An Introduction to Smooth Manifolds
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Lecture 47
Pull Back Form

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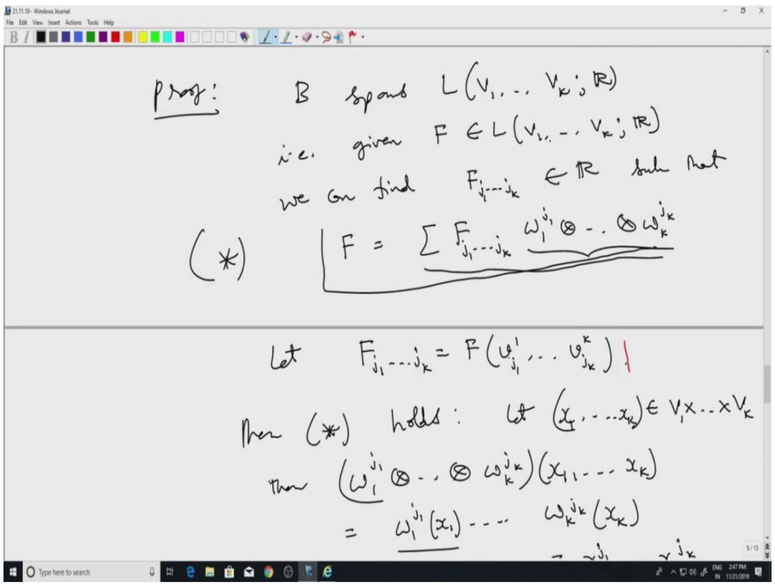

Welcome to the 47th lecture in this series. So, towards the end of the last lecture, we had constructed a basis for the space of multilinear maps to R, in terms of the dual basis of the underlying vector spaces. We will discuss, talk about some general properties about, of, and yeah, before I move on, so I should remark that, that the construction of the basis showed that the space of multilinear maps, v1 up to, from v1 up to vk to R, is actually just the dimension of v1 multiplied by dimension of v2 et cetera, dimensional. It is a product of the dimensions of the underlying vector spaces.

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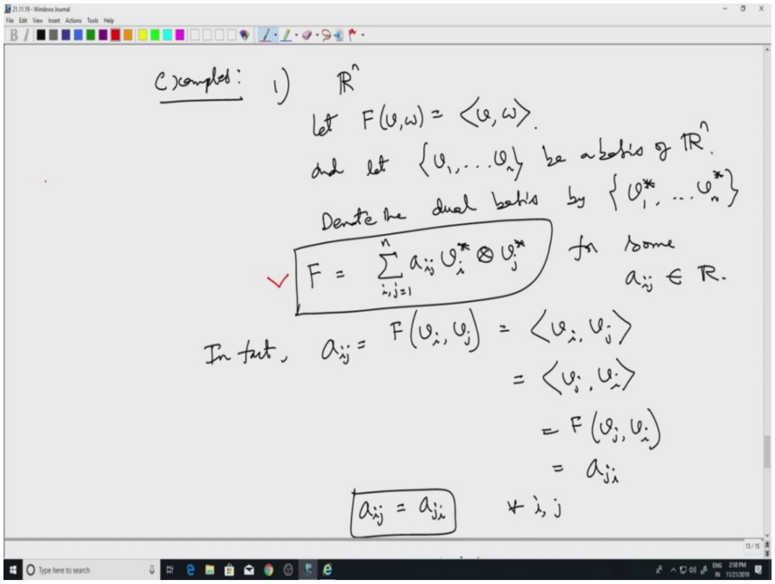

