An Introduction to Smooth Manifolds

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Lecture 41

Lie Algebras of Matrix Groups (part 1 of 2)

Hello and welcome to the 41st lecture in the series and I had to stop in the middle of a computation last time.

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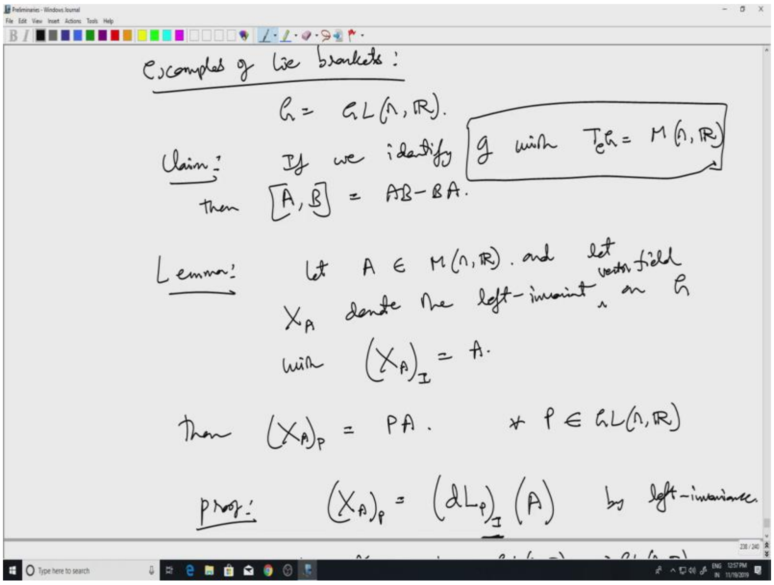

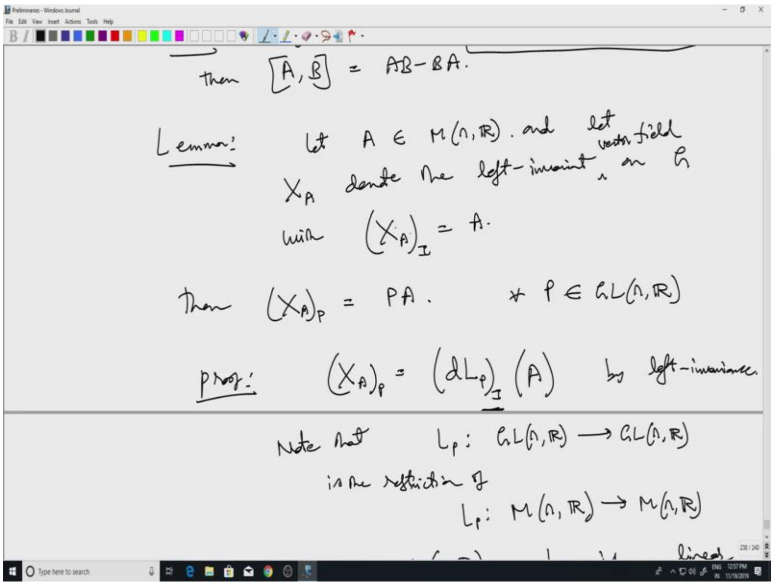

So, I was trying to calculate the Lie bracket on the Lie algebra of G L n R. And we wanted to see that if we identify the Lie algebra with the tangent space at identity which is M n R then the Lie bracket of the 2 left-invariant vector fields corresponding to A and B is just the matrix Lie bracket A B minus B A.

And the for that, I started with this lemma which describes all left-invariant vector fields on G. Namely if we start with A in the tangent space at identity then the left-invariant vector field corresponding to A which I denote by X A is just given by it is value at P is PA. And we computed, we did the computation of this.

