An Introduction to Smooth Manifolds
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Lecture 48
Symmetric Tensors

Hello and welcome to the 48th lecture in this series. So we were talking about multilinear maps and tensor products and so on.

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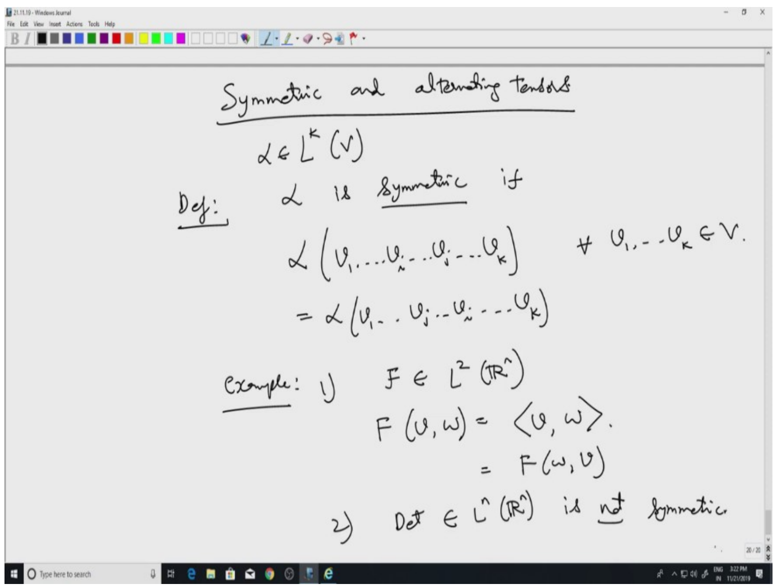

Now, let us this, in this lecture, let us focus on two special classes of tensors. So, by the way, I should mention that here I have written forms, so perhaps I should change it to tensors, symmetric and alternating tensor. Now, so let us start with an element alpha and Lk V with the same notation as last time. So alpha is symmetric, if the following condition holds. The condition is that alpha v1, I go up to vi and some other index vj and then v all together k.

This should be the same as, I keep all the other vectors fixed, but just swap, interchange vi and vj, vj vi and vk. And this should hold true for all vectors v1 up to vk, V. Right. So, if I interchange any two vectors keeping the rest as there, the value of the tensor should not change. And we have a very basic example of this, namely the inner product F and L2 of Rn.

This notation, yeah, well, L2 obviously does not mean the function space here. It just means, I am just using this notation, F and L2, F of v,w is v in the product w. So this is symmetric since F of v,w is F of w,v. So the inner product is, however, the determinant, this is in Ln of Rn is not symmetric because we know that, when we interchange two columns in the, in a determinant, the sign of the determinant gets reversed, so that is the exact opposite of, yeah, of a symmetric multilinear tensor. So the determinant is not, and before proceeding further, I, let us express this, this condition of being symmetric in a slightly different way.

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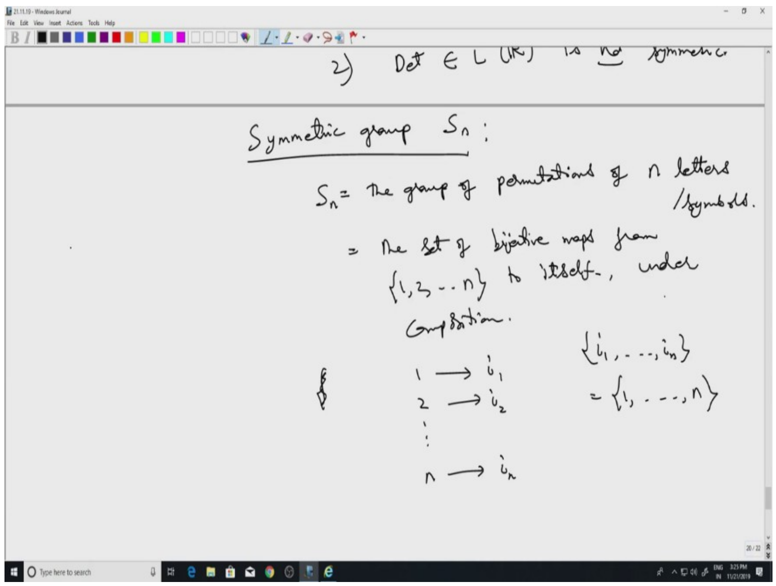