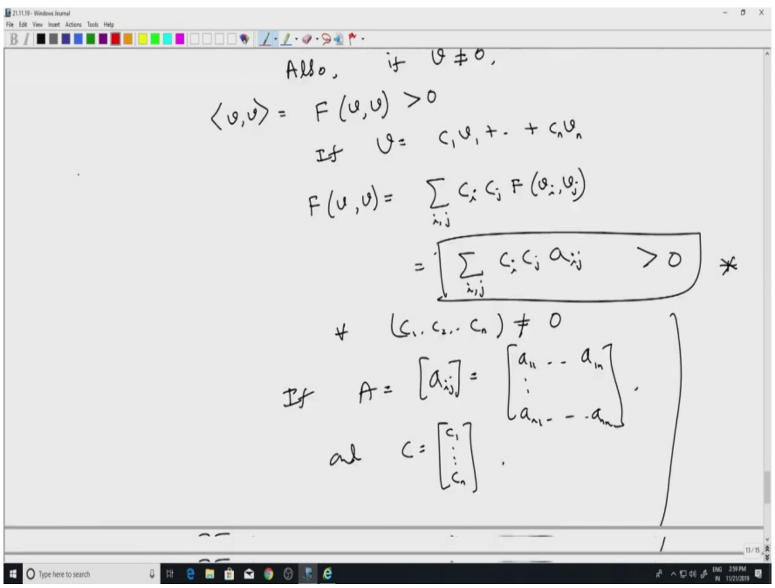

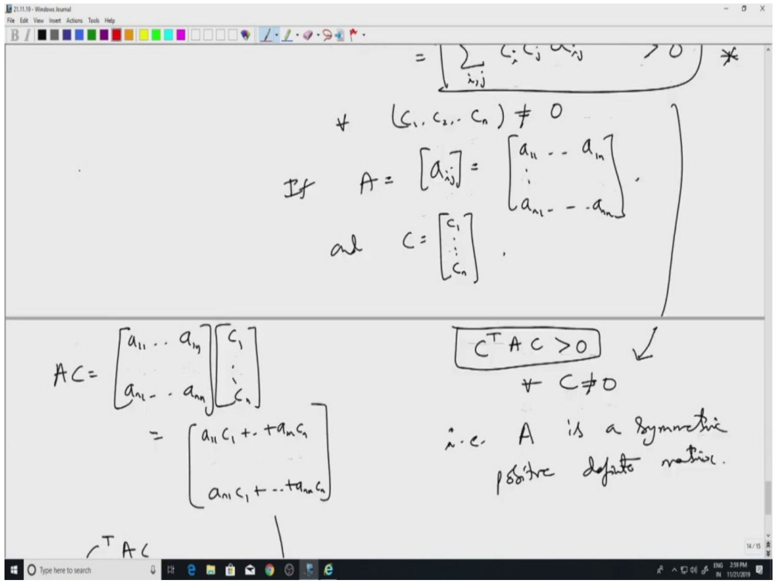

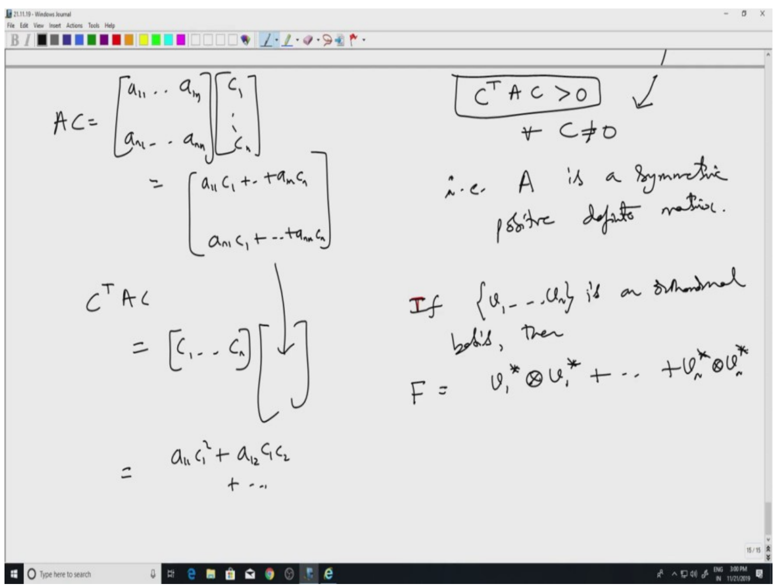

So, as an example of this, so let us see, so let us try to express, now that we know that this tensor products are one forms, form a basis, we can try to express some familiar multilinear form in terms of the basis. For instance, in Rn, let us take F of v,w to be the inner product of v,w which we talked about earlier. This is a bilinear form on Rn. So, let, so I can start with any basis of Rn. Denote the dual basis. Earlier I denoted it by omega 1 omega n, now I will just put a star. Dual basis by this.

