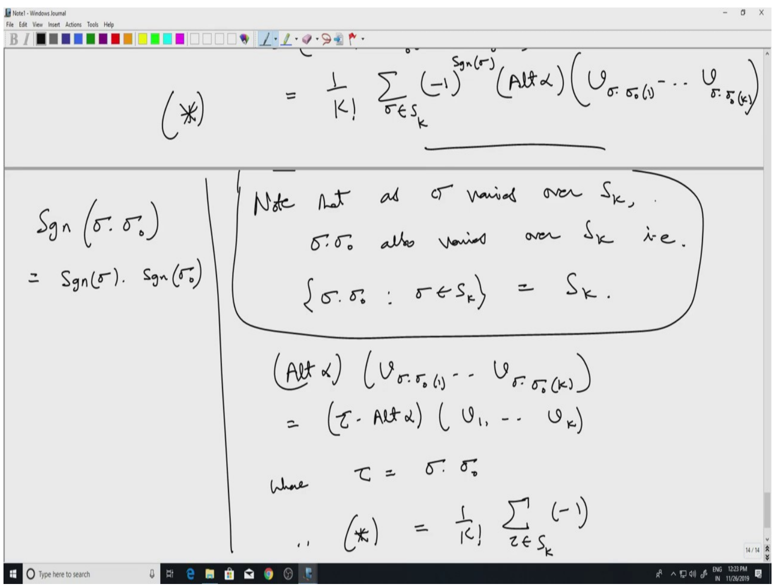

Proof: Let $\sigma_0 \in S_k$.

$$\sigma_0 \cdot (\mathrm{Alt}\,\alpha)(v_1, \dots v_k)$$

$$= (\mathrm{Alt}\,\alpha)(v_{\sigma_0(1)} \dots v_{\sigma_0(k)})$$

$$(*) \qquad = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{Sgn}(\sigma)\, (\mathrm{Alt}\,\alpha)\big(v_{\sigma\cdot\sigma_0(1)} \dots v_{\sigma\cdot\sigma_0(k)}\big)$$

$$\mathrm{Sgn}(\sigma\cdot\sigma_0) = \mathrm{Sgn}(\sigma)\cdot\mathrm{Sgn}(\sigma_0)$$

Note that as σ varies over S_k, $\sigma\cdot\sigma_0$ also varies over S_k i.e. $\{\sigma\cdot\sigma_0 : \sigma \in S_k\} = S_k$.

$$(\mathrm{Alt}\,\alpha)\big(v_{\sigma\cdot\sigma_0(1)} \dots v_{\sigma\cdot\sigma_0(k)}\big)$$

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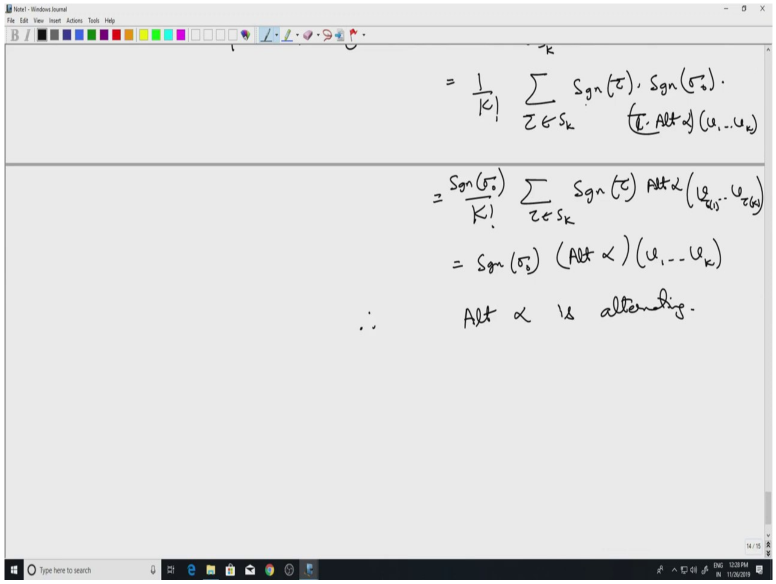
$$(\mathrm{Alt}\,\alpha)\big(v_{\sigma\cdot\sigma_0(1)} \dots v_{\sigma\cdot\sigma_0(k)}\big)$$