So, for that I need for that and subsequently as well, I need to save, assume a few things about the symmetric group. Let us talk a bit about symmetric group Sn. So, Sn is by definition, the group of permutations of n letters or symbols and distinct objects basically. So in other words what is a permutation, it is just a, permutation is a just a bijective map. Permutation on n letters is, is the set of bijective maps from 1 to n to itself, under composition. This is the group operation.

So one can, and normally one writes. So a bijective map, what will, so I can want to and 1 will go to something i1, i2, in. Where this, this set i1, i2, in, is the same as the set 1 up to n. So the bijective map just is prescribing where 1 goes to and where 2 goes to and et cetera. And this is the way one normally thinks about a permutation.

What we need to know about the symmetric group is that first of all, the point about, of thinking of them as bijective maps is then the group operation becomes immediately clear. It is just composition and moreover the fact that it is a group is also clear. Composition of two bijective maps is bijective. The inverse of a bijective map is bijective.

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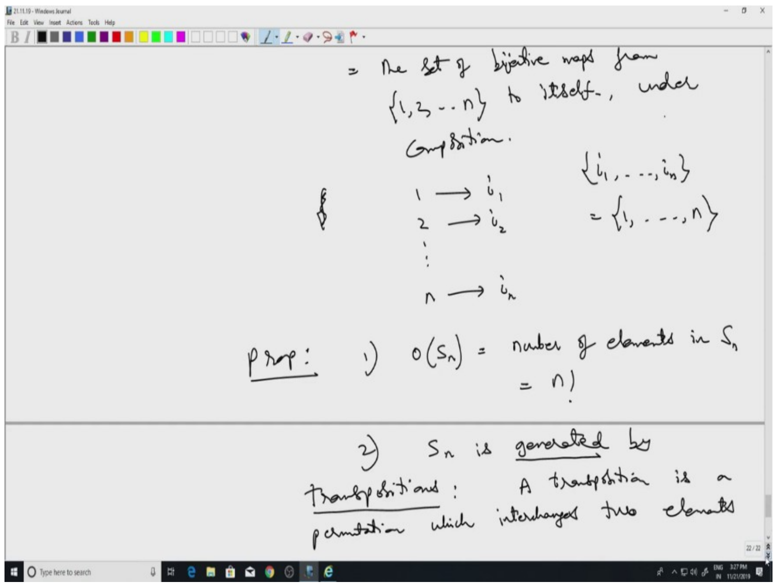