We know that by the theorem, I can write F as summation aij vi star tensor vj star. Right, so here this i j equal to 1 to n. So, in general, so this is the best one can, if for a arbitrary basis, this is all one can say. I mean, I have, I really cannot use the fact that I am working with the inner product to say anything more specific. But let us say, for, so this is for some aij in R. The point is what are what are these aij's?

So, as I said, in general, for a general basis there is not much which can be said. One can say something which is that since yeah, actually the proof gave us exactly what this aij's were. They are obtained by evaluating F on corresponding basis vectors, F where. So in fact aij equal to F vi vj. Now, since this is the inner product, we started with the inner product, the standard inner product on R n, we know that we can switch vj and vi and so this is vj,vi. But F vj,vi is aji, the coefficient aji.

So, what we can, we can say a certain few things, namely that aij equal to aji for all i and j. And moreover also, if I, if v is not equal to 0, F since it is in a product F v,v is greater than 0. So this is just the product of v with itself is greater than 0. But if we write, if v equals, let us say c1v1 plus cnvn, then we can, this F v,v, as we are done earlier, we can expand it in terms of this is equal to ci cj F vi,vj summation over i and j, which is the same as summation over i j ci cj aij. So, the point is that this is greater than 0, since this is F v,v.

And this is, this should hold, this inequality should hold for all c1, c2, cn not equal to 0, not the 0 vector. So as long as these coefficients are not 0, in other words, I started with assumption v is not equal to 0. That is the same thing as saying the coefficients are not 0, so therefore I have this. Well, so this (in) inequality here is something familiar, which arises when one studies matrices.

So, if A equal to the matrix given by the coefficients aij in other words a11 a1n an1 ann and if you write C as a column vector instead of a (())(08:09) like this, cn, then this inequality star, this inequality star just tells us that look at A times C, this vector C, column vector C and multiplied by C transpose. When I do A times C, let us just check it. So A times C is a11 a1n an1 ann. This multiplied by C is the column vector given by a11c1 plus a1ncn, an1c1 plus anncn.

And if I do a further, so, so this is AC. C transpose AC would be, AC equals, so C transpose, c1 cn. Now it becomes a row vector multiplied by this column vector here and that will be, well, a11c1square plus a12c1c2. Actually it is just a number I do not even have to put this bracket etcetera plus dot dot dot. So, aijcicj is what essentially what one gets, cicj. So, this is, this thing here what I have here is just saying that C transpose AC is greater than 0 for all C not equal to 0.

In other words this aij matrix where aij is F the inner product of vi,vj is a positive definite and here we had symmetric, it is a symmetric positive definite matrix. So, when I, and in fact the converse is also true if I start with a symmetric positive definite matrix, I can define a bilinear form by this prescription here and I can define F by this formula here, then one can check that F is an inner product.

An inner product in the sense that it is, it is a bilinear form which is positive definite, which is as the usual (prop), the properties of the usual inner product. Well, that is one thing, the, but let us, if we choose, so here I started, started with an arbitrary basis, v1 v2 vn. However if I start with a, if v1 v2 vn is an orthonormal basis, then this inner product, this becomes, well, again one is working with the same formulae. But except that now, the aij which is F vi vj would be , if i is not equal to j, this inner product would be 0, if i equals j, the inner product would be 1. So, one would just end up with v1 star tensor v1 star plus vn star tensor vn star. Right.