$$= (\tau\cdot\mathrm{Alt}\,\alpha)(v_1, \dots v_k)$$

where $\tau = \sigma\cdot\sigma_0 \Rightarrow \mathrm{Sgn}(\tau) = \mathrm{Sgn}(\sigma)\cdot\mathrm{Sgn}(\sigma_0) \Rightarrow \mathrm{Sgn}(\tau)\cdot\mathrm{Sgn}(\sigma_0) = \mathrm{Sgn}(\sigma)$

$$\therefore (*) = \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{Sgn}(\sigma)\, (\tau\cdot\mathrm{Alt}\,\alpha)(v_1, \dots v_k)$$

$$= \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{Sgn}(\tau)\cdot\mathrm{Sgn}(\sigma_0)\cdot(\tau\cdot\mathrm{Alt}\,\alpha)(v_1, \dots v_k)$$



Let, now let us call this, rather than putting it like this, let us say that note that as sigma varies over Sk, sigma dot sigma dot also varies over Sk, i.e. this set, this sigma dot sigma dot over all sigma n plus k is actually equal to Sk again, which is a, just another way of saying that. I mean, I am, all I am doing is, I know that Sk is a group, multiplying every element of Sk by, on the right by some fixed element of Sk, so naturally I get the whole group again.

So, now, because of that, what I can do is, I can write this. Let us look at this term here, an individual term. So Alt alpha acting on v sigma dot 1 is equal to tau the permutation tau times Alt alpha acting on v1, vk where tau equals sigma multiplied by sigma naught, this permutation. So, I can rewrite the whole sum in terms of this tau. Therefore, this star, star equal to 1 over K factorial. Now the remark that I made here is that, this the values, the permutation (ove) over covered by tau is a full symmetric group as sigma varies over the symmetric group.

So, I can write it as tau in Sk and then what I had here is minus 1 raised to sign sigma. Now here is an aside this. It is an easy consequence of the way the sign of a permutation is defined, that the sign of a product of 2 permutation is just the product of the signs, sign sigma sign sigma naught. And this is quite straight forward, since the way we defined sign was, you write sigma as a product of transpositions. Then you count how many there are. The even number the sign is plus 1, otherwise minus 1, same thing here.

So, if you have a product decomposition of sigma, if you have a transposition decomposition of sigma and a transposition decomposition of sigma naught, you get a, since this is a product of 2 permutations, you can just multiply both these decompositions together and from that it immediately follows that. This, if both for instance, if both are even, then the product will also have even number of permutations. If both are odd, the product will have again even number of permutations. If one of them is even, the other one is odd, the product will have odd and so on.

So, you only, it is one gets this essentially. So because of that, oops, again I am resorting to this unfortunate thing about writing it like this, sign sigma and here it is sign sigma and then the only thing I have changed is, instead of, I will write it as tau Alt alpha v1 up to vk. Now, notice that because of this comment that I made here, sign tau is the same as sign sigma times sign sigma naught. If I multiply both sides, the sign is either plus 1 or minus 1 so I can multiply by sign sigma naught on both sides, and I get sign tau times sign sigma naught equal to sign sigma.

And I can put it here and I am left with tau and Sk. This is tau, sign tau sign sigma naught, tau times Alt alpha acting on v1 up to vk. Right, so, now I can write it as actually, I need not. Okay, that is fine. So 1 over K factorial, so the point is, I can get the sign sigma naught out that is something which is independent of tau. So, and then, tau and Sk sign tau and then I just write it as Alt alpha v tau 1, v tau k, where tau varies over all permutations. And this we know is the same as so, 1 over K factorial. Again I combine with this and this is the same as Alt alpha acting on v1 up to vk.

So, essentially we started with sigma naught of Alt alpha acting on v1 up to vk and we ended up with sign sigma naught Alt alpha v1 up. So, therefore this Alt alpha is alternating. Now, if alpha is already alternating, one would like to check that this, the value of, right, where was I? Yeah, if alpha is already alternating, then I would like to check, I end up with whatever I started with.

