An Introduction to Smooth Manifolds

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Lecture 45

Tensors and Differential Forms

Welcome to the 45th lecture in our series. So today I will start discussing tensors and differential forms on manifolds.

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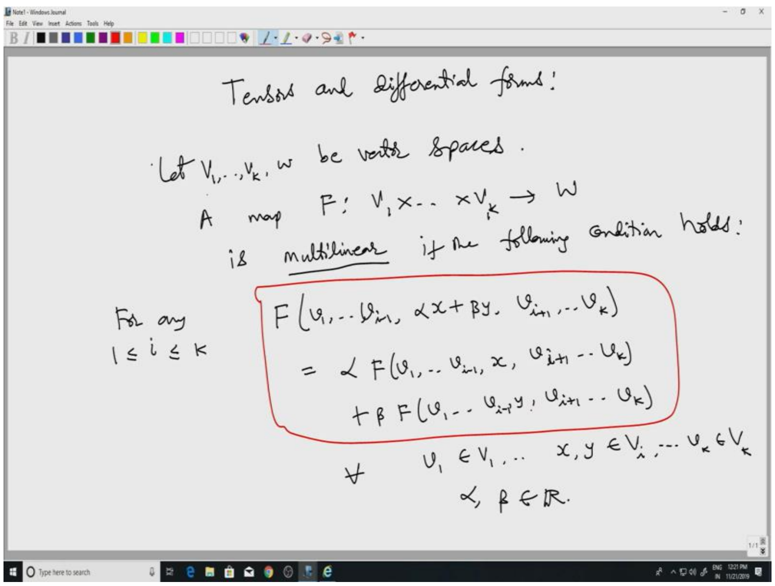

So, to begin with, let us discuss these concepts for vector spaces and then we will move on to manifolds, where the underlying vector space will be just the tangent space. So, let us, so let V be a vector space. To begin with, I do not need this to be finite dimensional. At some point I will assume that V is finite dimensional. So, let me be actually, yeah, right, in fact, what I will, to be more general, let us take, so let us take K vector spaces and in fact I need one more, sorry. So, K and I will also throw in a W, be vector spaces.

A map F from V1 to VK to W is multilinear if the following condition holds. What one wants is if I fix, if I fix K minus 1, so the input is K minus 1 vectors from different vector spaces. If I fix K minus 1 of these and just consider it as a function of the remaining slot then it is linear in that remaining slot. So, the way I write it is V1 V all the way up to let us say V i minus 1 and then in the ith slot, I take alpha X plus beta Y and then again Vi plus 1, then VK.

This should be equal to alpha F of V1, Vi minus 1 x Vi plus 1, VK plus beta F, Vi minus 1, y, Vi plus 1, VK. For all V1, so these belong to different vector spaces for all V1 and V1 etc. So, x, y in Vi. So here I should say, for any i in between 1 and K, I have this condition. So, Vi and then VK in dot, dot, dot. Then VK in VK alpha beta in R.

So, this equation is just saying that if I fix V1, Vi minus 1, Vi plus 1, VK and take a linear combination of two vectors in the, in some ith slot then it behaves like a usual linear map. And this should hold true for all the slots so. And as soon, as we will soon see there are lots of natural examples of this.

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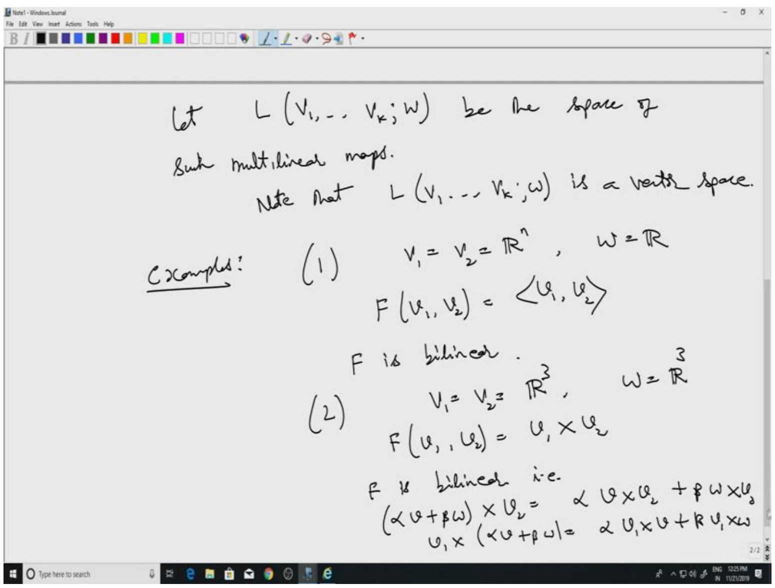