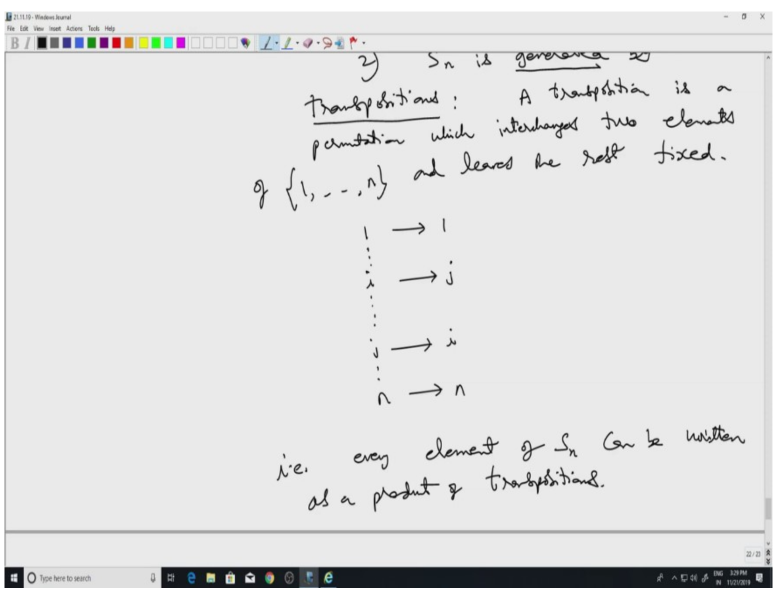

So what we need are the following facts. One, order of Sn, number, order of a finite group is now finite set as or number of elements in Sn is n factorial. We know that number of permutations, n factorial. The second thing is that Sn is generated by transpositions. So there are two words here which have to elaborate on.

First, what is a transposition? Transposition, permutation which interchanges two elements of 1 up to n and leaves the rest fixed. In other words, if I were to think in terms of bijective maps, it would be something like this. It would take 1 to 1.Ssome i would go to j. The others are again left fixed. j would go to i and goes to n. Of course this, if this, i happens to be 1, then 1 would not be left fixed.

And similarly with n but the idea is that all but two elements are left where they are and those two elements are sent to, one is sent to the other and the other one is to this. The, the two elements are interchanged. Right. So, and so, that that is a transposition and what do we mean by, generated by, i.e. every element of Sn can be written as a product of transpositions. So given any permutation, I can break it up. I can, here, of course product is composition in terms of bijective maps. So given, any element can be written as a product of transpositions.

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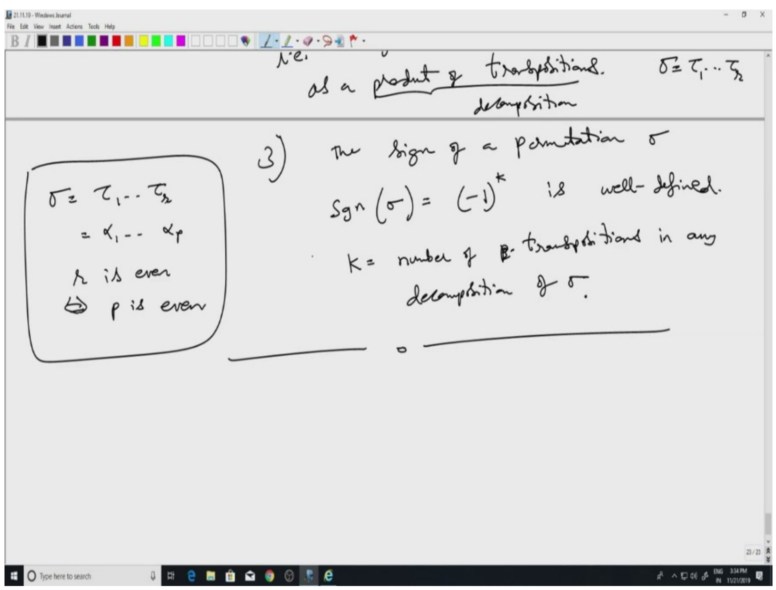

Now, the thing about writing it as a product of transpositions is that, this, it, there may not be a unique way of writing it as a product of transpositions. One can possibly have, one can has several ways. And, but what is true is that, the sign of a permutation, normally permutation bijective map is denoted by sigma. And the sign sigma, Sgn sigma is minus 1 raised to the number of permutations, so let me write minus 1 raised to k, k equal to the number of permutation, number of transpositions in any decomposition of sigma.

So every element can be written as a product. So let us call this a decomposition of permutation. So I am writing sigma is tau 1 dot dot dot product of tau r. So this r is what I am talking about here. I have called it k. k is number of transpositions in any decomposition of sigma. So when I look at, when I, the way that I have defined the sign, all that matters is whether k is even or k is even or odd and the claim is that the sign of a permutation if I define it like this is well defined.

