An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science Bengaluru
Lecture 44
Frobenius Theorems

Hello and welcome to the 44th lecture in this series.

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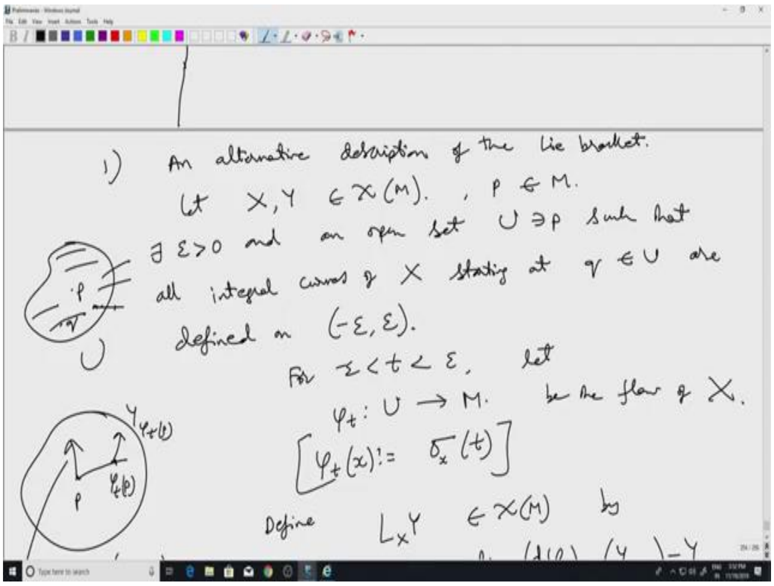

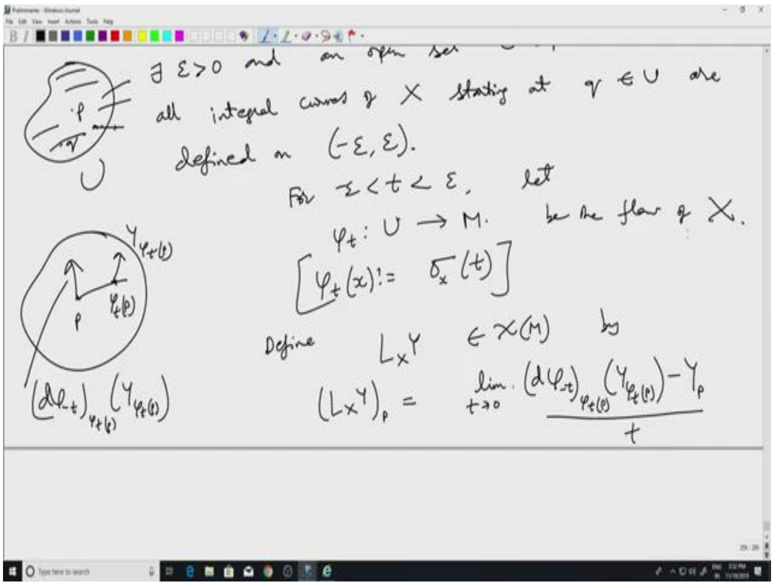

Last time I had stated with a description of the lie derivative of one vector field with respect to another. So this was done using the flow, so I have two vector fields to start with X and Y. I use the flow of X and actually the derivative of the flow map to pull back the vector field Y at some point back to this base point P and then compare with this YP and then take the difference quotient and then take limit as t goes to 0.

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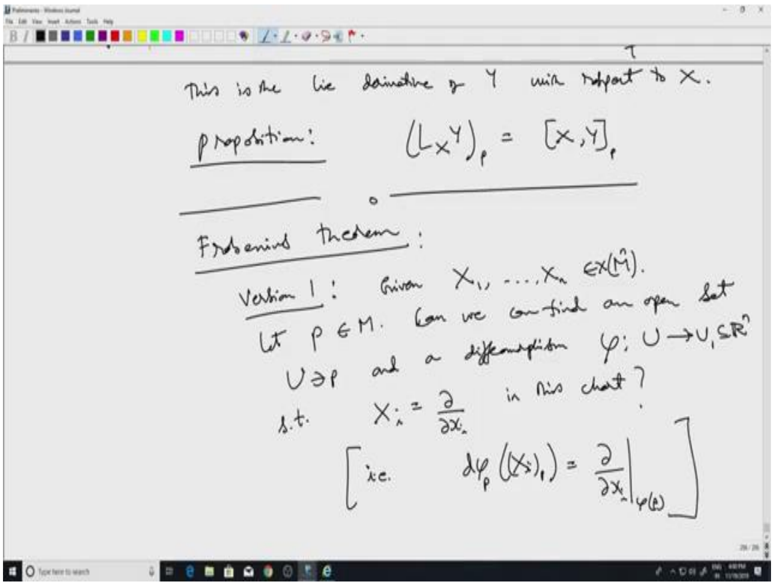

