An Introduction to Smooth Manifolds
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Lecture-05
Inverse Function Theorem

Now let us start our discussion leading up to the inverse function theorem.

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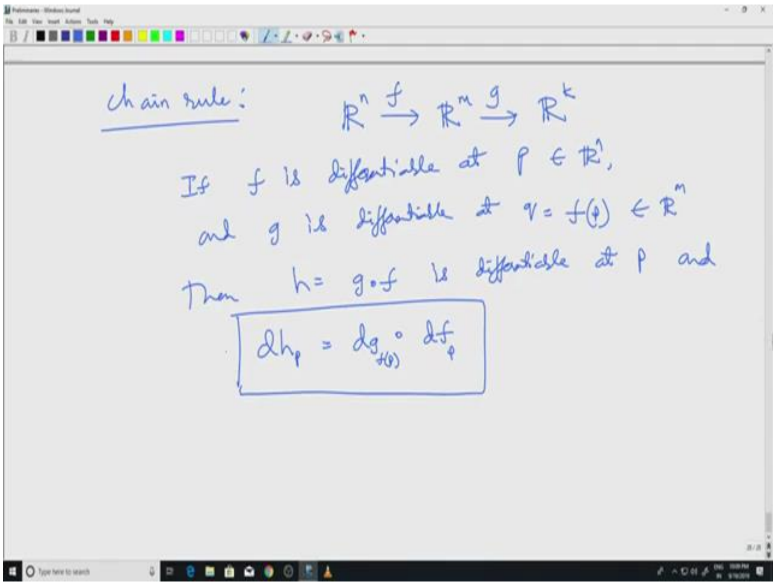

But before that, before stating the inverse function theorem, I would like to briefly discuss another thing, which is an analog of what one sees in one variable calculus, namely the chain rule. So, here I have two functions, Rn to Rm and let us say Rk, so this is f, this is g. So the hypothesis is that here I have taken the domain to be full Euclidean space, but it is only and it is enough to have open sets. So I can start with an open set in Rn and an open set in Rm. And then everything makes perfect sense.

Just for simplicity, I took everything to be, the domains to be all of Euclidean space. So chain rule is that if f is differentiable at P in Rn and g is differentiable at Q equals f of p, this is in Rm, then the composition h equals g composed with f is differentiable at P and the derivative of h at P, which is a linear map from, h goes from Rn to R k, so the derivative is also linear map from Rn to Rk.

And this linear map is a composition of two linear maps one is d g evaluated at the point f of P composed with d f at P. This is the analog of chain rule in higher dimensions. So, it is just a composition of what we, essentially what it says is if you have a composition of differentiable functions, then the derivative is a composition of the two separate derivatives of the functions and here composition as linear maps. So, in particular if one wants to look up, look at the matrix Jacobean matrix of this linear map d h. Then I would look at the Jacobean matrix of this and matrix multiplied by the Jacobean matrix of df.

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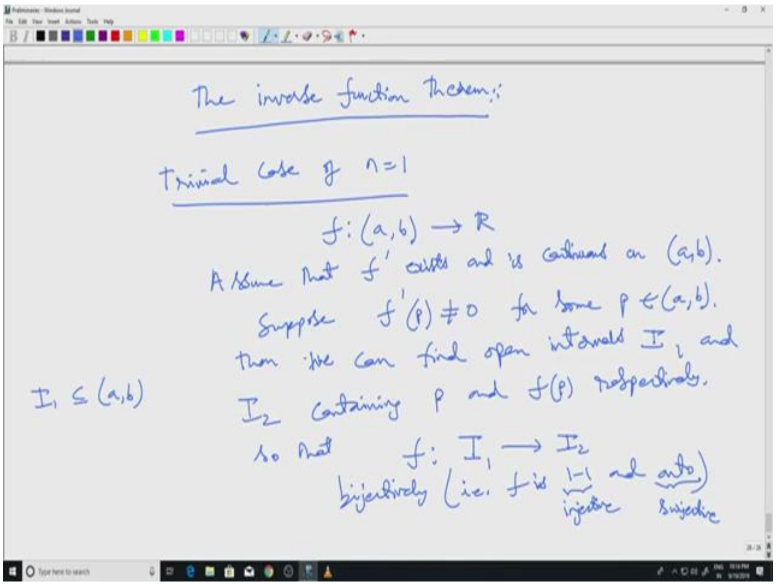