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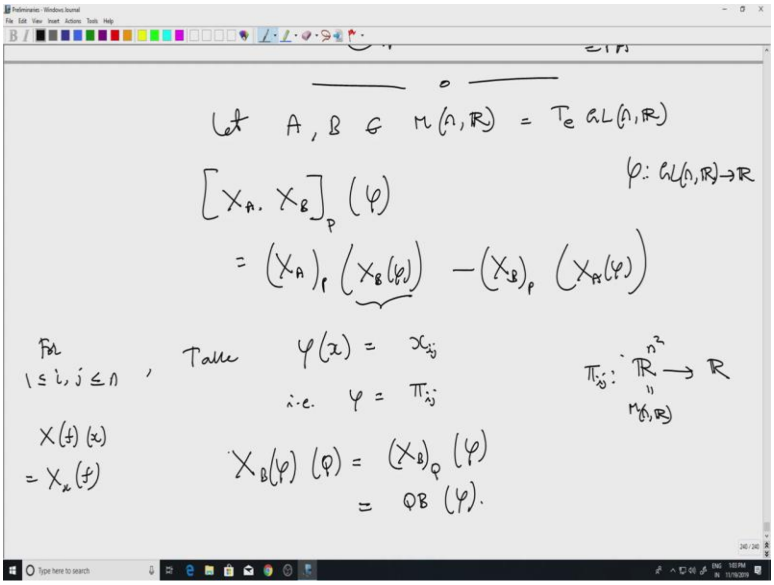