So this is called the, this is the lie derivative of Y with respect to X, even to make sense of this definition I would of course need to know few things that, I would need to know things like phi of minus t composed with phi of t is the identity and so on. So one, this the definition I have given here one has to be do slightly more, look at it bit more carefully to make show that everything is properly defined and so on. But all that can be done and any way I am not going to go into details about this. So let me just leave it at this.

So the proposition is that the lie bracket of XY at P is the same as lie derivative of Y with respect to X at P is the same as the lie bracket of X. On first site this equality is not obvious at all because, when we define the lie derivative of Y with respect to X, X and Y seem to play quite different roles. After all we are using only the flow of X and pulling back Y not the other way round.

While the lie bracket definition X and Y more or less have the same roles except that one comes with a minus sign, the other comes with a plus sign. There are two terms but the roles are more or less the same. So here and in fact that is also reflected in things which are obvious for the lie bracket or not, so obvious for the lie derivative if we do not have this equality.

So for instance the fact that the lie bracket is antisymmetric. In the context of lie derivative is not clear at all that LXY is equals to minus LYX. If we have this equality, then it becomes clear. Similarly, the fact that the lie bracket is bilinear in the first variable X is not clear in the lie derivative. So there is something to be done here to prove this but I will not go into the details of this.

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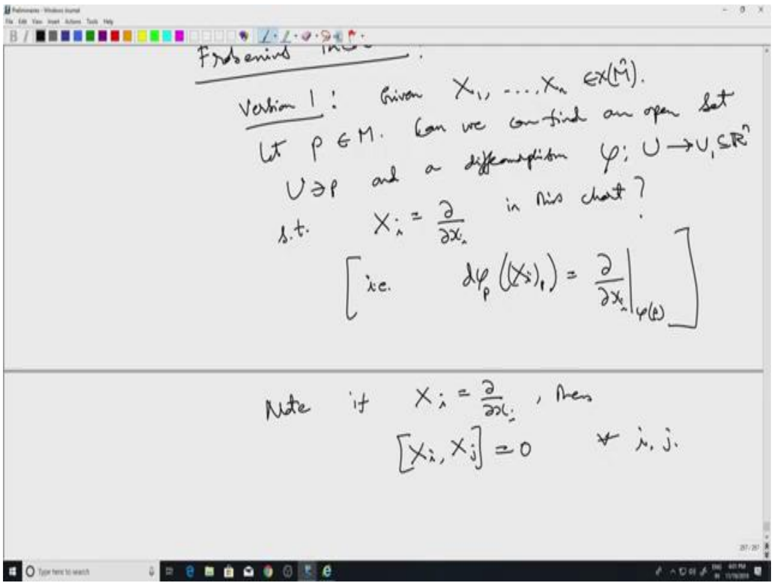

Rather what I would want, what I want to talk about is the final topic regarding the lie brackets in vector, lie bracket and more generally vector fields is this so called Frobinius. So there are two versions, so the first one is version 1. Here what we want to do is, here again the lie bracket plays a crucial role in this, both these theorems and the, in version 1 one asks the following question.

We know that when we have a coordinate chart we get a frame on that chart. In other words, we get n linearly, n vector fields which forms a basis at each point in the chart. So we ask for the converse. Given n vector fields on a manifold, manifold or an open subset of a manifold it does not matter, given n vector fields. So the manifold dimension is n as usual, when can we, so given this let P belong to M. Can we find an open set U containing P and a diffeomorphism phi from U to U1 in Rn such that Xi equals del by del Xi in this chart?

So in other words, of course as usual del by del Xi means the, del by del Xi denotes the pull back under the derivative of phi inverse of the usual del by del Xi operators in Rn. So those, can we find a chart so that these vector fields, just starting with some vector fields such that this happens I can write it like this, so the question is about the existence of this diffeomorphism phi.