Now the inverse function theorem. So basically this says that if you have a function from Rn to Rn, so here it is crucial that the both the domain and target spaces are the same dimension. And if you have a function, differentiable function from Rn to Rn, and if the derivative is invertible as a linear map, then the function is locally one-to -one. So well, it says a bit, quite a bit more than that. But let us, let me just go back to the trivial case.

The trivial case is the one dimensional case. So here I have a smooth differentiable function from an open interval to R and the assumptions are as follows. Assume that f prime exists and is continuous on a, b. So, not only do we assume that the derivative exists, but we also assume the derivative is continuous on a, b. Then the claim is suppose, the derivative f prime p is not equal to 0 for some P in a, b, then we can find open intervals I1 and I2 containing P and f of P respectively. So, that f, this I1 is actually, I should remark here, this I1 is actually contained in a, b.

So, we can find, so that f is defined on I1. So if maps I1 to I2 bijectively i.e. f is one to one and onto. So I instead of saying one to one or onto, I will use the words injective and surjective. And if both of them hold, I would say bijective. So this is one to one is synonymous with injective and this is called surjective. And if both conditions hold, it is called bijective.

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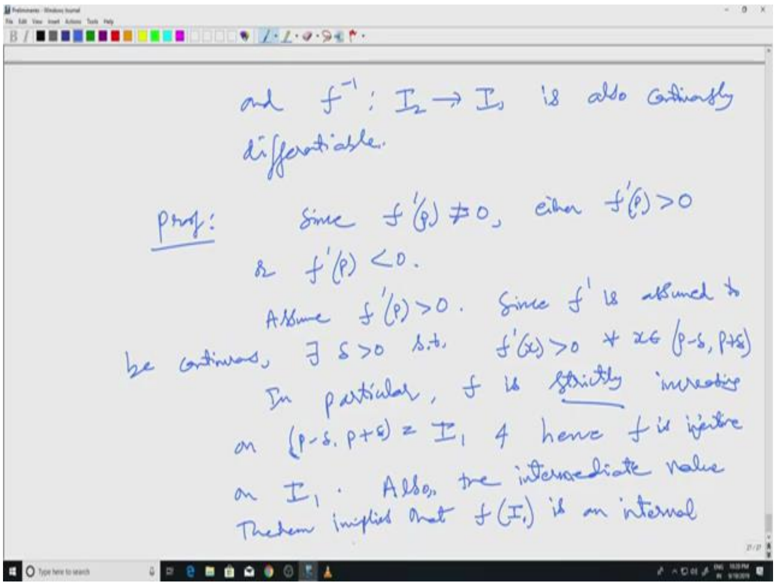

So what we are saying is that f maps this one interval bijectively onto the other interval. And f inverse, the moment I know it is bijective, I have the inverse map from I2 to I1, is also continuously differentiable. So, the derivative of f inverse exists and is continuous. So, this is the one variable statement of the inverse function theorem. What we are saying is, if the derivative is not 0 at a point, then we can, if we stay close enough to that point, we can find an interval around that point and interval around the image point, so that f maps the smaller interval onto the, onto I2.

And f is actually one-to-one there, so, we get an inverse function for f. And this inverse is also differential. So, that is the, so the existence of the inverse function. And but it is only locally, this is I1 is maybe a just a very small sub interval of this bigger interval a, b. So we cannot say anything about what happens to f on the whole interval a, b. It is only close to P that we can say something.

