An Introduction to Smooth Manifolds
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Lecture 40
Lie Brackets

Welcome to the 40th lecture in this series. So, I will continue with my discussion of vector fields. In particular, we will talk more about Lie brackets of vector fields.

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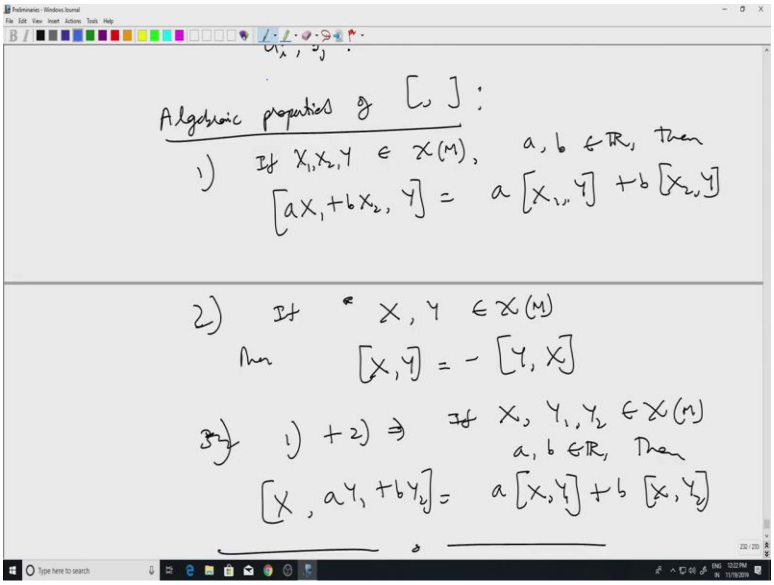

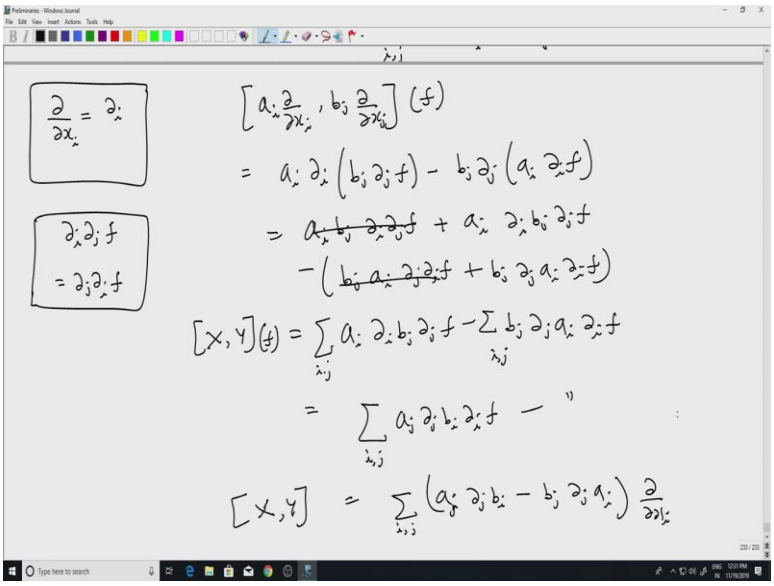

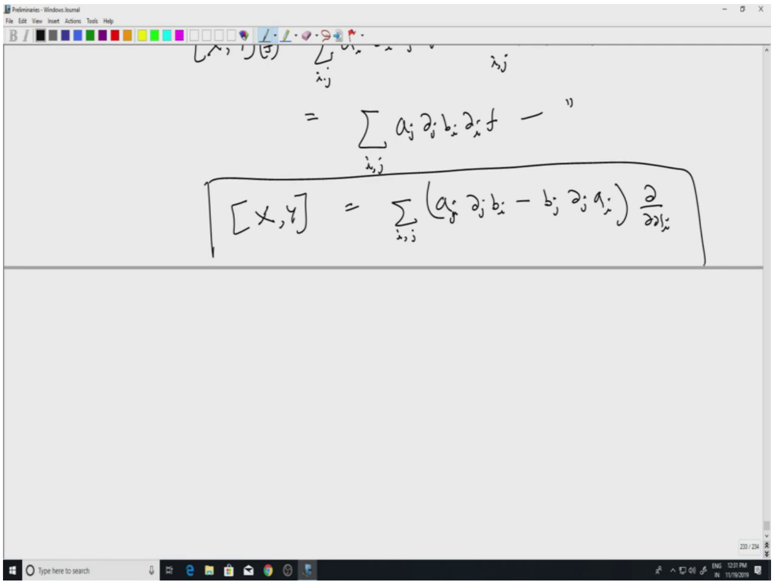

So, last time, towards the end of the last class, I the talked about this simple algebraic properties that the Lie bracket is bilinear, in the sense that if I fix one vector field and it is linear in the other slot. So, here I have return it just for the first slot, but then it is also antisymmetric that will imply it is linear in the second slot as well. There is another crucial property of the algebraic property of the Lie bracket of vector fields which are called the Jacobi Identity, which I will mention in this lecture or perhaps the next one.