And let L V1, VK, W be the space of multilinear, such multilinear maps. So, notice that I have said space it is, since it is clear that if I have two such functions F and G, I can add them and get again another multi linear map. And if I have a scalar, I can multiply F by scalar and get a multilinear map again. Note that as a vector space throughout as we have been doing in this as has been the convention in this course, when I say vector space, it is always a vector space over the real numbers. So, the scalar field is just R.

So, note that this is itself a vector space. So here are some examples. So here I will look at V equals V2 equals Rn. W is just R. So, I am, and my F is going to be F V1, V2. It is just the inner product of V1, V2 or the inner product or the dot product. It is the same thing. So, the dot product of V1, V2. This we know is, this is multilinear. In this case when they are only 2, K is 2, we say that the map is bilinear.

In the second example, for the second example, let us again take V1 equals, well I do not take an arbitrary Rn, V1 equals V2 equals R3 and W is R3 again. And I define F of V1, V2 to be the cross product of V1, V2. This again is a bilinear map. So, this is again something that we know elementary coordinate geometry, that the cross-product, the way it is defined, it is i.e. alpha V plus beta W cross V2 equals alpha V cross V2 plus beta W cross V2. And similarly, in that, if I fix the first slot V1 cross alpha v plus beta W V1 cross V plus beta V1 cross W.

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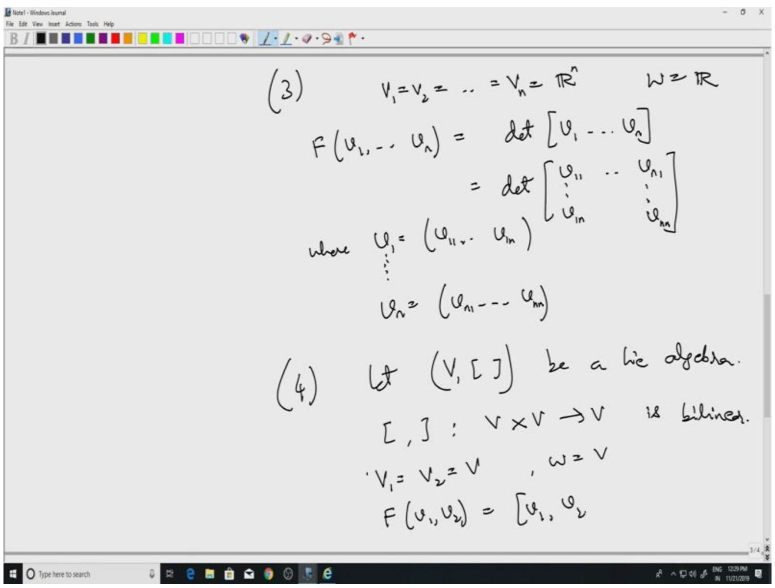

Another important example of a multilinear map is the determinant. So here V1 equals V2 equals Vn. I need exactly n and I have Rn and W as R. So, F of V1, V2, Vn, I defined to be determinant of the matrix obtained by, so as is the usual convention each of these vectors Vi can be regarded as column vectors, elements of Rn can be regarded as regarded as column vectors. So, I just write the columns.

So, this is determinant of V1 1, V1n, Vn1, Vnn where V1 equals V1 1, V1 and Vn equals, we will write it in next page, dot, dot, Vn equals Vn1, Vnn. This again, the fact that this map is bilinear is a restatement of the fact that when we take the determinant of a matrix, if we take a linear combination of two columns then the determinant can be splits as a sum in the appropriate way.