In other words, even though the same sigma can be written as a product of transpositions, I mean, it is sigma can be written as tau 1 tau r and let us say alpha 1 alpha p, where the taus are transpositions, so are the alphas. If, but even though it can be written in two different ways, what we are saying is that, the, whether this r is even or odd, is the same thing as asking whether this p is even or odd, r is even if and only if p is even. That is the gist of what I am saying here.

So this is the so called parity, even or the evenness or oddness of the number of elements of transpositions is independent of how we write the permutation in terms of transportation. So these are fact. I mean the, this. all these three things, they are facts in elementary group theory or elementary combinatorics. And I am not going to prove this, like, I will just assume this. And it is kind of, it is necessary to be somewhat familiar with this, dealing with that, dealing with permutations, multiplying permutations and so on to proceed further.

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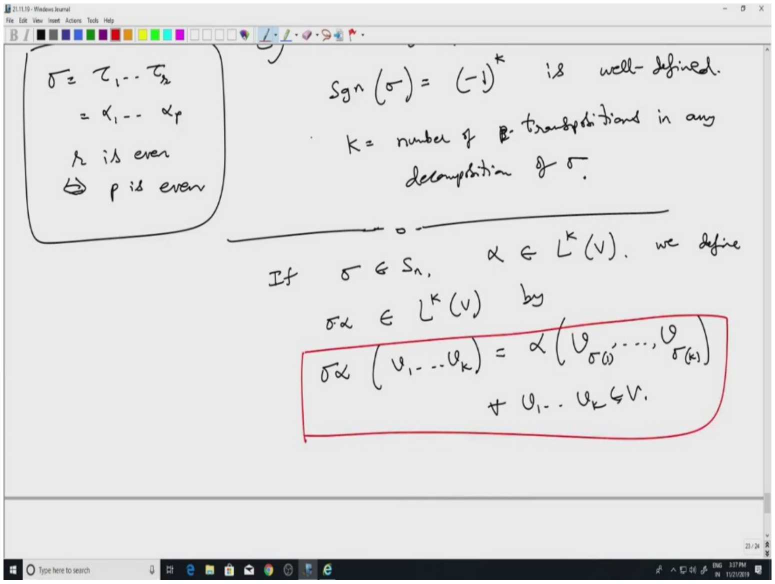

So with this in hand, now, let us see this, go further and talk about symmetric, we are in the process of describing symmetric tensors in a different way. So here is the proposal, yeah, before that I also need, yeah, well. Let me put it like this, if sigma is a permutation and alpha is an element, it is a k, k tensor, then we define sigma times alpha. Strictly speaking, these are, these, there is no multiplication here. After all, this is a permutation and this is a alternate, this is a multilinear map. They are in two different sets.

But the definition of this is, this is define, I already wrote we define. So, we define this thing here, it is going to be again an element of Lk V, by sigma alpha acting on v1, vk is just alpha acting on, all I do, is I use this permutation to move these vectors around. After all, these vectors are from V. So it it makes perfect sense to put vk in the first slot for instance, v1 in the third slot and so on. So, we will move them around as dictated by this permutation. So sigma 1 dot dot dot v sigma k. And this is for, for all vectors, we went up to vk. So let us remember this notation. I will be using that, this again.

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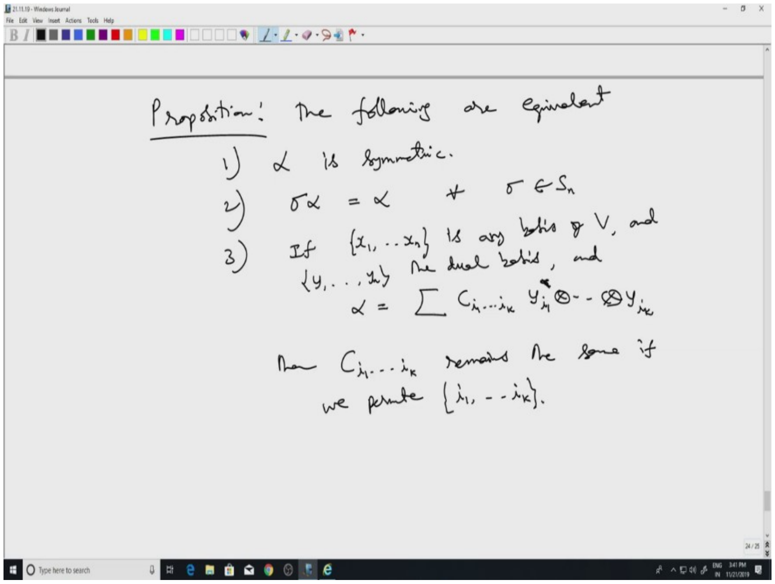