So actually, let me be more explicit, i.e., d phi of Xi at any point Xi at P is del by del Xi at phi. Well can we find an open set under diffeomorphism such that this happens in this chart. Now we know that these, if such vector fields, note that if Xi can be written as del by del Xi then since the del by del Xi in Rn commute this the lie bracket of del by del Xi and del by del

Xj is 0 in Rn same thing should happen for this Xi. Then Xi, Xj equal to 0 for all i and j. This turns out to be the crucial thing.

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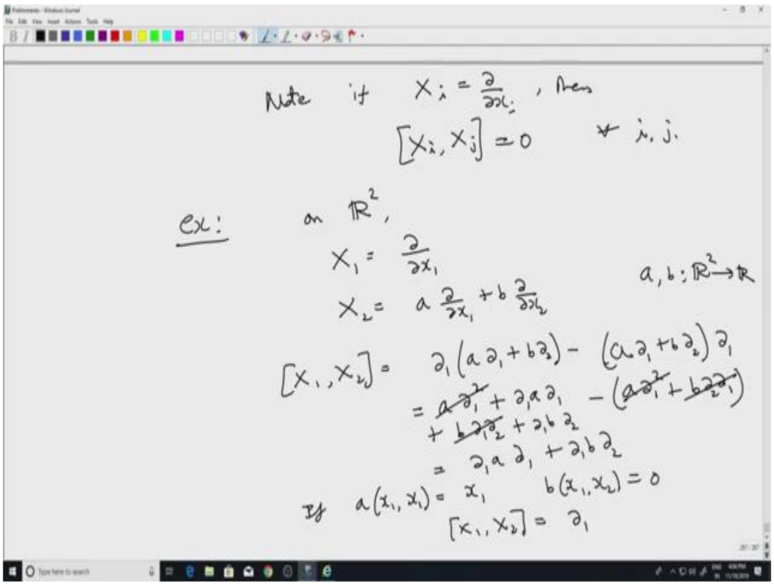

So this is a, this gives a necessary condition, not all, it is a, so as an example it is quite easy to write down two vector fields, so on R2 suppose I take X1 equals, well I just take del by del X1 and X2 I take it to be some function let us say a times del by del X1 plus b by del by del X2. Then the lie bracket of X1, X2 so here a, b are functions on R2 which has 2 short.

So it is again using the earlier notation it would be del1 of a del1 plus b del2 minus a del1 plus b del2 of del1, this will give me, we have done this computation once but anyways it is a times del1 squared plus del1 a del1 plus b times del1 del2 plus del1 b del2 minus this one just gives me only two terms a times del1 squared plus b times del2 del1.

Now this thing here will cancel, a del1 squared will cancel off with this and b del1 del2 will cancel off with b del2 del1, so finally you are left with del1 a del1 plus del1 b del2. All I have to do is just choose a function, so if for example if a of x1, x2 equal to x1 and b of x1, x2 is identically 0 then the lie bracket is would be just del1 a which is 1, del. So the point is that, the lie bracket is not 0.

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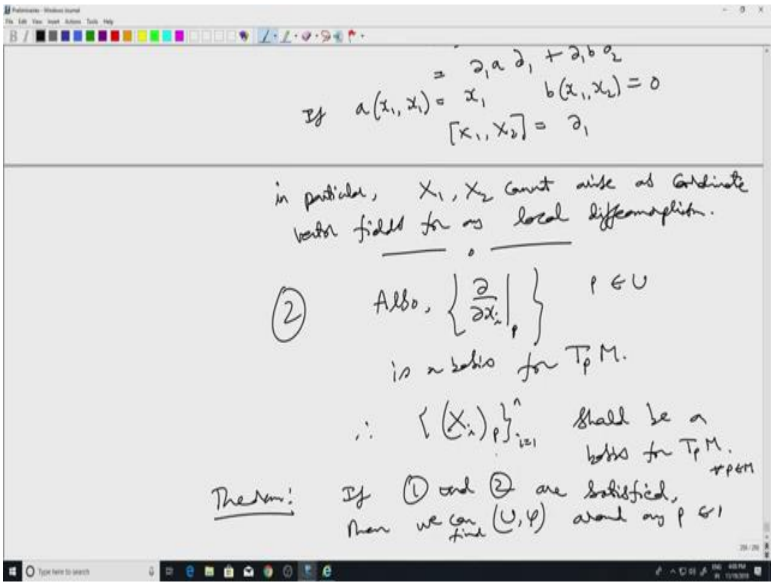