Now, what would be Yeah, so the proof in this case I will just briefly mention what the proof is? It is extremely simple and unfortunately, it does not shed any light on how to prove the higher dimensional version. But let us see how the proof would go. The thing is, if you have, since f prime p is not equal to 0, either f prime p is strictly positive or f prime p is strictly negative. So, assume the first, the proof is similar in the second case.

Assume f prime p is positive. Now, the thing is that we have assumed that the derivative function is continuous. So if this is if prime is positive at some point p, it is going to be positive in a neighbourhood of P. Since, f prime is assumed to be continuous, there exists delta greater than 0, so that f prime x is positive for all x in p - delta p + delta. This, that the fact that f prime is positive on this interval means that f is strictly increasing on this. In particular, well sorry f is strictly increasing, in particular f is strictly increasing on p - delta, p + delta.

So, this is going to be our I1, so that I1 which I asserted in this theorem is this interval on which f prime interval containing p on which f prime is positive throughout. So it is strictly increasing therefore, it is one to one and hence f is injective on I1. Now, if you look at the image of f, image of f follows from, so also the fact that it is strictly increasing and Intermediate Value Theorem. The Intermediate Value Theorem.

Intermediate Value Theorem implies that f of I1, the Intermediate Value Theorem for continuous functions implies that if I start with an interval and look at the image of that interval under a continuous function, it is again going to be an interval. It does not, it will not say that if I start with an open interval, the image will be an open interval, but it does say that it is an interval. Implies that this is an interval.

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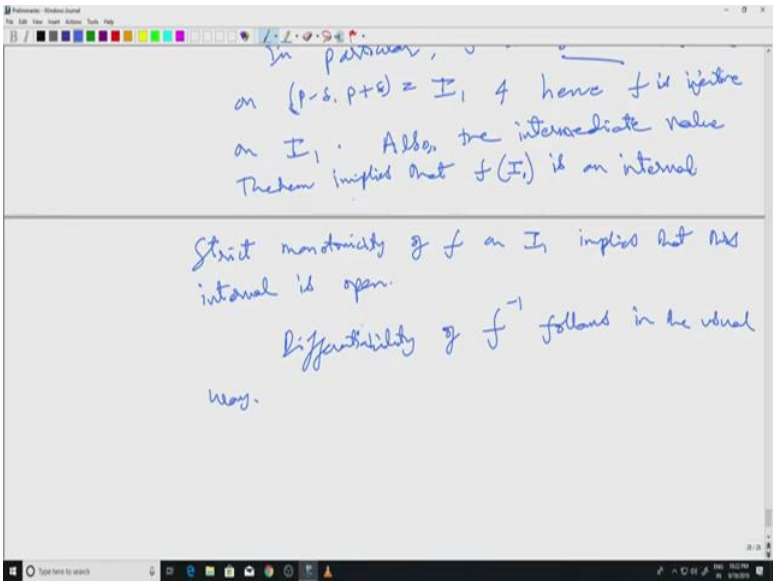

Again strict monotonicity, strict monotonicity of f on I1 implies that this interval is open. So, that completes the proof. Well not quite, I mean the differentiability of f inverse, differentiability of f inverse follows in the usual way. So in the usual way meaning, we learned in one variable calculus that if you have a function which has an inverse and the derivative of the function is not 0, then one can, the inverse is also the differentiable. And in fact, the derivative of the inverse is given by the inverse of the derivative. So that is, that is what I mean by the usual way.