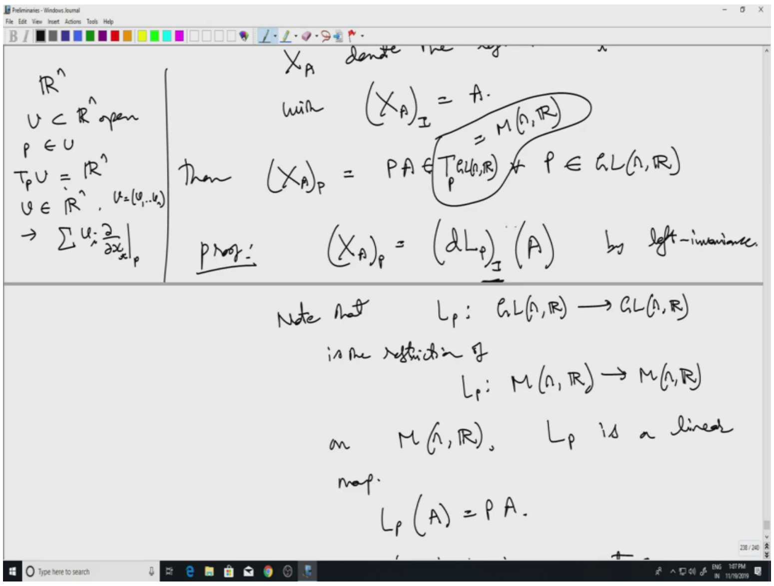

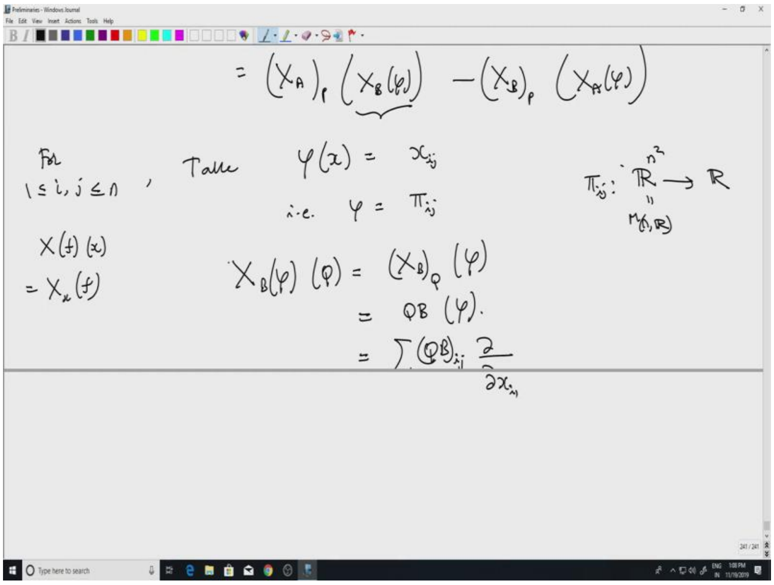

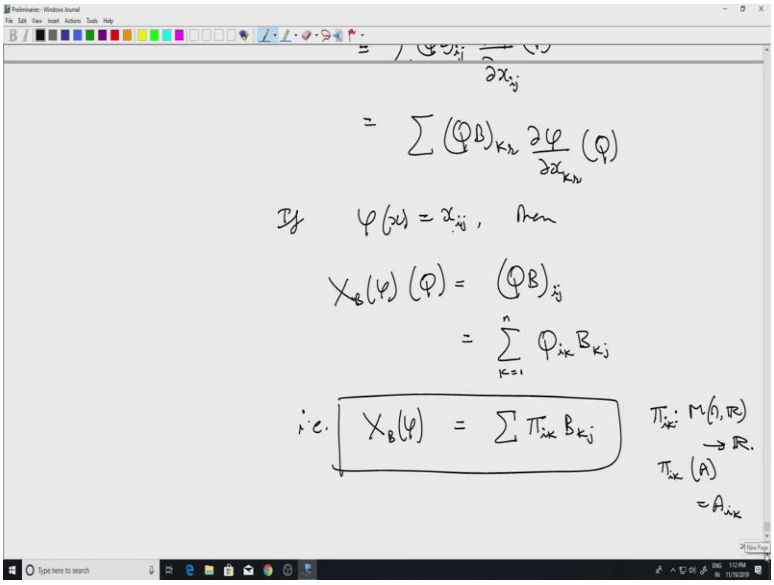

So, now let us proceed with the main computations. Now, let us come to Lie bracket. Let A B belong to M n R and let us keep in mind this is tangent space at identity of G L n R. So, we are I will continue with the same notation. X subscript A, X subscript B. Now what I want to do is yeah, let us do XA XB. So, I want to understand this left-invariant vector fields, I want to see what it is value at a point P is. And I will act it on a C infinity function phi.

So, phi is a C infinity function on G L n R on the manifold. So now this is, I will just go by definition, so this is XA at P acting on XB phi – XB at upon P XA acting on phi. So, one wants to understand this XB of phi. Before that now what I would like to say is that instead of considering all functions phi, what I will do is, it is enough I claim that it is enough to understand it is action on.

So we will restrict ourselves to take phi to be the functions phi of x equal to just xij. So actually, I should say phi i j rather than because it is dependent on i j. So here I will say, for i j line between 1 and n take phi equal to phi of x equal to xij i.e. phi is the projection map, phi equal to Pi ij. So, this is a projection map from Rn2 to R. The this is M n r. So, I will project to the ij'th co-ordinate. I mean this is yeah well, with the appropriate numbering.

So, the it is enough to take, so let us see what happens with these the action of X A X B on these things xij. So, for that, now let me look at this term here, rather not even that. I will just look at the thing inside the brackets, I am just going to look at this X B phi, remember that this thing is supposed to be a function on the manifold G L n R, so I want to understand it is value at a point Q.

And we know that this by definition is remember that X f at x is X, this was the way the function arose. So, this is the same thing as XB at the point Q acting on phi. Now we know exactly what XB at Q is. This is by the lemma that we just proved, this is P A, this is Q, in this case it would be Q B acting on phi. Well when we say Q B acting on phi, one has to keep certain conventions in mind for instance here when I said XA at P is P A. now A is just an M n R element and P is G L n R.