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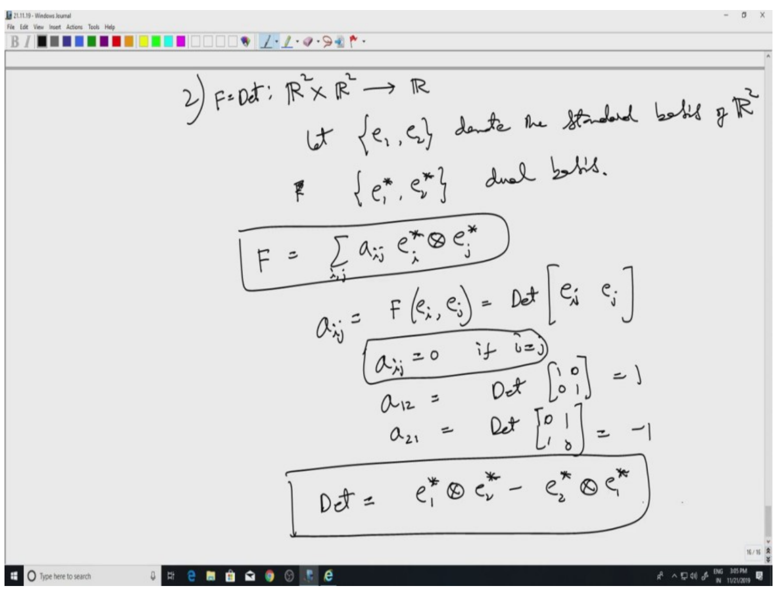

Now, the second example is, let us look at on R2. In the case of R2, by linear form on R2, so the determent as a function from R2 cross R2 to R. So, let us try to express. So this is my F. Again I would want to express F. So here, let us start with the, for instance I can have the standard basis, e1, e2 denote the standard basis of R2. Now, I would like to express this F as, we want to write F as, so yeah. And again, the e1 star, e2 star dual basis. So I know that F can be written as aij eij star tensor ei tensor ej star. So I, summation over i and j.

So, one wants to find out what these coefficients aij are this case, for the case of the determinant. So, we again go through the same process. So aij is just given by F of ei, ej which is determinant of the matrix given by e1, no rather ei ej, where I write ei the, as usual, these are written as column vectors. Now, if i equals j, then these 2 columns will be the same. Determinant is 0. So aij is equal to 0 if i equals j.

And i is not equal to j, well, that, then we get 2 possibilities. If i is not equal to j, there are essentially two things to consider, a12 and a21. Now a12 would be determinant of e1 which is 1 0, e2 is 0 1. This is just the identity matrix and this would be 1, while a21 would be, the first thing, first column would be e2 which is 0 1. Second column would be e1 0 1, 1 0, so this would be minus 1. So, what I end up getting is, now going back to this, I end up expressing the determinant as, so a, there are 4 (possibil), there are 4 terms in this sum. i and, i can vary from 1 and 2, j can vary from 1 and, j varies from 1 to 2.

So, but amongst these 4 terms, I already know that a11 and a22 are 0, because of this. And then a12 is this, so e1 star tensor e2 star minus e21 e2 star tensor e1 star. So we can write the, the determinant can be written like this. In fact, we are going to come back to this again when I talk about alternating tensors or multilinear forms.

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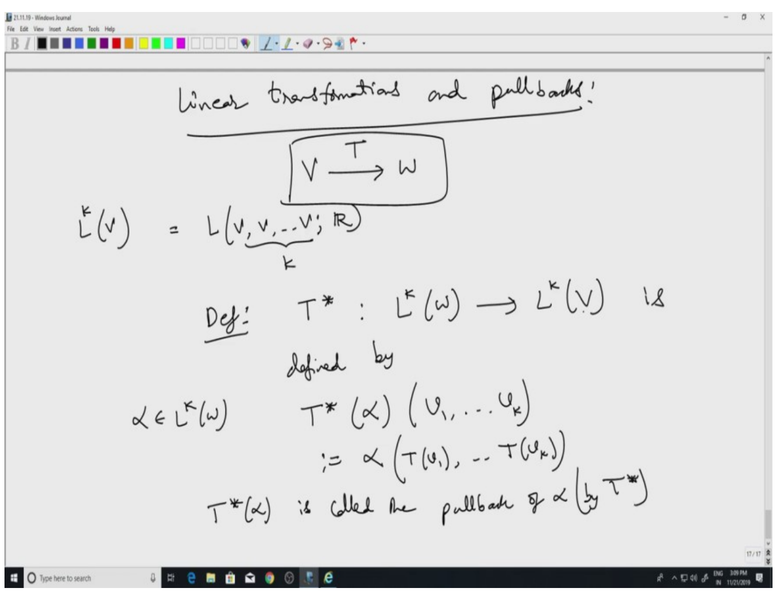