And the next example is, let V with a bracket Lie algebra. One of the defining properties of this Lie bracket, of a bracket on a Lie algebra as we have seen, is that this map from V cross V to V, this is bilinear. So, and so the, here, in with our notation V1 equals V2 equals V and W is also V and F of V1 comma V2 there is V1 comma.

Right, so for instance, this would give an example, here V can be an infinite dimensional vector space. For example, if V is the space of all vector fields on a smooth manifold then we have this Lie bracket and then this, that's, that itself is a bilinear map.

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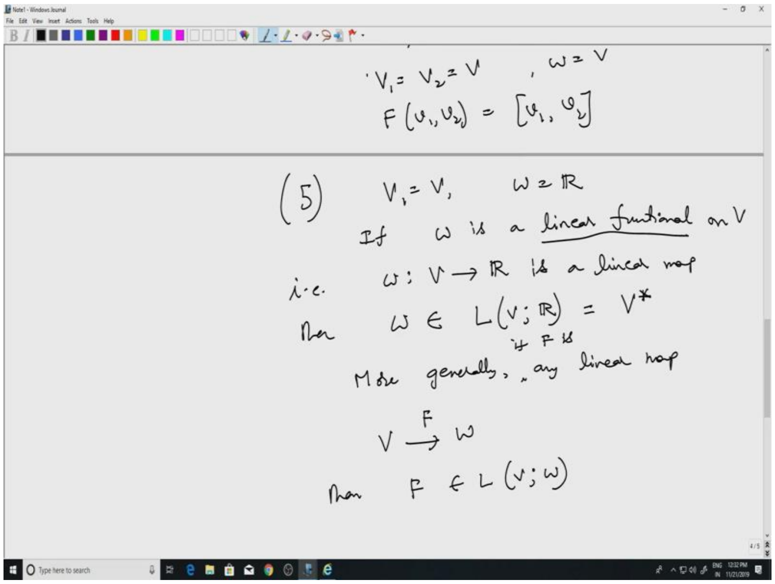

A trivial and not exactly trivial, but a familiar example of a multi linear map is actually just a, this would be just a linear map to R in fact. So V1 equals V, W equals R. So, in this case any linear functional, if alpha omega is a linear functional on, on V, then so, a linear functional i.e. omega from V to R is a linear map then omega belongs to L V, R.

And of course, any linear transformation, this is a special case. More generally, any linear map F from, more generally, if F is any linear map from, between two vector spaces, then F belongs to L V, W.

Right, now the goal is this. I want to focus on these, the fourth example, no, the, the case of linear functionals. So right here at this stage, I should say that this L V, R is also denoted by V star, the dual space. The space of all linear functionals is usually denoted by V star. And so, what I want to do is I want to use this, elements of V star and somehow combine them to get arbitrary multilinear maps.

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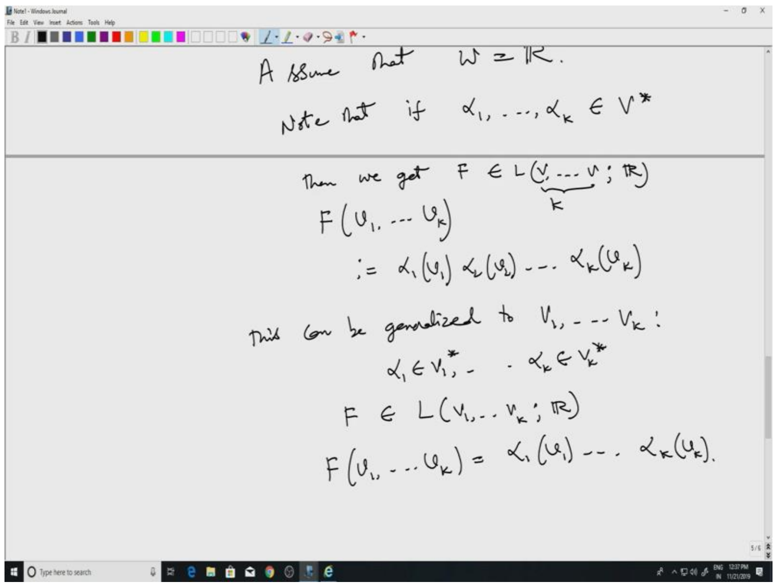