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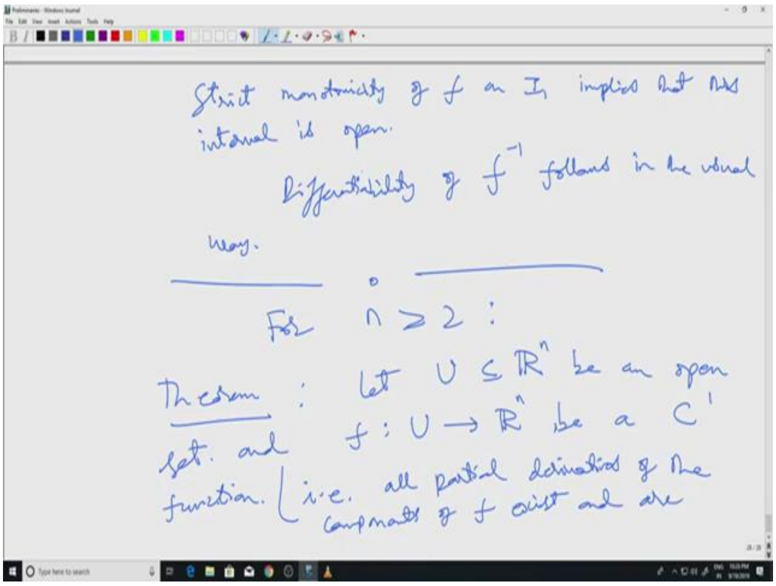

So now let us turn our attention to higher dimensions, for n greater than or equal to 2. We have the inverse function theorem which states, so let U contained in Rn be an open set and f from U to Rn, be a C1 map. By map or let me just say function since that is what I have been saying all along, C1 function. So i.e. all partial derivatives of the components of f exist and are continuous, this is the meaning of C1. All partial derivatives of the components of f exist and are continuous.

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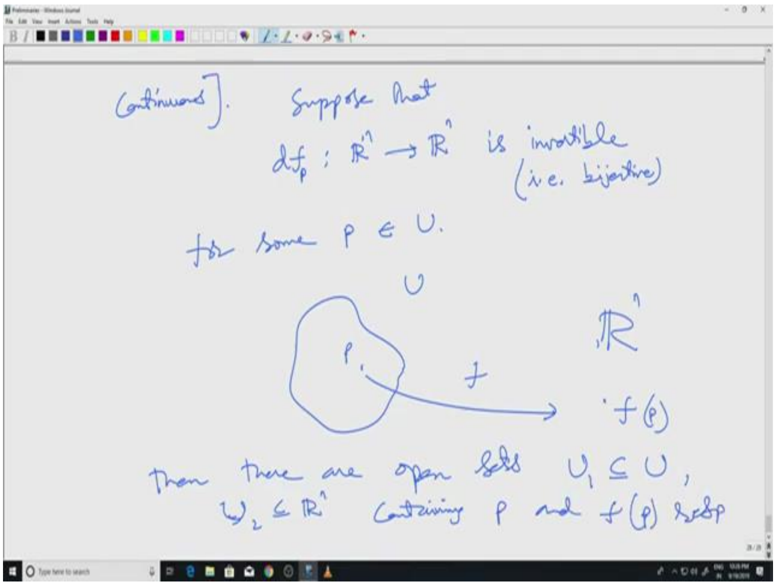

Now suppose that, this is the main assumption d fp which is a map from Rn to Rn, is invertible, which is the same thing as saying, i.e. bijective, suppose the d fp is invertible i.e. bijective for some p in U. So here is a sort of schematic picture. So this is my open set U, I have f and this is Rn and have some point P inside this, this is getting mapped to f of p. Circulate at only this and this is f and I am assuming that the derivative of f at this point is an invertible linear transformation from Rn to Rn.

If this happens, then there are open sets U1 contained in U, U2 is something, some open certain Rn containing P and f of p respectively.