Now in the end of last class, I had started deriving an expression for the Lie bracket in open coordinates. So, let me just complete that. So, we had reduced it to X can be written as sigma ai del by del Xi Y can be written as bj del by del Xj, where ai, so let us recall that ai and bj are functions from the open set the chart to R. So, and using these linearity, bi-linearity properties one reduces the expression to this. It is a double summation over i and j then one has this. So, let us see what this is.

So, ai, let us look at an individual term, bj del by del Xj. Now let us act it on a function and see what we get. So, this by definition is, I will use for simplicity of notation instead of del by del Xi I will put del i. So, this is, let us keep this notation in mind. So, this is this becomes ai del i and then of bj del j f minus bj del j of ai del i f. One uses the Liebniz's rule 2 times and then one gets ai, so here I will get ai bj del i del j f minus, plus ai del i bj del j f and the second thing which is bj ai del j del i f plus bj del j ai del i f.

This the first term in this and the first term in this cancel out. Notice that I have del i del j f and here I have del j del i f, but these two are equal. What one should remark is that, you know that mixed partial derivatives are equal in Euclidean space. However, these are not, when I say del i or del by del Xi, I do not mean quite mean the Euclidean partial derivative, I am working on the open set U in the manifold.

So, del by del Xi as usual is the pullback under the chart map, derivative of the chart map of the usual del by del Xi. However, this thing (contin) this equation equality of mixed partial derivatives continues to hold in the manifold as well. This is true in Euclidean space and this is true in the chart. So, these two things will cancel out and I am left with ai del i bj del j f plus bj del j ai del i f.

So, this is I will write summation over i j and this is summation over I j as well. Now I want to have the same recall that we obtained all this by acting on f. So, here I have del j f and here I have del i f, I want to get same index for both of these, so I can just switch i and j in this does not matter or I can switch i and j in the first one, let me do that, summation over i and j because both i and j one run from one to m.

And this. Sorry, yeah this is not, what I have here is not quite, so this is just the individual term. So, what I have written here the summation is the full Lie bracket not the individual term. So, X Y here I should right X Y equal to this, the thing again, so this is X Y and this is this and here I will get, I interchanged I and j so aj and this was in minus, aj and then here it was del j bi del i f minus the same term and then finally I wrote it as a j del j bi minus bj del j ai then del by del Xi.

So, this thing here it was acting on f and if I would do not want f in that, I will just write it like this. So, this is the final expression that one wanted. This expression gives, so with this in hand one can compute. Now, as I said there is no particular significance of this, it is just a way of explicitly one is given a vector field in turn local coordinates, one can use this to calculate the Lie bracket.

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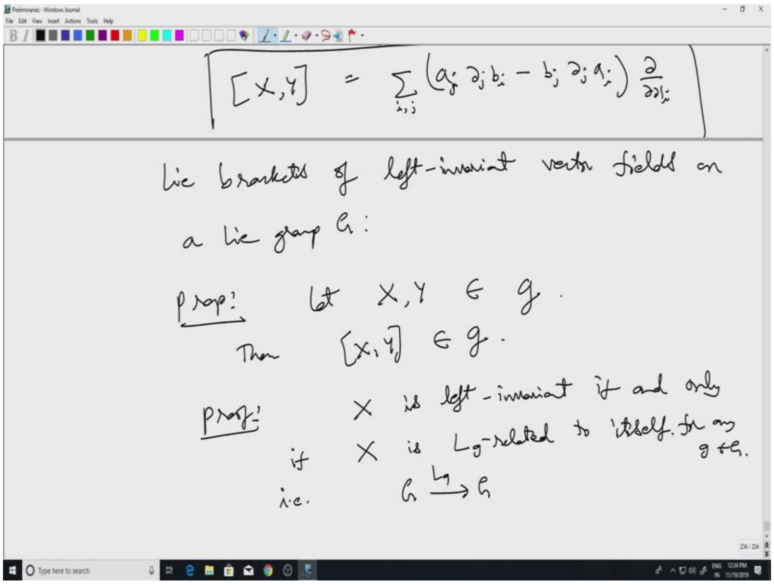