Right. So, then, here is a small proposition. So I want to say that the following are equivalent. The first thing is, alpha is symmetric to, alpha is, again I am assuming that alpha is an element of Lk V as as I did here. So alpha is symmetric should be the same as saying that sigma alpha equals alpha for all sigma in Sn.

The third thing is, if I write, if alpha is, if x1, xn is any basis of, any basis of V and y1 the dual basis, and if we write, alpha equals, we know that we can write it as summation Ci 1, Ci k Yi 1, no there is no star and then tensor. And then tensor yi k. We know that, we already seen that, this tensor products of one forms, forms a basis for Lk V. So I have just expanded alpha in terms of this basis.

Now, the thing is that if alpha is symmetric, it is equivalent to saying that Ci 1, ik remains the same and the same, if we permute i1, ik. So this coefficients themselves, if I move this, for instance if i1 put into i2 slot and i2 is put in i1 slot et cetera, I should, nothing should change. So this last condition here is a sort of practical way of detecting, using the symmetric, alpha is symmetric condition namely, we write it in terms of the basic, in terms of the basis of Lk V that we constructed. Then just by looking at the coefficients, we can find out whether alpha is symmetric or not.

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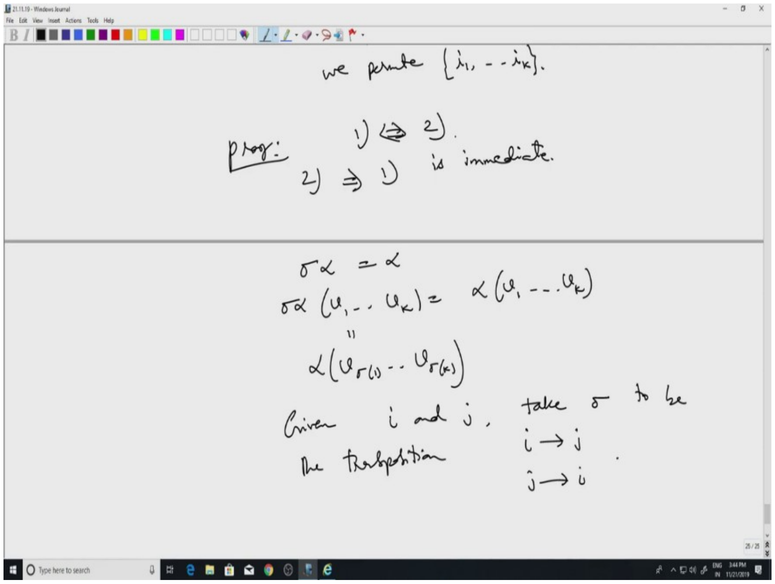