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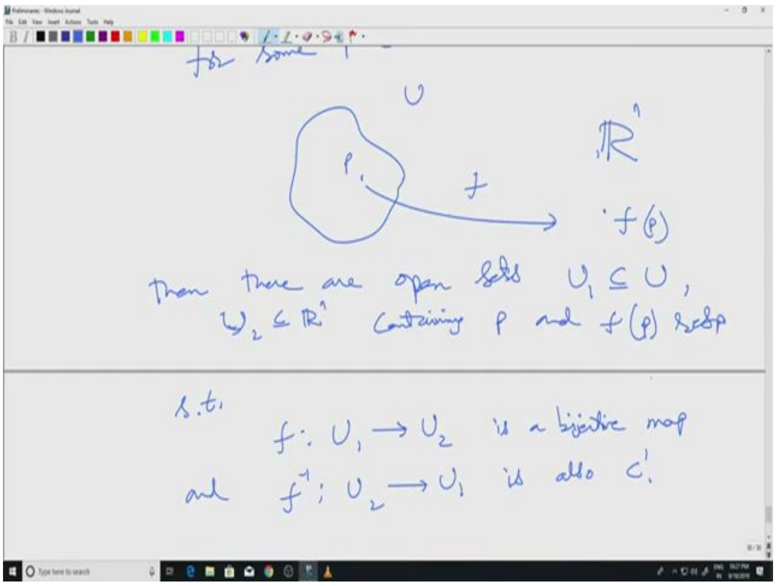

Such that f from U1 to U2 is a bijective map and f inverse from U2 to U1 is also C1. So this U1 is going to be some smaller set here U2 is some open set containing f of p. When I restrict f to U1 when I get a bijective, the conclusion is that when I restrict f to u1 I get a bijective map on to u2 and moreover the inverse is also a differentiable, continuously differentiable function. So I stated it so that it is exactly this, it looks exactly the same as the one variable statement.

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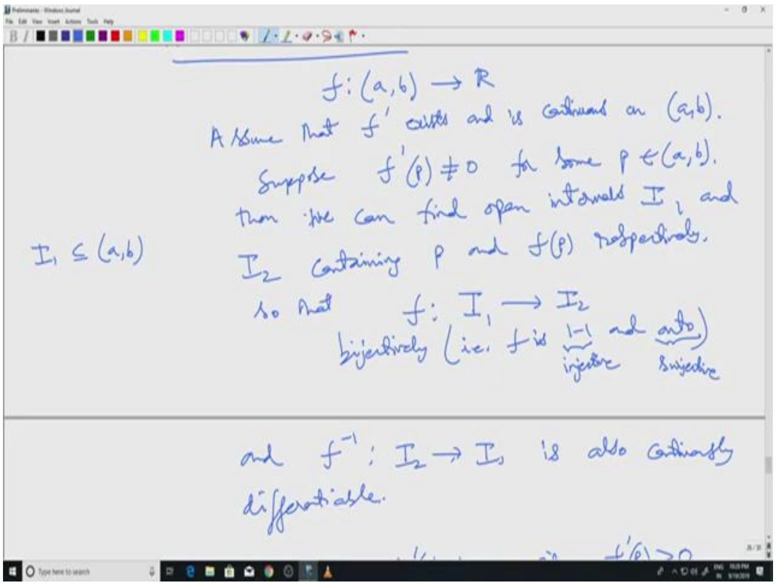

Here in the one variable case, the derivative as a linear transformation is just a number. So it is one by one matrix, one can think of prime p as a one by one matrix and to say it is invertible is just saying that it is not 0. And what simplifies things in the one variable case dramatically is that we have this nice condition, that the moment the derivative is not 0, we know it is either strictly increasing or strictly decreasing. While here there is one has to give a completely different argument. Let us, this has lots of interesting conclusions or corollaries.

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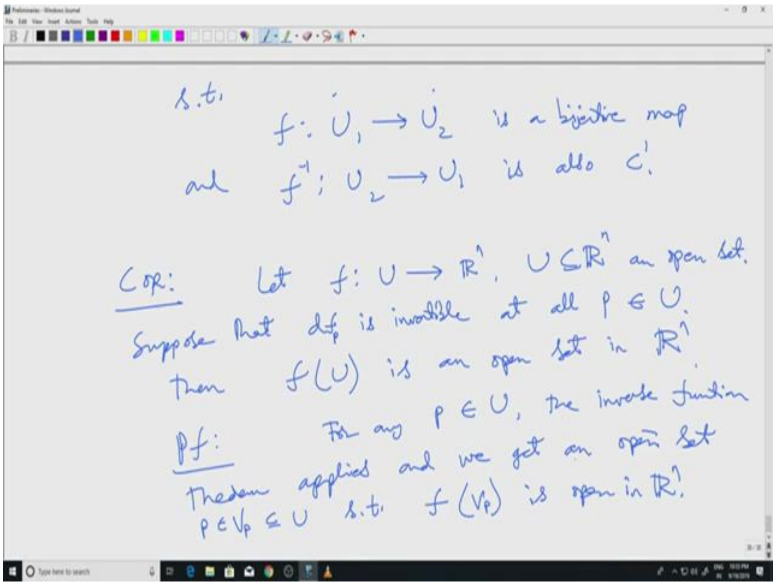