So, let us see how that can be done. So here I assume that, assume that W is R. So, we are looking at multilinear maps with values in R and let us notice one thing, note that if, if I take omega 1, actually maybe my, instead of omega let me use alpha. If alpha 1, alpha K are elements of the dual space of V, so V is arbitrary.

So, if I take any K elements of the dual space then we will get a, we get some, an F and L. Actually here, I do not even require, okay that is, it is just a motivating example, I do not even require all of these to be elements of the same, I can take different vector spaces as well. So I, but I will come to that shortly.

For now, let us just take K elements of V star. I claim that out of these, using these, I can come up with an element of the multilinear map on V. And this is done in a very easy way as follows. So, I define F of V1 comma VK. So here of course I should specify that K the number of vector spaces is K as well. So V1, VK is, I just take alpha 1 V1 times alpha 2 V2, alpha K VK. So, this is just the product, when I, this is just a product of real numbers. Alpha 1 V1 is a real number and so on. So, I am just multiplying this K real numbers.

So, this is the definition of F generalized to V1, VK. So, K different vector spaces. So, if I take alpha 1 in V1 star, alpha K in VK star, I get an element F then L V1, VK R by F of V1, VK equals alpha, the same formula. It is an easy exercise to check that in this prescription, this way of defining F, is actually multilinear.

So, and in fact it is quite straight forward. If I fix, for, for instance, if I want to check if it is linear in the first slot, I have to fix V to VK. When I fix all these V2 all the way up to VK, I just get a real number here this product and it just becomes a function of the function alpha 1 of V1. Well alpha 1 of alpha 1 is by definition a linear map. Therefore, it would be linear as a function of V1.

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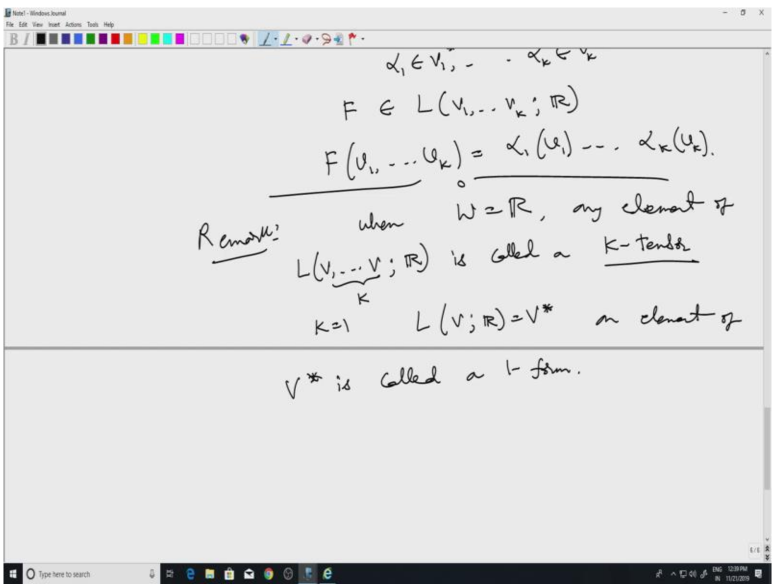

And the point is, yeah, this we have come up with a way of generating, generating multilinear forms. So out of, starting with multilinear maps, oh, I should actually mention one thing. Remark, when W is R in element of L V R K of this is called a K form. Actually, it is called a K tensor.

In the special case that K equals 1, this is just the dual space. In this case, an element of this, is called a, of course I can call it a 1 tensor. But it is also called a 1 form rather than a 1 tensor. And the reason why this suddenly switch, the change in terminology will become clear later on when

I talk about forms rather than tensors. So, an element of 1's, V star is called a 1 form and I will use this language.