So, the product is again just an element of M n R. So, this is just an M n R. However, this is supposed to be thought of as an element of the tangent space at T, at T of G L n R. Which is still M n R, this is still M n R but except that partial (deri), so essentially so for to make sense of this, what I wrote here, I will just write in aside. So, V is a vector space, we know that the tangent space at any point P of V, actually and here in to be more, even more explicit.

I will take an open set, just like this here G L n R is open inside M n R. Let me do that here as well. U contained in V open, P in U. We know that the tangent space at any point is equal to V. Now this equality means (tha) means a following. So, if I start with V in V, the corresponding element of TP U is, also in fact let me just take Rn Euclidean space for clarity.

So, instead of dealing with any arbitrary, after all we are in the realm of Euclidean space, standard Euclidean space here. So, Rn here and then Rn and here too it would be Rn, oops. Rn and Rn. So, but if you start with any V in Rn, if I the tangent vector at the point P is would be so I write V co-ordinates, V1 V2 Vn then this goes to the Vi, the derivation given by del by del xi.

The main thing here where P occurs is in the derivation part. So, I evaluate that del by del xi at the point P. So, even the I mean the this V in Rn of course has nothing to do with P, but this isomorphism will keep track of the point P in this way. So, it is Vi del by del xi at P. So, here too when I said T P G L n R is M n R, it is a same thing here. So, if I start with an element of M n R, I get a derivation at a point P via this prescription.

So, Vi del by del xi at P. So, let us do that. Here of course the 1 is dealing with Rn square rather than Rn. So, for instance when I right Q B at phi, one has to remember that this Q B is supposed to be a derivation at the point Q. After all this would this, I am supposed to get a tangent vector at Q. Let us do, so this is for any phi. Now I will plug phi equals, phi of x equal to xij. Namely the projection map pi i j.

So in that, but first let me right the general expression, so this is Q B, the co-ordinates of this element of Rn2, QBij. It is exactly like what I wrote here, the same thing that I am, whatever I wrote here, I am just using the same thing here. And it would be del by del xij of phi. Now the main point here is that this is evaluated at the point Q. So, so this, since I am going to us i j for this function phi, I do not want to use the same i j as summing index.

So, let me rewrite it in terms of some other indices k r del phi by del xkr of Q. Now, let us do this business of taking specific functions phi and see what we get. Let us take phi equal to xij. If phi of x equal to xij then all the partial derivatives of phi will be 0 unless this index pair k r is exactly equal to i j then and only then will I get this to be, then I will get it to be 1 otherwise it will be 0.

Then XB phi at the point Q will be exactly Q B at the point, at the ijth entry of Q B, the matrix Q B, because all the terms here will vanish. So, which using the definition of the matrix product I can write as Qik and the Bkj, k equal to 1 to n. So, in short, another way of writing this is i.e. XB of phi, remember Q is supposed to be the variable here and we are trying to understand this function XB of phi.

So, this, if I do not want to mention the variable it is XB of phi is, now this function Qik is just the projection, the ikth projection acting on Q. So, this is pi ikBkj, so I get something like this. So, pi ik is the ikth projection map from M n R to R. So, all it does is, if I input the matrix A, it gives me the ikth entry.

So, this, so fine that is what we have and then finally let us return back to, so this was just one term we had picked out. We are trying to understand this term here, of course if you understand this, you understand the second one as well, the computations for this will be exactly the same as this. So, now that we have a formula for this, I can do one more derivation action and let us see what we get.