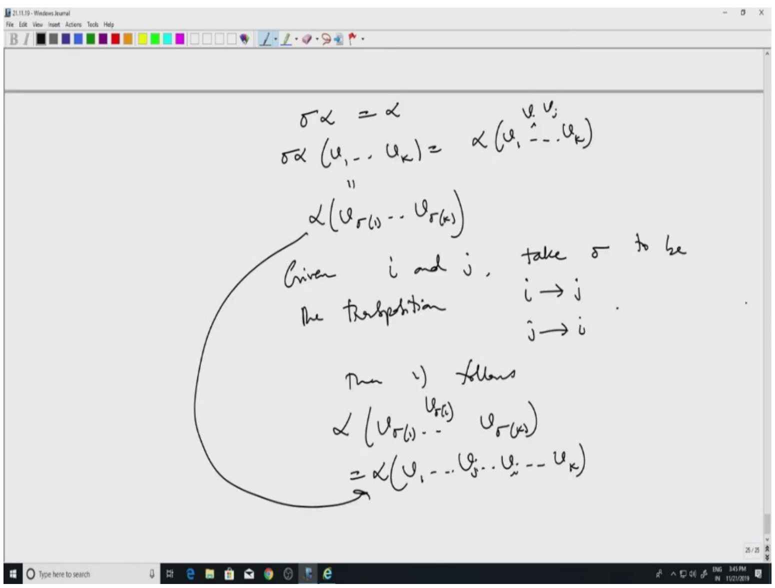

Now, I will just focus on 1 and 2, so rather than, it is not that, 3 is also not (anym), not any harder than 1 and 2. So proof 1 is equivalent to 2. Let us just see this one. Well, if 2 implies 1 is immediate, because 2 requires that, so after all what is 2? 2 is saying that sigma alpha equals alpha. This means, so if I evaluated on any k vectors, this should be this. But this thing here is by definition alpha of v sigma 1, v sigma k.

So this quantity should be the same as this for any permutation sigma. Now 1 is just that if I interchange any two vectors keeping the rest wherever they are, the value of alpha should not change. So to see that this condition follows from this permutation condition, just take a specific permutation, namely the transposition. Given i and j, these are the i and j which occurs in the definition of a symmetric tensor. Take sigma to be the transposition, which takes i to j, j to i and the others are left fixed.

Then 1 follows because alpha of v sigma 1, v sigma K would be, assuming i and j are not 1 or k, what one gets is, right, where was I? Yeah, so this is v1, now here there would be v sigma i somewhere here, but i has gone to j. So that would vj and similarly j is gone to i, vi and this vk. So the left hand side, so this thing here. is vj, vi this and the right hand side is vi, vj would occur here. vi, vj, assuming i is less than j. So essentially the idea is to take a transposition and then you get it.

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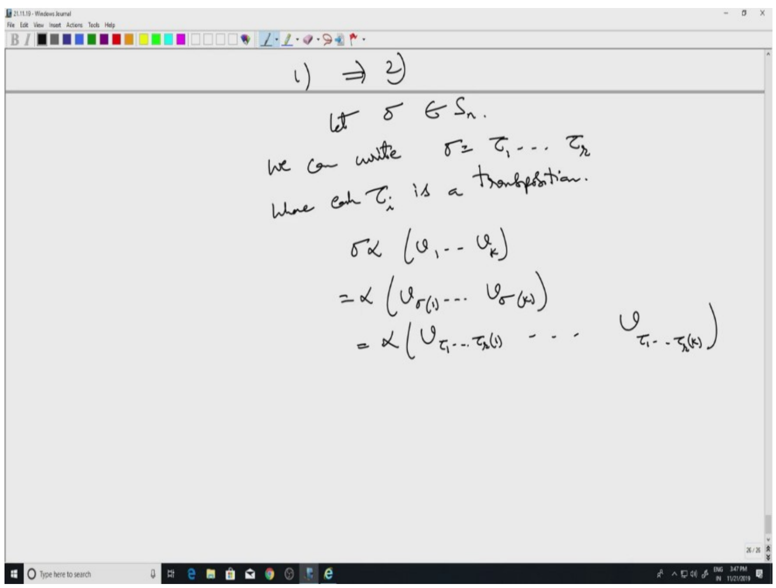