So, what I have described now is using 1 forms, I was able to get a K tensor. In fact, then, but one can ask if this gives us all possible K tensors or more generally all possible elements of L V1, V all the way up to VK. So, that is in fact true. So, let us do that.

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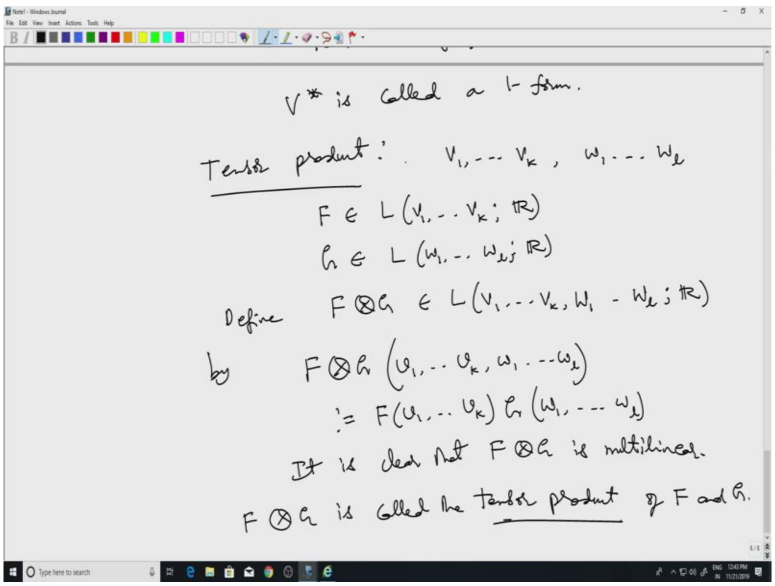

Before I move on to that and this operation is called the tensor product, what I just discussed, the way of obtaining. And it need not be, here I started with two 1 forms in the first take, the first case. But I need not start with 1 forms.

If I start with, so our vector spaces V1 up to VK, W1 up to WL. And let us take F and L V1, VK R G in L W1, WL R. Then I can again do the same kind of thing that I did earlier, namely just multiplying the values, that works here as well. So, I define, define a new multi linear map into R and this is going to be an element of L, the input as V1 up to VK, W1 up to WL and target is still R by F tensor G. This is going to act on, so K vector is coming from the Vis and L vector is coming from the Wi's.

So, this is V1, VK W1, WL. Maybe I should use W in capital, W1, WL. This is small w. So, define equal to, again as I said, it is just a matter of multiplying F V1 up to VK. This is a real number since F was an element of this, then G times W1 up to WL. It is quite easy again to check that this is multi linear. It is clear that F tensor G is multilinear. Right, so this is called the tensor product. The tensor product of F and G.

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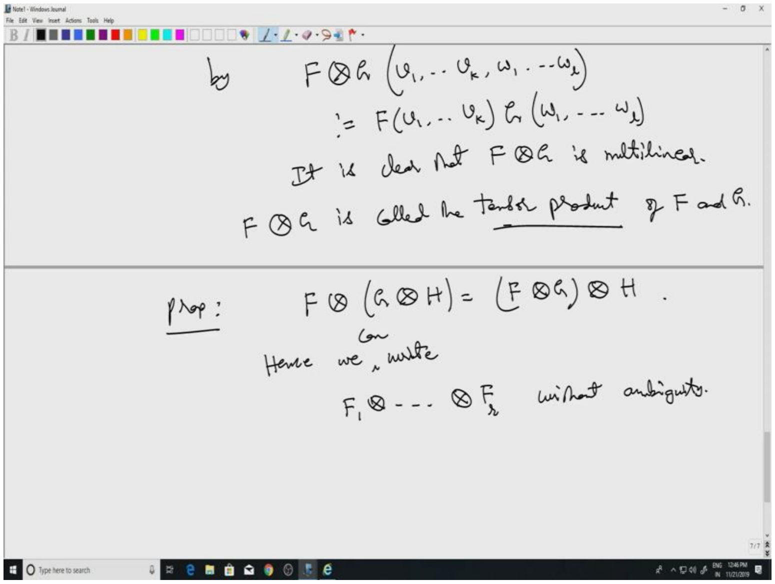