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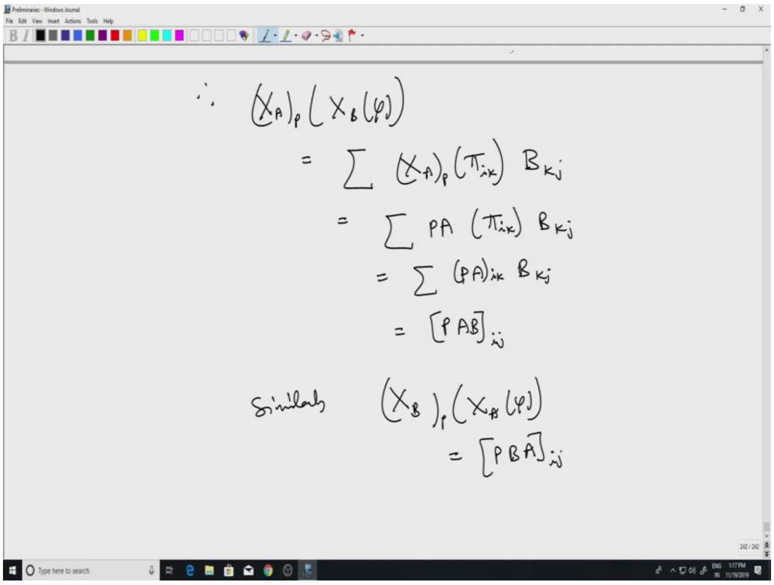