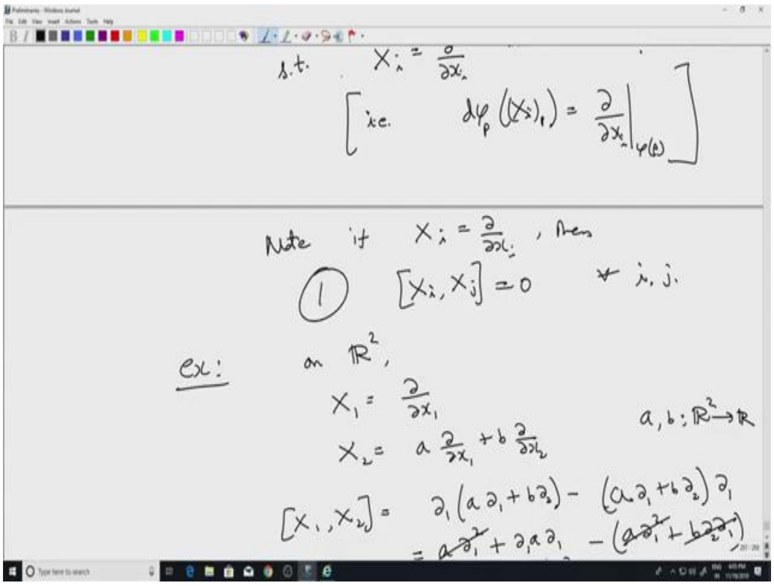

Well, so in particular these two cannot arise as coordinate, in particular X1, X2 cannot arise as coordinate vector fields for any local diffeomorphism, locally defined diffeomorphism for any chart. Now, so this is one restriction on X1, X2 that the lie bracket should be 0. There is another restriction, so this is the first one. The second restriction is that, also we know that this del by del Xi at any point P, P in U is a basis for TPM.

Therefore, but if these vector fields happen to be coordinate vector fields then it is necessary that these things Xi at P are equal to 1 to n should be a basis. In other words, this the (raw) exactly n on them, same thing as saying that they, there should be linearly independent or that the span, either one will do should be a basis for TPM. So in other words there should be what we called frame.

So these vector fields should be a frame, should form a frame for an open set and the lie bracket, there wise lie bracket should be 0. Then the at least some necessary conditions, 2 necessary conditions are satisfied. But the question is are these 2 the only necessary conditions or them to arise as coordinate vector fields? The Frobenius theorem says, yes these are both necessary and sufficient. If 1 and 2 are satisfied, should be a basis for TPM, so here for all P in M 1 and 2 are satisfied then we can find a chart.

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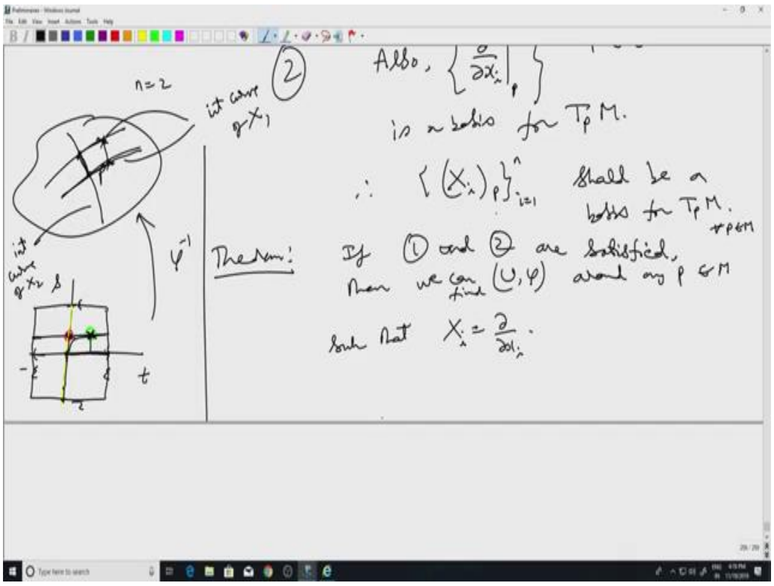

We can find a chart U, phi around any P in M such that Xi equal to del by del Xi at this chart. And this is a useful thing to know in various situations and I should remark that the way proof of this goes, it is actually uses a flows and lie derivatives. It is a, the proof is and how the lie bracket comes about is, so the proof, this is not even an outline but a very big sketch of how one would go about trying to find, what on means attempting to do is, so let us say we want to find this, the goal is to define this map phi, find an open set U and a map phi such that this happens.