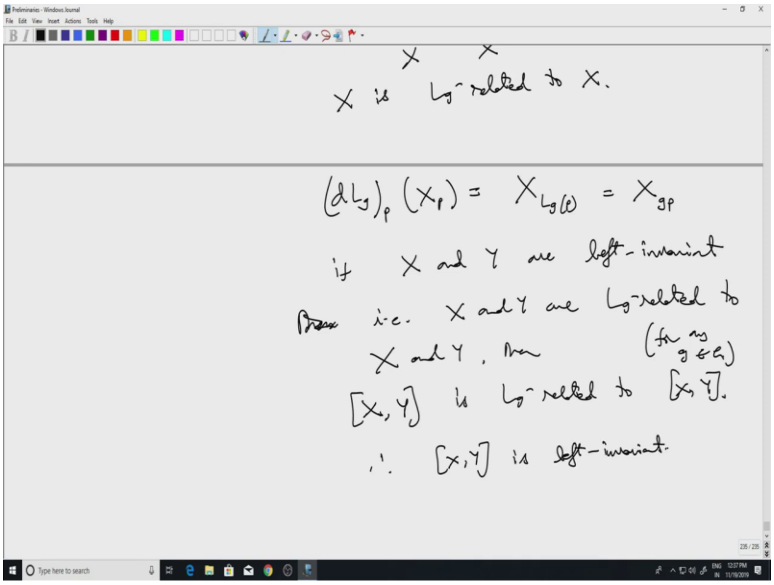

Now returning back to Lie brackets of, let us talk about Lie brackets of left-invariant vector fields on a Lie group G. So, we have seen that the crucial property is that we have seen that if 2 vector fields are f related under some smooth map f then the Lie brackets of those 2 vector fields are also f related. Let us use that to prove the following. Let X and Y be in the space of left-invariant vector fields on g, which I have denoted by this script g.

So, let X and Y belong to this then the Lie bracket is also belongs to this. In other words, the Lie bracket is also a left-invariant. And this follows immediately from the fact that the property of being f related transfer over to Lie brackets as well. So now notice, so the main point here is that X is left-invariant if and only if X is Lg related to itself i.e. see the left for any g, for any g an G.

So Lg is a map from the Lie group back to left translation is a map from the Lie group back to itself. And what we are saying is that, so there is X here and there is X here as well. So, X is Lg related to X. So, what does this mean, actually going by the definition of f related, so this just means that dLg at a point p acting on Xp equal to X at Lg of p.

This is the meaning of being a Lg related. So, vector field is Lg related to X if this happens. But the right hand side is the same thing as X left translation is just g p. So, now if you look at this equation dLg and p acting on Xp equal to Xgp, this we have seen as the same thing as the vector field being left-invariant. We have used this several times.

So, therefore this property of left-invariants can be expressed in terms of the fact that Lg related to itself. Therefore, if X and Y are left-invariant then i.e. X and Y are Lg related to X and Y again then the Lie bracket X Y is Lg related to X Y. Actually, the therefore here is not appropriate. So, this is yeah, so Lg related to X Y. Here, therefore X Y is left-invariant. And this is for any g, for any g and G.

Therefore, X Y is left-invariant. So, this is a very important thing because now the, we have already looked at the set of vector fields on a Lie group and left-invariant vector fields and we have seen that it is finite dimension vector space which is naturally isomorphic to tangent space at identity. In particular, it is a finite dimension vector space, but this preposition shows that it is a finite dimensional vector space with an additional algebraic structure on it.