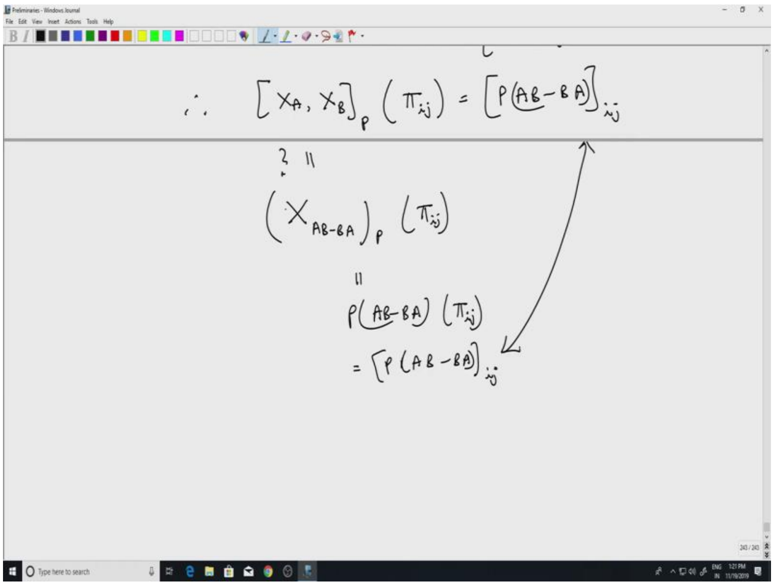

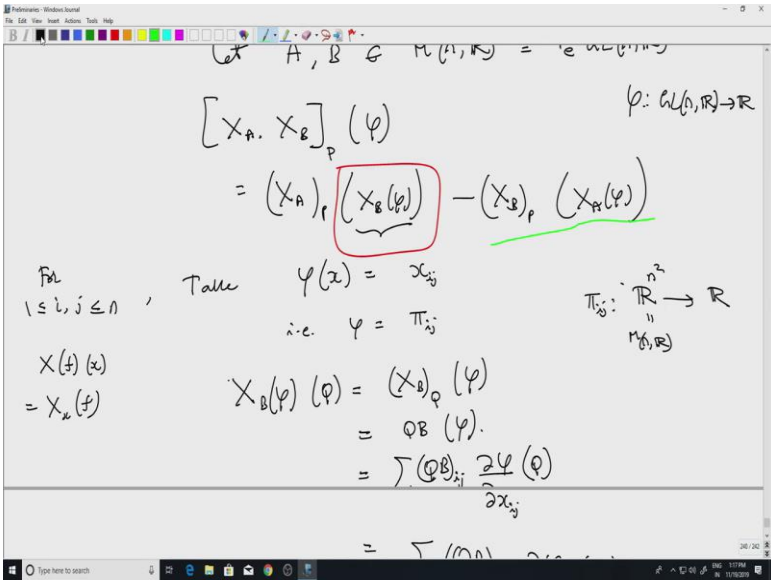

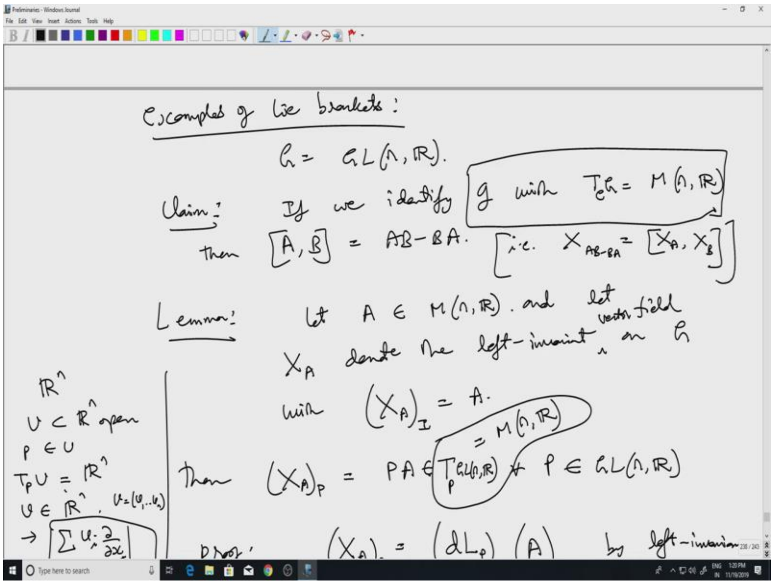

So, with this in hand, so XA at P acting on XB phi. Yes, of course all this was done for the specific the simplification was done the specific function phi equal to x, so phi the function phi is pi ij. For this thing, the computation turned out to be this. And let us continue with the same function phi. And so this will be again, so whatever I have here so it is XA at P.

So essentially I just plugged this expression here inside this and take XA inside the summation sign. So, it will be XA at P acting on pi ik, Bkj is a constants any way, (())(16:30) Bkj. And here again I will use the fact that XA at P is nothing but PA of pi ik Bkj and by the same logic that we went through here, here that this XB at.

So whatever we did for this Q and B, now we again do it for P and A, because we are again now acting on a projection map just like we did here. So ultimately, will the same logic will leave us with, here we were left with, when we acted it on this we were left with this expression pi ik as a function but the only difference is that, between this computation and here is that we are just interested value at the point P, rather than, we do not have a function.

So we just take a specific point and we will get this expression here. So, what I will get is P A at i k because here, this is the part I am using. Similar thing here and so I will get P A at ik and then I have Bkj. And I will get so this thing Bkj is the same. So and this of course is the ijth entry of the matrix P A B, the ijth entry of this matrix P A B.

Now if I do the same the same thing, same considerations will apply for the second term here except that A and B are switched. So, it is clear that the second term will give me similarly XBP X A phi as PBAij. In fact, I do not have to do the computation, I just, this A and B were arbitrary, so the same thing will apply here as well.

So, finally what one gets is, therefore, the Lie bracket of XA XB at P acting on the special function Pi ij turned out to be, so the ijth entry of P A B minus B A this, so this is what we get. What we wanted to say is that we want this to be actually equal to, so that this is, I will put a question mark here. Is it the same thing as the, we wanted to claim that this is the same as the left-invariant vector field corresponding to the matrix product XP – XA at P acting on pi ij, because that's the statement, the original statement was that.

In fact, going the original statement, which I wanted to say, when I said under this identification, if we identify G with this then this happens in the language of this XA business it is a, i,e, XAB-BA equal to XA XB. So, this is what we are trying to prove. So we have this equal to pi ij. Now this, so I want to know whether this is equal to this and I have reduced it to this expression and we know that this is equal to P A B minus B A.