Now, before I do that, so let us, linear transformations and pullbacks. So here the situation is as follows. So let us say, a linear transformation from V to W. And I want to look at, so far I have been using this notation L. If I want to look at multilinear maps on V, I will have to take certain copies of V which I mean, the case we have been considering involve different vector spaces. Here, now I am going to assume all vector spaces are this, in the domain are the same. So I will be looking at V cross this space, so k times, k entries of these. So this, I will denote by L superscript V. So, and of course V can be any vector space, same thing for W.

So, now what I want to talk about is, that when I have a map of the underlying vectors spaces, I can get a corresponding map from the space of multilinear maps on W to V. So, define T upper star, called the pullback map. This is going to take Lk W it gets reversed. T is from V to W but this will take Lk W to Lk V, is defined by, so the input for T star, T upper star is going to be an element of Lk W, let us call it alpha.

And the output T star alpha is supposed to be an element of Lk V, so this should act on v1 all the way up to vk. And I define it to be, and here, so I started with alpha in Lk W. So, I define it to be alpha of T of v1, T of vk. So this is called T star alpha, is called, is the, called the pullback of alpha by T upper star.

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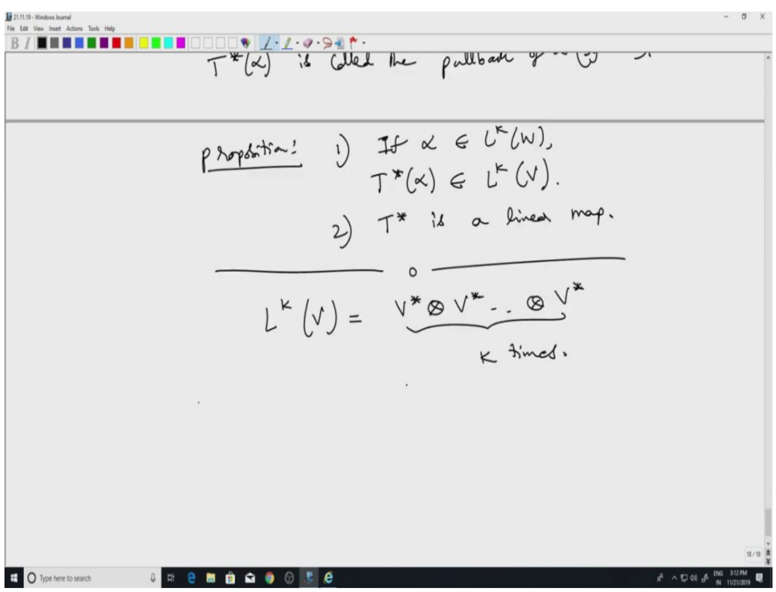