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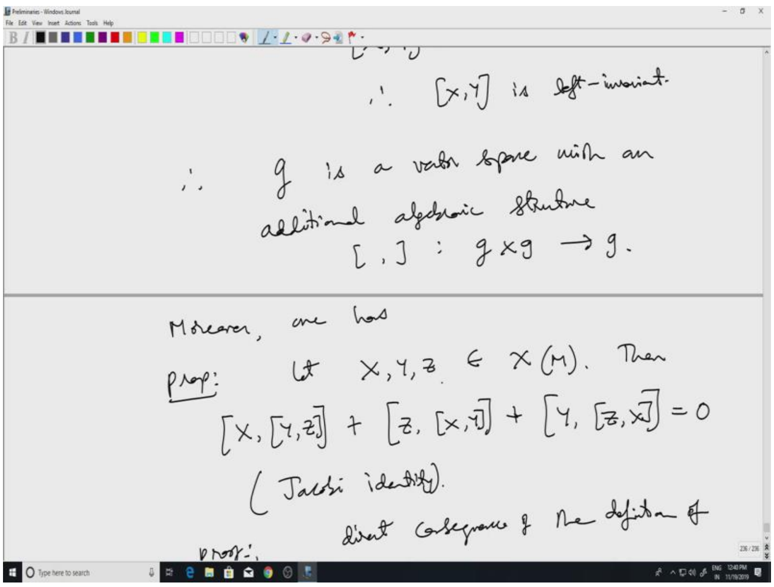

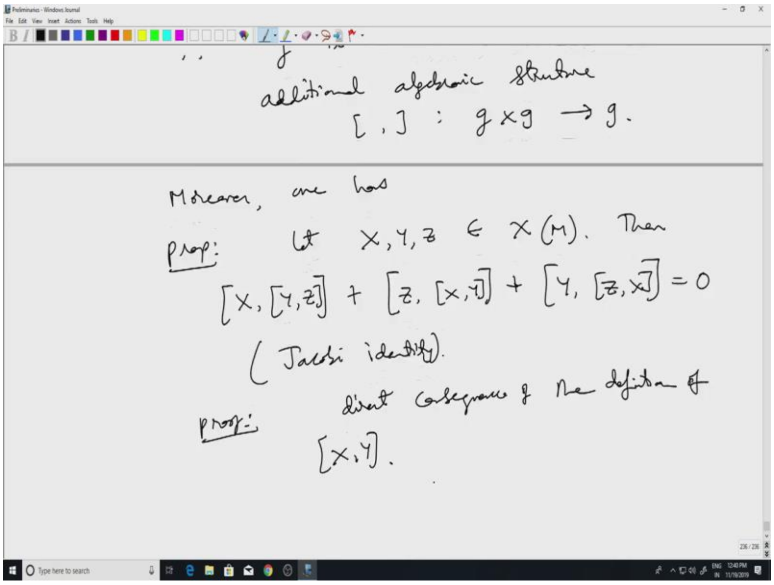

Namely that therefore this g is a vector space with an additional algebraic structure, namely the Lie bracket. So, this is a map from g cross g to g and in fact moreover this is the Lie bracket of vector fields, one has. Let X Y Z belong to vector fields on any smooth manifold, not necessarily a Lie group then X, so I can take X Y Z. It is like associativity does not hold.

So, the Lie bracket of the vector field is not associated, but instead one has this property X Y Z, you cyclically permute these things, Z X Y plus then Y Z X equal to 0. So this is called the Jacobi identity. This equation is, yeah, it is called the Jacobi identity. So, the Lie bracket we already seen that it is antisymmetric and bilinear. Now here is one more additional property of Lie bracket of vector fields. And this is true on any smooth manifold.

So, this, I will not prove this, this is a direct consequence. Direct consequence of the definition of Lie brackets. So, one can act it, the whole left hand side on some smooth function f then simplify the expressions. So, but the point is that returning back to the case of left-invariant vector fields, we have the set, space of left-invariant vector fields on a Lie group is a vector space with a bracket operation which is bilinear, antisymmetric and satisfies the Jacobi identity. Now such a thing has a name.

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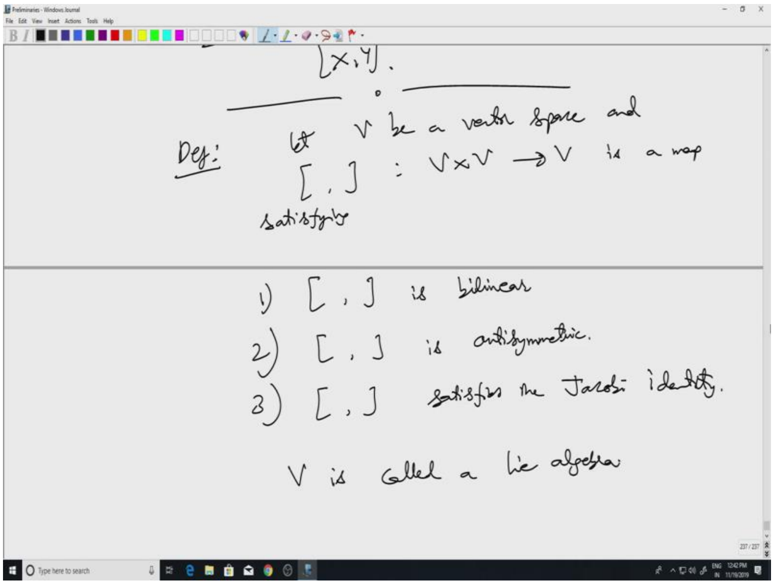