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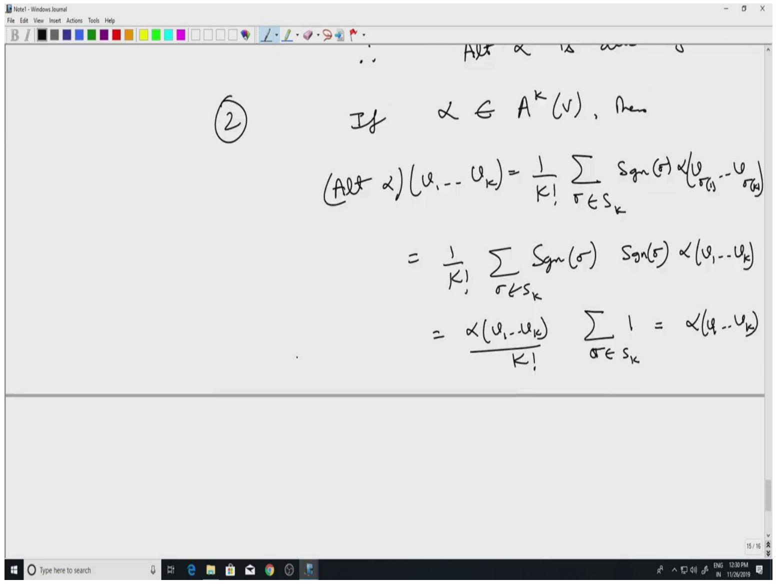

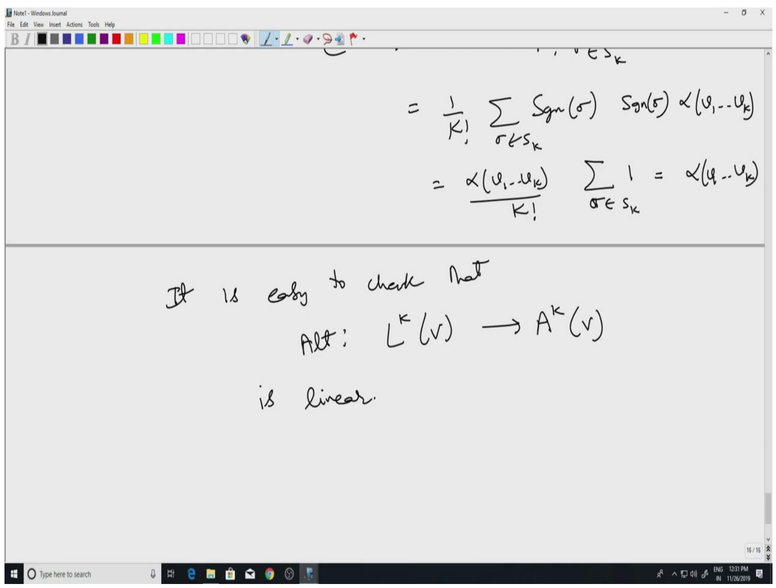

And that is clear as well, simply because, well, if alpha is already in what is this, alpha is already in Ak V, then Alt alpha acting on v1 up to vk is 1 over K factorial sigma in Sk sign sigma alpha v sigma 1, v sigma k. And this 1 over K factorial sigma in Sk sign sigma, now here, alpha v sigma 1 et cetera v sigma k, since I have assumed that alpha is alternating, what I get here is, again another sign sigma and then alpha acting on v1 up to vk.

So, I can take out this alpha acting on v1 up to vk, then I have a K factorial here. I am left with sigma in Sk, now the sign sigma whether it is minus 1 or plus 1, we have a square here, so I just get 1. So, I am summing, and this sum contains as many terms as there are sigma in Sk, which is exactly K factorial terms. So, the number of elements in Sk is K factorial. That K factorial cancels with this K factorial and I am left with alpha v1 up to vk.