Well, so let us try one thing, so here is a manifold and here is a point P. Now, let us take the case n equals to 2 dimensional, so here I just have 2 vector fields. So now we know that around P all integral curves of X1, we can find a neighbourhood such that all integral curves are on P are defined in this thing that for at least time epsilon. And similarly we can find by shrinking the neighbourhood if necessary we can assume both X1 and X2 have integral curves defined for some common epsilon, shrinking the neighbourhood and taking a smaller epsilon if needed.

So in some neighbourhood both integral curves are defined. So, let us, one way to find this diffeomorphism would be, so let us call these two as S and T. So this is the T variable, this the S variable. So what I will do is I map this X axis to the integral curve for the X1 vector field passing through P. So the 0 goes to P and this part at least for everything is defined all the way up to minus epsilon, epsilon here to minus epsilon, epsilon.

So this part I map to this curve and now if I want to, the idea is enough. Now, suppose I want to get hold of other points, so I want map this entire square in R2 into the manifold. So actually instead of finding phi, I am trying to find phi inverse, so I am going to define phi inverse. So first I map this central line which is the X axis, now if I want to map another horizontal line, so I have to start at some point here.

And again the idea is to take the integral curve through the corresponding point here but where should this point go? So I want to send this somewhere, then natural thing to do is just map this Y axis to the integral curve of the second vector field X2. So like this, so this yellow line segment is mapped to, so this is integral curve of X1 and this would be integral curve of X2 with the initial condition both of them pass through P.

Once I have the integral curve of X2, I can try to fill out and see where all elements of this square go to because I just look at the horizontal line passing through that. So then map this to the corresponding again integral curve of X1 passing through this point. So this is also an integral curve of X1. So one can certainly try to do that and one can define this in terms of flows, so I flow along.

So if I want to see where this point goes to, I use the flow of for Y2 and move here and then move and the use the flow of X1, sorry not Y2, X2 use the flow of X2 to move here then use the flow of X1 to move here. But a problem arises, because here the same, in R2 the same point this red rather I have already used red so this green point can be approached this curve lands me here or rather I can go like this and go like this as well to the same curve.

If my map is to be well defined, then I will be forced to the conclusion that, forced to have the condition that moving along the flow of X2 and then X1 should be equal to moving along the flow of X1 and then X2. This condition that the flow of X1 and X2 should commute, it turns out as precisely equal to the condition that the lie bracket of X1 and X2 is 0 and way one sees that is via the lie derivative.

At all ties in up, that ties up very nicely but the idea of construction of coordinates is rather simple via this is picture that I have drawn here. After one shows that the map is well defined, then it is not that difficult to check that this phi actually gives a diffeomorphism from this open square, phi inverse gives a diffeomorphism from this open square on to an open subset of the manifold.

And of course the original goal of getting hold of Xi as del by del Xi is automatic because after all we are dealing with integral curves. So this condition when I differentiate, this coordinate vector fields here are going to the standard coordinate vector fields in R2 are certainly going to X1 and X2 here because we are working with integral curves.

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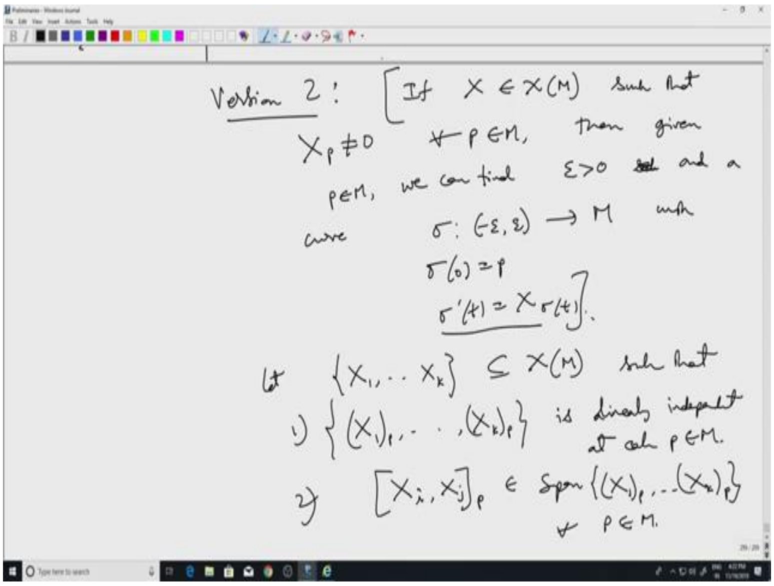