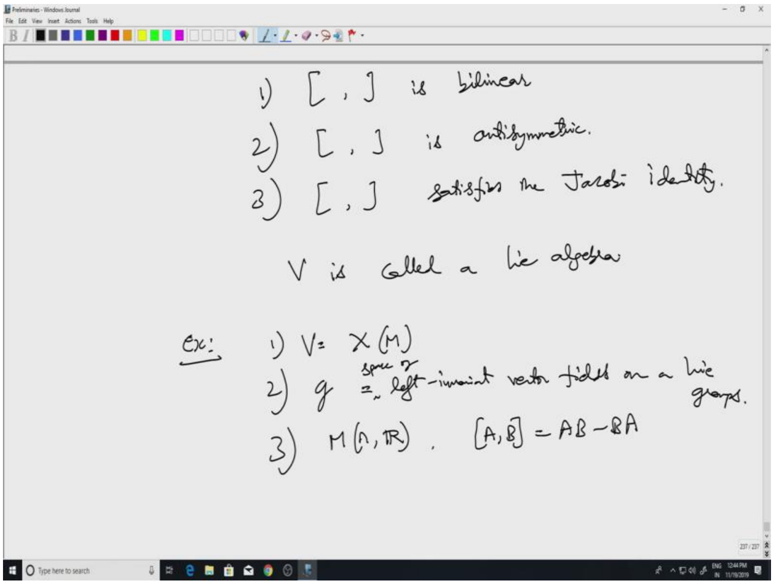

Definition: Let V be a vector space and suppose I have a bracket then map which is satisfying the following three properties, one is, first is this map is bilinear. Second thing is that this map is antisymmetric. Lie bracket of X Y should be equal to minus Lie bracket of X Y X. Third thing is, so any such factor space, V is called a Lie algebra. So, in other words we have talked about the Lie algebra if a Lie group, but this notion makes sense in every general setting.

Whenever you have a vector space with a this bracket operation, we can define notion of Lie algebra. And the two examples we have in mind are the Lie V equal to the set of vector smooth vector fields on a manifold. And this is not a finite dimension vector space, but the second example we have is g, the left-invariant vector fields space of left-invariant vector fields on a Lie group.

What we just showed is that this left-invariant vector fields, if I take Lie bracket we again get left-invariant vector fields. The left-invariant vector field so it is closed under there, Lie bracket operation which comes from the more general Lie bracket of vector fields. The third example is, let us look at the set of in cross in matrices and let us define the Lie bracket of X Y to be just A B minus B A.

Then one can easily check that this is a Lie algebra as well with this. Now, what I am going to show now is that, this actually the third example is not anything new, in fact the third example arises from the second one, the space of left-invariant vector before appropriate choice of a Lie group, this is a special case of two. In other words, M n R.

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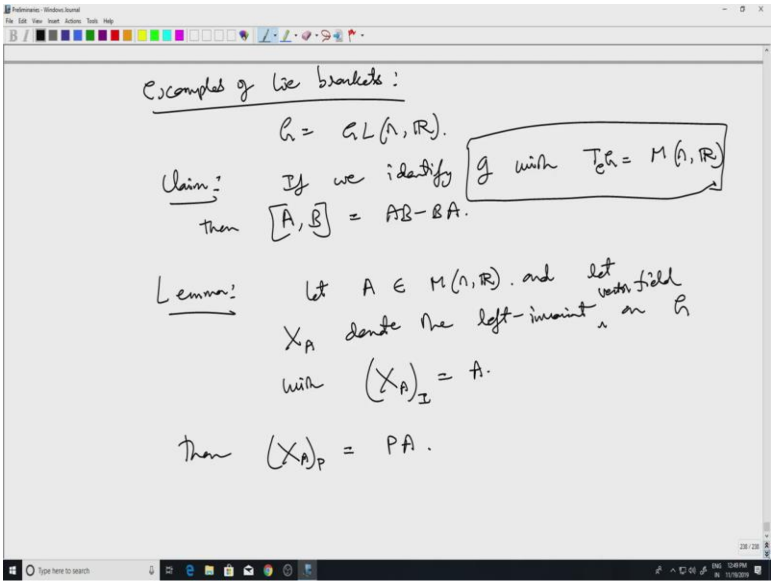

Claim: ... Then $[A, B] = AB - BA$.