After all we have very explicit description of the left-invariant vector field like this and this is supposed to act on pi ij. And we already seen this when a matrix left-invariant vector field when a matrix acts on a, when a left-invariant vector field acts on the special function pi ij what one gets is essentially QBij. And here it is PA-BAij. It is a same calculation that was involved in this green box here. So we have used it twice. So and this and this are the same.

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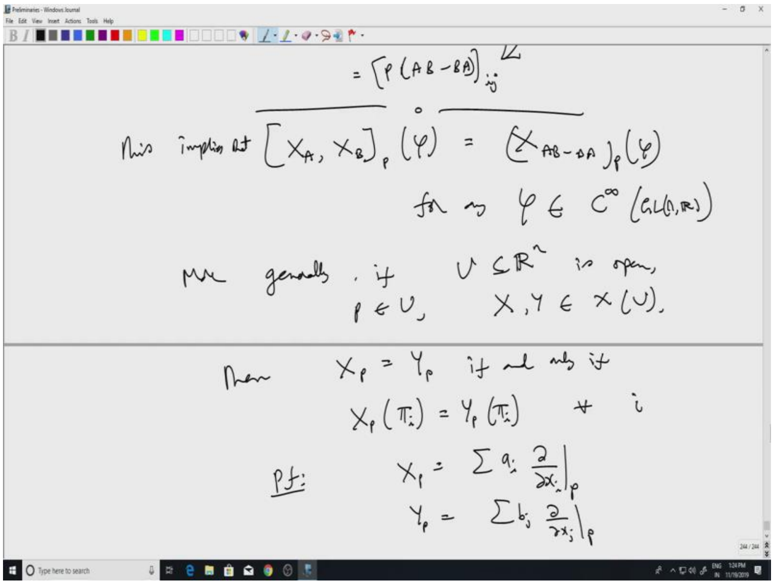

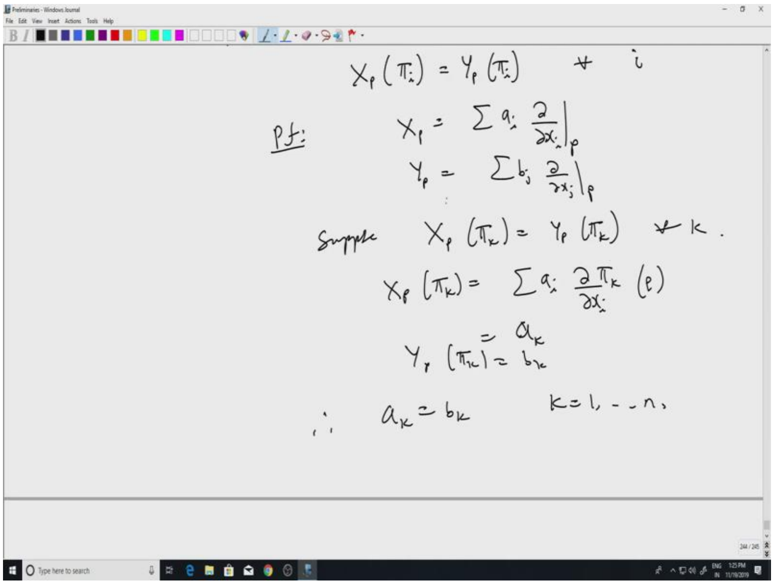

So, but however, what we have done is we have checked it just for these XAXBP acting on this function pi ij for any I n j we are getting the same values. What about the arbitrary functions? So, XA XB, so I say that, I claim that this implies that. This equal to XAB-BA at p at phi for any phi in, this C infinity G L n R. Now, this has nothing to do with G L n R or anything. So, it is a more general fact.

So, let me just state it, like this so, more generally, let us just take an open set and Rn is open, you take a point inside that and I take two vector fields on this open set. If I want to claim that this equal to this then if and only if, I just have to check that they are equal on the co-ordinate functions. So, if and only if XP projection maps equal to YP pi i for all i.