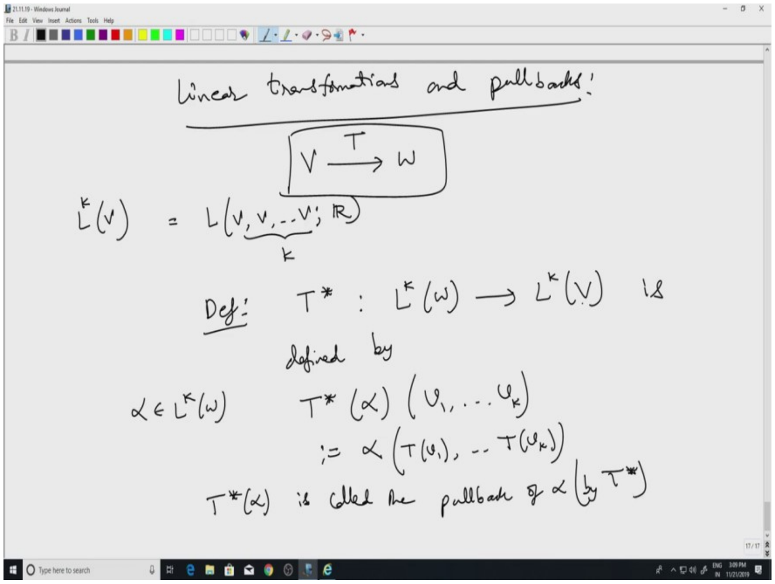

So, then small proposition. Right, before I do that, yeah okay. So I can just state it as a proposition. Now notice that I have written, what I have done here is, I just, so what T star alpha acting on v1 up to vk, I come up with a number. So this is once I give, we are given v1 up to vk. This right hand side here is a number. But what I really want to know is whether T star alpha is an element of Lk V.

So, that is in fact true. That is, in other words, it is multilinear. So first thing is, T's, if alpha belongs to Lk W, T star alpha belongs to Lk V. This is extremely straightforward, using the fact that this original map T was linear, so I would not prove this. And second, the second thing is that these L spaces, these sets Lk W and Lk V are themselves vectors spaces, as I remarked earlier and I have a map between two vector spaces. So one can ask if, whether T star is linear.

Again, very easy to check that. T star is a linear map from this space to this space. But, in fact, one can say more. So these are elementary observations. But, since we know that Lk V, we proved that this is actually equal to V star tensor V star tensor V star k times. And same thing for Lk W as well. One can ask what this T star does to tensor products.

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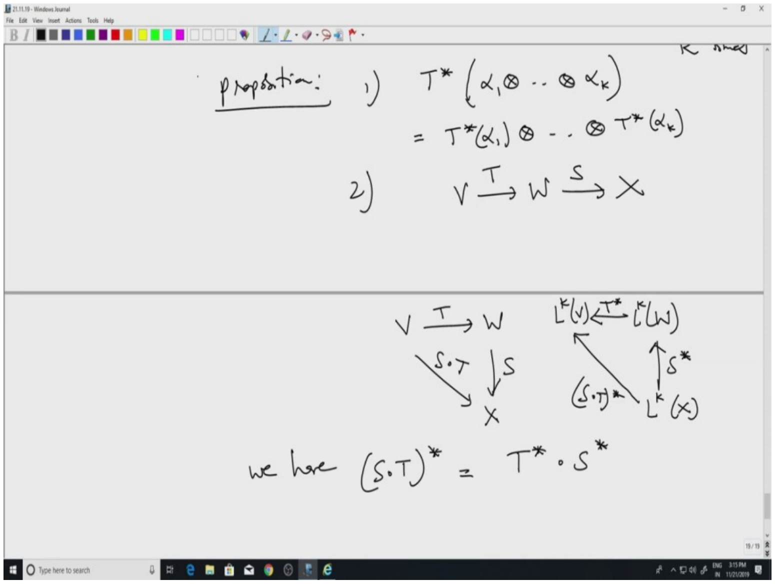