So, that proves the 2 properties of Alt that I wanted and as I said, it is easy to check, as usual, the map itself, the Alt map. It is easy to check that Alt from Lk V to Ak V is linear. So, Alt of the sum of 2 forms is the sum of their Alts and is similarly, Alt of a constant times a form is constant times Alt of that form. I want to write it down, these are fairly straight forward. Now, what I want to do is, just like we had a nice basis for Lk V, if we start with, starting with the basis for 1 forms, we, and the tensor product, we could get a nice basis for Lk V.

Similarly, I would like to get a basis for Ak V starting with 1 forms. However, tensor products will not do the job. So, tensor product of two 1 forms, for example is not an alternating 2 form, just a two tensor. So, one would like to combine 1 forms in somewhat different way and get a basis for Ak V. And that is where this process of Alt is going to come in handy.

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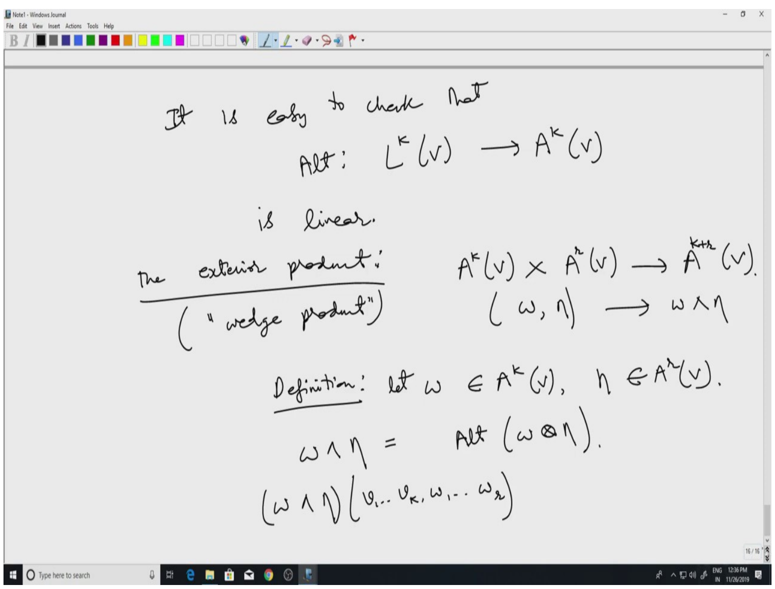