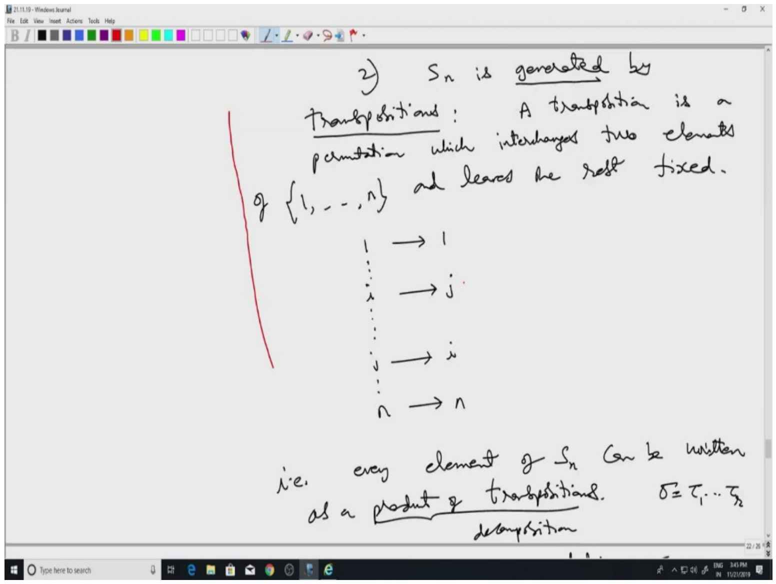

As for 1 implies 2, that if pair wise swapping does not change the value, we are given that. Now we want to know that permuting all the variables will still not change the value. So, essentially it follows from this second property that I wrote down here, namely that, Sn is generated by any permutation can be written as a product of transpositions.

So, if we use that, then it is clear that 1 implies 2. So let us start with any, some permutation. Let sigma belong to Sn. We can write sigma as tau 1, tau r where tau, each tau i is a transposition. So if I look at sigma alpha acting on v1, v2, vk, this is by definition v sigma 1, v sigma k. And this is tau v, tau 1, tau r of 1 et cetera, v tau 1, tau r of, right.

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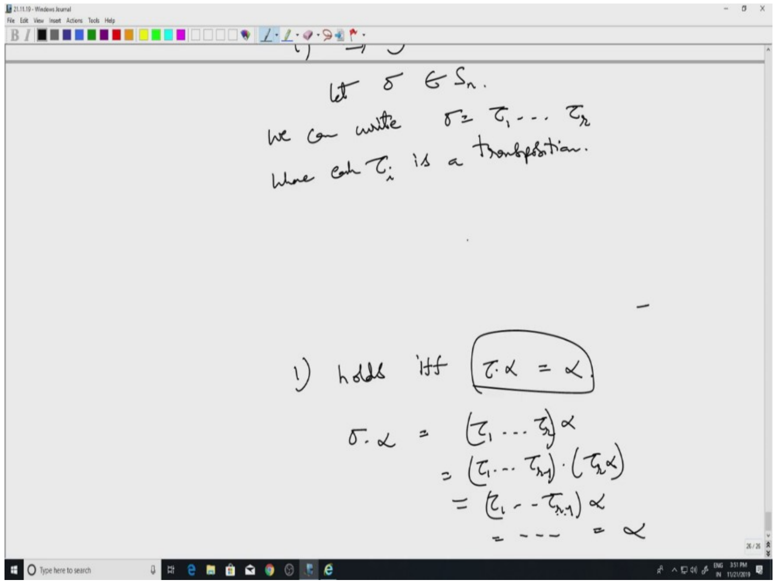