So let me begin with one corollary. Again, I am start with an open set in Rn, let f from u contained in Rn an open set. Suppose in the previous theorem, I just made the assumption about the invertibility of the derivative only at a single point. But now let us assume that, suppose that d fp is invertible at all P in U. So, then the conclusion is, then the image of this open set U is a subset of Rn is an open set in Rn, this is the conclusion.

And this follows immediately from the previous thing because so let me quickly outline that. Well so let us start with, let us see what how to describe this f of U. So I start with first any point in U, for any p in U, the inverse function theorem applies because we assume that the derivative is invertible at all points in the domain. So, start with any point in the domain and I can apply the inverse function theorem. Applies and we get open sets, let us call it Vp and W f of p. Well, Vp and so in fact, let me just, let me deal with just one open set, anyway the other one is going to be image of one of them, so there is no need to mention 2.

And we get an open set Vp which contains P and which is contained in U such that this f of Vp is open in Rn. This is what I called U2 in the previous theorem. U2 is since this map is bijective from here to here, U2 will be just f of U1, so that f of p is open. Okay fine.

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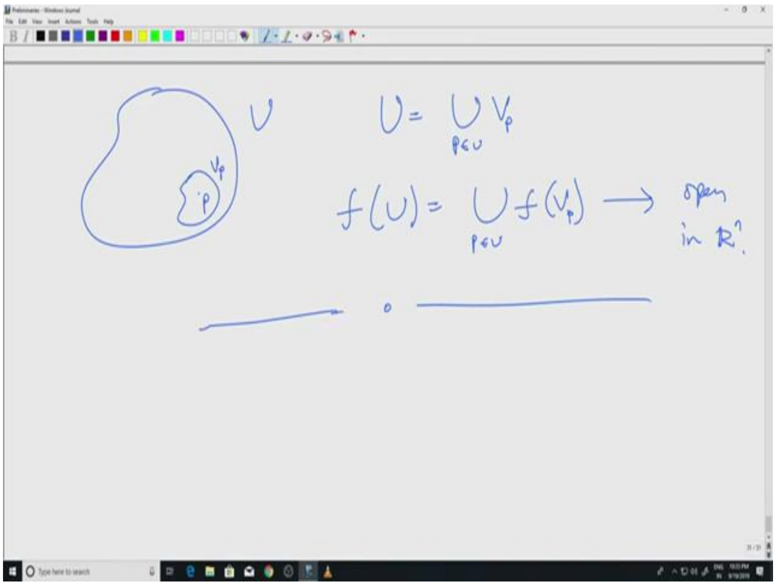

So now, the thing is that every point is contained in some such Vp. So, if I look at this open set for every P there is a Vp and that Vp itself is contained in U. So, therefore, I can write U as the union over all P in U of these Vps, okay just take the union of all these Vps, I do not get anything more than U and everything in P is of course contained in one of these, so, I get this.

if I write it like this, then f of U will be just union over P in U f of Vp. F of Vp, I just remarked that f of p is open in Rn. So, I have a arbitrary union of sets, each of which is open in Rn. Therefore, we know this, the union itself is open. So, this thing is open in Rn, therefore f of U is open in Rn, which was the conclusion that we wanted. So, this is one of the consequences of the inverse function theorem. Now. In my next lecture, I will expand on another important consequence. So we will stop here.