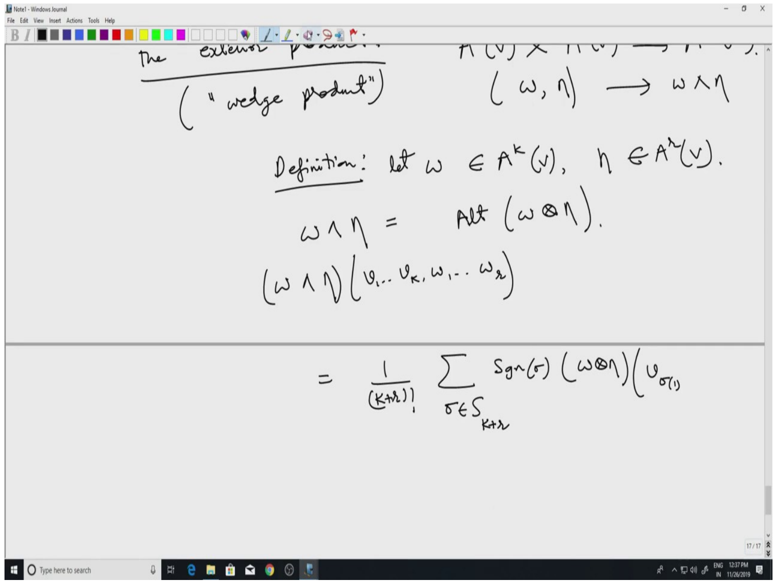

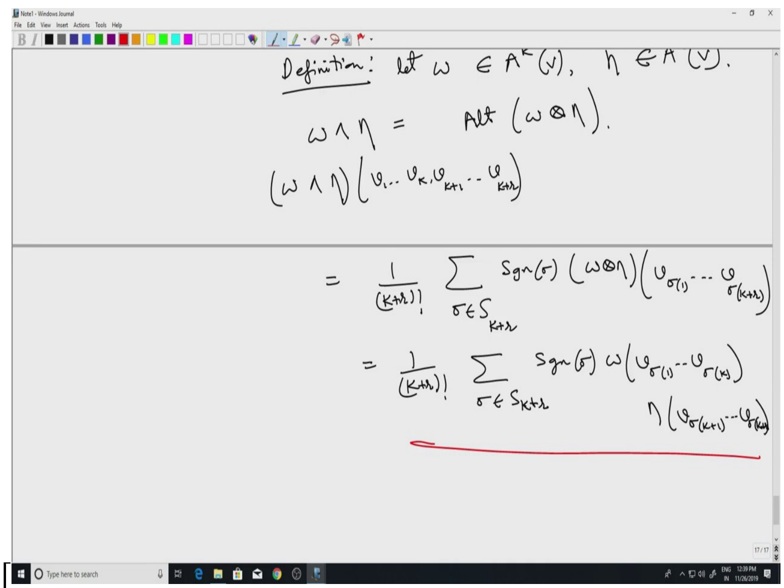

So, this, now we come to this important operation of the exterior product. So, in the world of alternating tensors, this serves the same role as the tensor product. In other words, we can combine a K form, alternating K form and an alternating P form and get an alternating P plus K form et cetera. So, and the way I define it is, I actually end up using the tensor product, but I have to do something more, so the exterior product is a map from, as I said Ak V, the input is to Ar V and I am going to get something in Ak plus r V.

So, omega eta and this will be denoted by omega wedge eta. This is also called wedge product. And if I, so if I start with omega and eta, well, I can forget that they are alternating. Just think of them as multilinear forms and take the tensor product. Then I will get a K plus r form. But then, now that I know that given any multilinear form, I can get something alternating out of it, we can apply Alt. So that is essentially the idea.

So, let us define the wedge product omega in Ak V eta in Ar V. So let this omega wedge eta, I define to be, for the moment I am going to ignore constants. The constants actually play an important role here up to a constant. So, it is essentially Alt of omega tensor eta, if one wants to spell it out, so one can see what its action is on.

So, this is going to act on v1, so I will give different symbols for the variables v1 up to vk, and then w1 up to wr, so altogether there are K plus r input variables, input vectors. And what I am supposed to do is, well, I will have to do, the Alt operation already involves a factorial k plus r factorial and then sigma and now this is, I am going to apply Alt to omega tensor eta.

Omega tensor eta is a k plus r multilinear form, so I will have to take S k plus r and then, now here I will be doing, I forgot the main thing which is sign sigma and then omega tensor eta of v sigma 1. Oh, here actually, sorry, so I have to be consistent with this notation. I cannot really use these separate variables. So let me just go back and call it v k plus 1 vr.

Then this will be v sigma, so here k plus r, not vr, v, v k plus r and then here it would be sigma k plus r, right. And one can expand this a bit more, so sigma in S k plus r sign sigma and by the definition of the tensor product, I will end up acting omega on v sigma 1 all the way up to v sigma k, the first k of these variables, then multiply it by eta acting on v sigma k plus 1 v sigma k plus r.

So, this is what it is, the last expression, this is the definition of the exterior product if we are simplified it as much as we can in general setting. Alright, so, now there are lots of important properties of this exterior product. Again, the first thing is that, that this product is again an alternating form. This is the first observation. That of course is true, because I have applied Alt and we have seen that, Alt takes any multilinear form and makes it alternating.

So, that is okay, given that there is a bunch of important properties of this exterior product that I will write down in the next lecture, that would be a good place to start. And then I would not be able to prove all of them. Maybe some of them I will discuss. We will stop here at this point with the definition of the exterior product. In my next lecture I will talk about its importance, the main properties of this. And then the plan is to move on to, all this while we have forgotten about manifolds. Now I will go back to manifolds after that and sort of carry over all these properties to the setting of manifolds, all these constructions to the setting of manifolds. Okay, thank you.