And, the reason is quite simple. It is just that we can express XP as summation ai del by del xi at P and YP as bj del by del xj at P. Oh perhaps, yeah. Now, let us I have already used the index i and j for these things, so for the projection map I will use pi k. Suppose XP pi k equal to YP pi k for all k. Well if you act XP pi k is summation ai del by del xi acting on pi k at the point P.

Now, this is the projection map, in other words the coordinate map into the kth co-ordinate. So, the only term which will survive is the kth one. So, this will just give me ak. So, similarly this YP pi k as bk, so what happens is that, you just end up with therefore, ak equal to bk, k equal to 1 to n. Therefore, the two vector fields, the two vectors, these two vectors are the same. And that is what we have used.

So, it is enough to check it on this projection maps. So, that proves that we have our explicit description of the Lie bracket of left-invariant vector fields as in Lie bracket of matrices. And in fact, this carries over, this description of Lie bracket; this carries over to the subgroups of G L n R that we have been looking at, namely O n R and S L n R. So, I will just say a couple of words about that.

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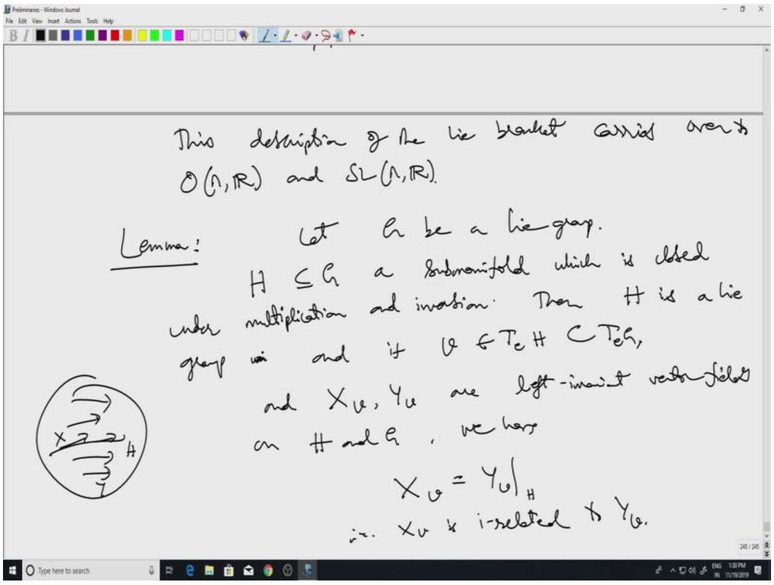

So, lemma, so let me write it like this, this description of the Lie bracket carries over to S L n, O n R and S L n R, which are the other two Lie groups we are looking at. They are the, they are, so the lemma is let G be a Lie group H contained in G and sub-manifold which is closed under multiplication and inversion.

Then H is itself, H is a Lie group, H is a Lie group and if V belongs to the tangent space at identity which is contained tangent space G, XV YV are left-invariant vector fields on H and G, we have XV equal to YV restricted to the subgroup H i.e XV is the inclusion map i-related to YV. So, I am saying something quite obvious here actually.

We were saying that if you have a sub-manifold which is also a subgroup then the first part is that, that H is itself a Lie group. So, this amounts to just saying that multiplication and inversion are smooth. We already know it is a sub-manifold and the fact that it is closed under multiplication and inversion means that H is a subgroup. But it is a Lie group so multiplication inversion operations are smooth simply because H is a sub-manifold.

So, that part is clear. Once it is a, and now V is in the tangent space at identity, I can extend V to be a left-invariant vector field on this Lie group H and also on the Lie group G, because the same V belongs to two different tangent spaces, this and this. I am saying that the left-invariant vector fields; the left-invariant vector field on this H is just a restriction of the left-invariant vector field Y on the Lie group.

And, with that we, I will be able to describe all those, all the Lie brackets on these groups O n R and S L n R. So, I will complete this next time. So, we will end this lecture on this, with this specific lemma. Okay, thanks.