And let us note that, let us note one thing which is kind of important, but it is quite easy to prove. Proposition is that this operation, the tensor product operation, is actually is associative. So, if I have three things it becomes a bit cumbersome to write down. So, let me just write F tensor G tensor H equals F tensor G tensor H. So here with this notation that I had earlier, F is a multi linear map on a bunch of vector spaces Vi and G is on Wi. So here we can allow H to be on another set of vector spaces let us say U1 U2 Up and then this equation holds. So, I won't elaborate on this but let me just remark that this is quite straightforward.

So, this is immediate from the definition. And the point is that once we have associativity, we do not have to, when we have multiple tensor product, we do not have to worry about which one we are going to do first. Hence, we can, we write, suppose I have, here I just had three. So, if I have a whole bunch of them, let us say F1 Fr without ambiguity. So, if I have a multiple tensor product then I do not have to worry about which one I do first pair wise. Because this will tell me that whatever way you put brackets here, it does not matter you get the same answer.

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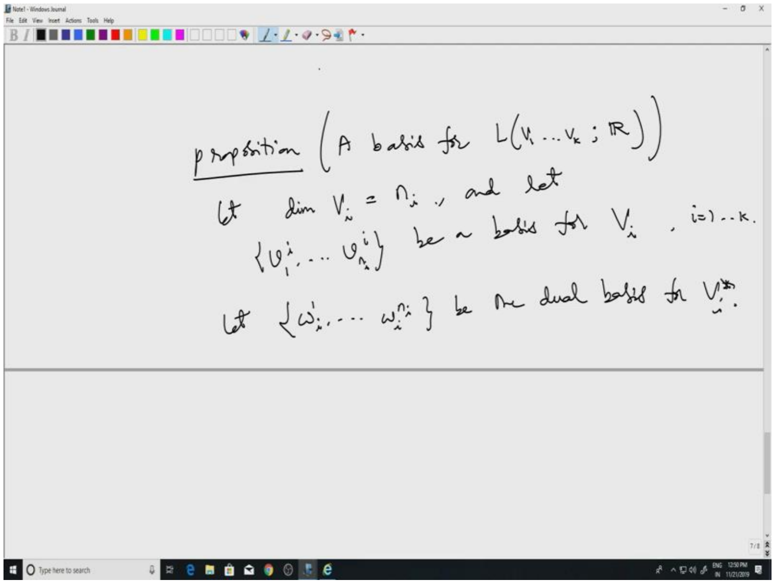

And in fact, this, finally we are going to not deal with the most general setting of these different vector spaces V1, VK,W1, WL and so on. We will just stick to the following case so all the V1 equals VK equals and just least V, a single vector space V. And similarly, all that Wis will also be V etc.

So, what we want to do, now here is the main proposition. This is going to give us a basis for L V, V R. In fact, let me just stick to, let me just, sorry, let me just change it slightly. So instead of taking all the vector spaces to be the same, what I will do is I will keep them different but this WIs that I had here, they will all be W1 equals V1 etc. WL equals VK. So, L equals K and so on. So, I will just deal with when I am going to take tensor product, I am just going to assume that all the multi linear maps have the same domains basis for this.

Let dimension of Vi equals Ni. So here I am assuming finally that these are finite dimensional vector spaces. And let V1i, Vnii be a basis for Vi. So, i equals 1 to K. So, I have this K vector spaces, for each one I choose a basis.

So, I keep track of which vector space by the superscript i and once I have a basis for Vi, once I have a basis for Vi, I get a corresponding dual basis. Let omega 1, now the i becomes a subscript. Ini be the dual basis for the dual vector space V star. So, let me stop here. The goal is, in next class I will, I will discuss how using these omega i's and then the tensor product operation, I can get a basis for L V1, V 2, VK. So, we will stop here. Thank you.