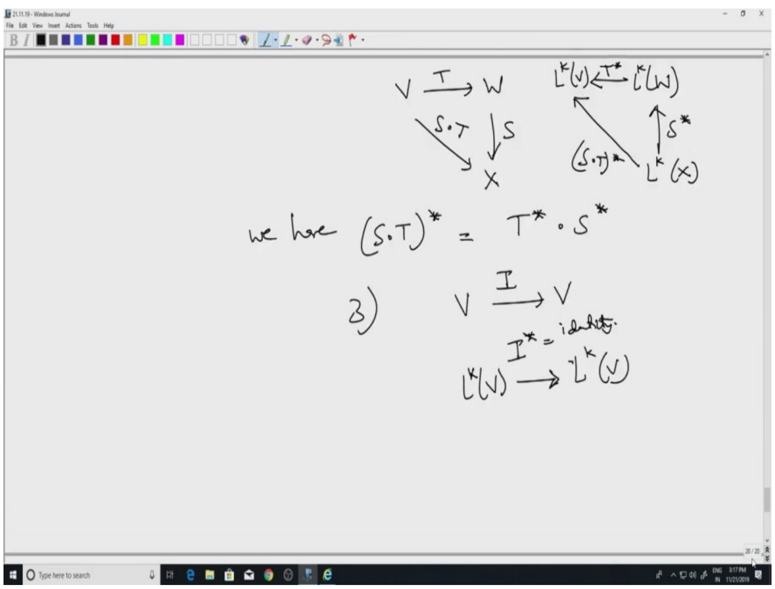

So, the second proposition is that T star Lk W is W star tensor again k times. So, if I take an, now an element of W star tensor W star is not as something of the form which is, it is a linear combination of tensor products. It is not, it may not itself be a tensor product, but I can look at (some) something like this. T star alpha1 tensor alphak is equal to T star alpha1 tensor T star alphak. These are one forms. This again is just a matter of going by definition. So I will skip this and more over, this more (imp), yeah, and equally important this, this composition law for T star.

So, if I have 3 vector spaces W, V to W to X, let us say. So this is T and this is another linear map S. One can of course compose S and T and get a linear map from V to X and then one can take. So let me write it, write it in a different way. So V W X this is S and this is T. So here I have S composed with T. Now, when I look at what the action on Lk, so I will have to go in this way, W's Lk W, Lk X and Lk V, so T star goes here and then S star goes here and then this is a S composed with T star goes here.

The question is whether going like this S star and then doing T star it is the same thing as S composed with T star and the answer is yes. So S composed with T star, we have S composed with T star is T star composed with S star. Again note that this order of composition gets reversed here. This was composed with T star but this has become T star composed with S star now. And another important property is that if I have the identity map, this is it actually quite trivial but one should mention this.

These 2, 2 and 3 sort of go together is (compar) together they make up what is called a (())(28:37) property of this pullback. So, if I have the identity map from V to V then the, this I star from Lk V, again to Lk V is also identity. All of these things are extremely straightforward to check, no, other than the definition nothing else is needed. So, we will move on and I will leave with you to check that these things are true.

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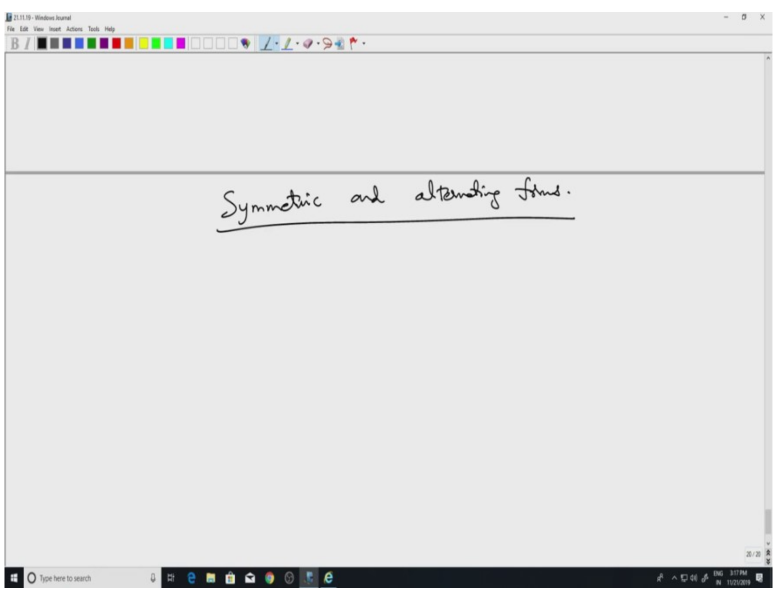

Now, I want to (mov) talk about (spe), two special classes of alternating, of multilinear maps, symmetric and alternating forms. So, perhaps this is a good time to stop and in the next class, I will begin discussion on these things. Thank you.