So that is one portion of Frobenius theorem, the other one has to do with integral of manifolds. So this is the chart version, version 2. Here we ask the following question, so let us first note that if X is a vector fields on M such that XP in not equal to 0 for all P in M then given P in M we can find epsilon greater than 0 such that, epsilon greater than 0 and a curve sigma of minus epsilon to epsilon with sigma 0 equals P and then sigma prime t equals X at sigma t.

In other words, all I said, written here is that we can find an integral curve passing through that point. Now, so what would be, so this is something that we already know, this is not the theorem, so what would be the case? Suppose I have two vector fields rather than one vector field, so rather finite number, let X1, Xk be a subset of vector fields, be a set of vector fields on M. Now this condition, for one vector field we had the condition that it does not, it is not 0 at any point.

We saw that if it is actually 0 at a point, the integral curve through that point is just a constant map, you do not get a one dimensional, you do not get a curve. So now here suppose I have this is inside this, b contained in (())(26:52) of M, such that the condition that XP is not equal to 0 we will replace it by, so that one X1 is linearly independent at each P in M. And now the, here there is a big difference between this one vector field and higher dimensional case.

So the goal is, so here a curve we like to regard as a one dimensional sub manifold. So what the one vector field case tells us is that if you have vector field which is non 0 I can find a one dimensional sub manifold which is tangent to the vector field. That is what this is saying, actually it is, actually it is equal to the vector field.

If you have bunch of finite number of vector fields, then we want to know when they sort of they form, the vector space they form at each point, when it is the case that this subspace of this vector space is actually tangent to a sub manifold. So the condition is that, such that this happens and second condition is Xi, Xj at any point P belongs to the span of X1P, etcetera, XkP for all P in M.

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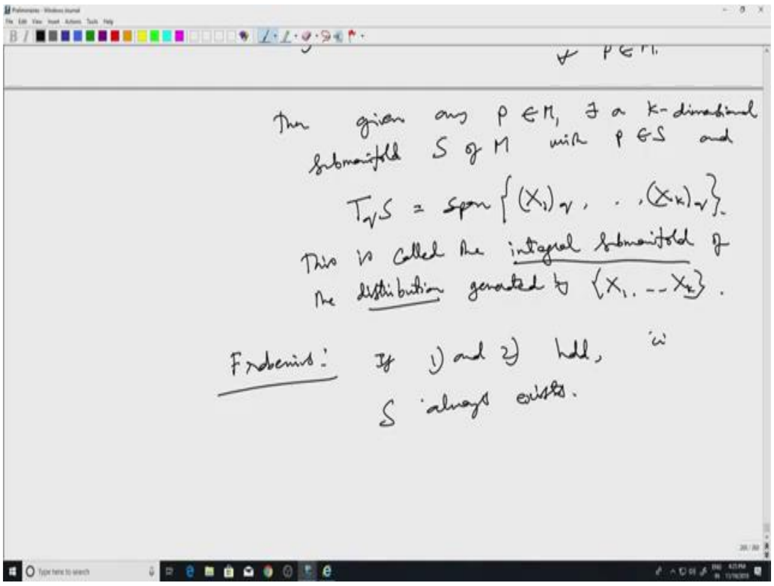

Then given any P in M, there exists K dimensional sub manifold S of M where P in S and the tangent space to S at any point is actually just the span of these vector fields X1q, Xkq. So this is and the sub manifold will turn out to be unique as well, this is called the integral sub manifold of, well let me integral sub manifold of the distribution, that is the technical term of the distribution generated by X1, Xk.

So the geometric idea is simple enough, just that given, if you are given a set of vector fields, we want to know when exactly there and we know suppose they are linearly independent at each point, so the dimension of the tangent space this span is fixed.

We want to know when there is a k dimensional sub manifold and the Frobenius theorem tells us, I will end by this the Frobenius theorem, again a necessary and sufficient condition is that, actually I have already written the necessary and sufficient condition. So, if 1 and 2 hold, S always exists. You can always find such as sub manifold. So we will stop here and in the next lecture I will talk about introduce differential forms. Thank you.