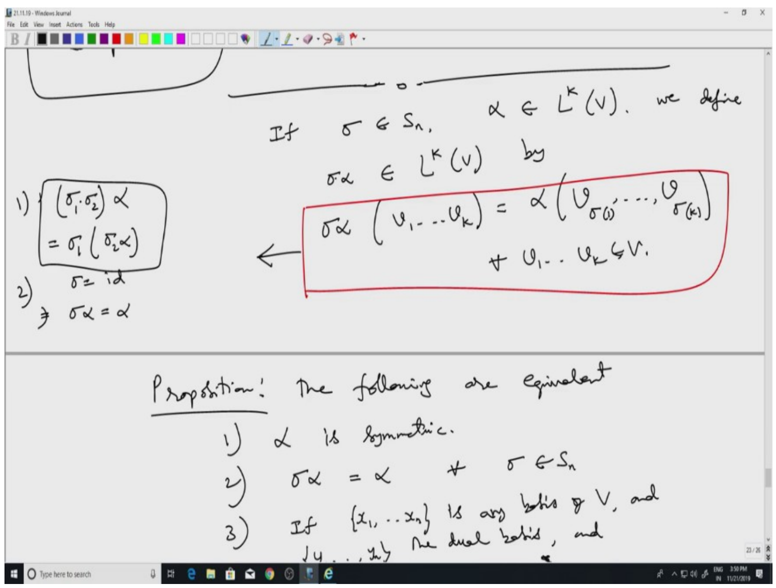

So, what, now the fact that one, one holds, now we are assuming one holds. One holds if and only if, as we have seen in the previous calculation, this calculation here, this calculation shows that one holds if and only if, if I start with any transposition, not arbitrary permutation. And if I do tau alpha, that should be the same as alpha. Actually, there is no need to write all this here.

So let me just, sorry, so, let me erase this. In fact 1 holds if and only if tau alpha equals alpha. But, so now we are interested in sigma alpha. We want to check sigma alpha is alpha. But sigma alpha is, sigma I have written as a product of transpositions tau 1, tau r alpha. Now, here I have to use one property of this operation that I, it is called, actually called group action property.

Namely, when I define this, this thing here one key (proper), two key properties are that, if I multiply two permutations, I would not prove this, but assume this. So if I do sigma 1 sigma 2 alpha, this is the same as, first I do sigma 2 alpha and then, a sort of associative property.

This is one thing and the second one is clear, which is if sigma is identity, the identity permutation, then sigma alpha is always going to be alpha. But this one requires a small check that if you multiply permutations first, and this is true, these, both these properties are true for whatever alpha you start with. You do not need to assume it is symmetric or anything, just the definition will imply these two things.

So if you are given this 1 and 2, rather what I am using is 1. So sigma alpha is, yeah, given that, then what I can do is, I can first do, instead of first doing group multiplication, first I can do r minus 1, Tr alpha. First do this and then do group multiplication. Now the rest, but Tr alpha is, now I use this, this is alpha. And again repeat the process. Now you do Tr minus 1 alpha and so on, repeat, after all finally you will get alpha. So both, so that holds.

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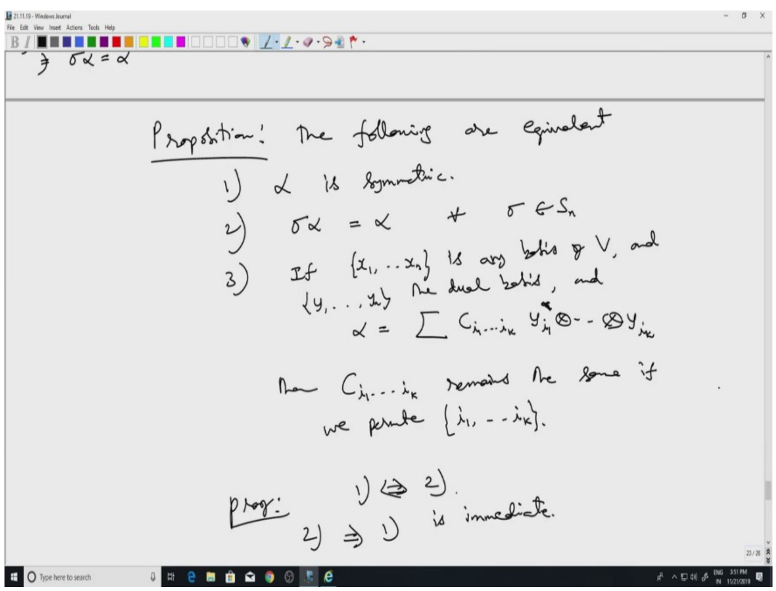

So, we have this, and the third one is also actually follows quite easily from the second one, second one implies the third one, so we will stop here.