Lemma: Let $A \in M(n, \mathbb{R})$ and let X_A denote the left-invariant vector field on G with $(X_A)_I = A$.

Then $(X_A)_P = PA$. $\forall P \in GL(n, \mathbb{R})$

Proof: $(X_A)_P = (dL_P)_I (A)$ by left-invariance.

Note that $L_P : GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$

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Note that $L_P : GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$

is the restriction of

$L_P : M(n, \mathbb{R}) \to M(n, \mathbb{R})$

on $M(n, \mathbb{R})$. L_P is a linear map.

$L_P(A) = PA$.

$\therefore (dL_P)_Q = L_P$ at any $Q \in M(n, \mathbb{R})$.

Since $GL(n, \mathbb{R})$ is open in $M(n, \mathbb{R})$

$(dL_P)_Q = L_P \quad \forall Q \in GL(n, \mathbb{R})$.

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$\therefore (X_A)_P = (dL_P)_I (A) = L_P(A) = PA$

So, let us do that now. Example of Lie bracket. So, let us do it for the basic Lie group that we know. Mainly G L n R, s o G equal to G L n R. The set of invertible and cross in matrices. We know that, so here is the claim, yeah so the claim is if we identify the Lie algebra g with the tangent space at identity of G which is M n R then, if you identify this then the Lie bracket of A B is equal to A B minus B A.

So, let us prove this. So, this is what I was, I said that, few minutes earlier that the third example is a special case of two. So, if we the Lie group to be G L n R, the set of left-invariant vector fields is which is I find with the tangent space and it is M n R. So, and the point here is that the Lie bracket operation coming from vector fields is the same thing as this matrix operation A B minus B A. So, for that, let us we will need a few Lemmas.

So, the first Lemma is that let us describe what a left invariant vector field looks like. Let, I want to understand this, this identification more clearly. So, let me take A and M n R which is the tangent space at identity and let XA denote the left-invariant vector field on G, left invariant vector field on G with the whose valued identity is exactly A. This is how this isomorphism arises that you start with a left-invariant vector fields, you want it is values at identities, so you get an element of the tangent space.

So, I am denote, conversely starting with this a left-invariant vector fields. So, which I have denoted by X A. So now what I want to do is, right. I want to describe XA at any point. This is at identity it is A, so I claim that XA at any point p then XA at p is nothing but p A. This is the first, so this gives a explicit description of any left-invariant vector field. So proof is quite simple. So, well by definition XA at p is d the left translation by p.

So, this here of course p is any (())(25:05) p in G L n R. So, b definition of the left-invariant vector field with value A, so X A at p is equal to dLp at identity acting on A. The value of X A at identity which is A. So, we have this, this is by left-invariants. Now, so let us look at this map (dL), so we want to understand the derivative of the left translation map.

We do not have to do any computations because what we observe is note that the left translation map which is a map from G L n R to G L n R is actually is the restriction of the left translation map from M n R to M n R. And the point of regarding it is a map on a bigger space is that this left translation map on M n R this left (tra) Lp is a linear map. So Lp of an, remember that Lp of A is just p A.

So of course, it is linear in A. So, it is a linear map, therefore the derivative of L p at any point Q is equal to just the map itself, at any Q in M n R. So, and this we have seen this from the definition of derivative in Euclidean space. Now, here we are in the (rel) and we know that the motion of the derivative when we are in the realm of Euclidean spaces, whether we use the original definition or the abstract manifold definition, we get the same thing.

So, this dLp at any point Q is just a Lp itself. Since G L n R is open in a M n R the derivative of Lp is the same as what it would be when I would disregard it as a map from M n R to itself. dLp of Q is equal to Lp d for all Q in G L n R as well. So, this is the advantage of regarding it as a map of vector space. And once we have this and this Q can be anything, not necessarily the identity.

Though here, I am mainly interested in identity, so, but it is a same thing. So, what I get finally is that therefore XA at p equals dLp at identity of A and this is the same thing as Lp of A and by definition this is p A. So, one is done. So that is the (clai). So, we have proved the Lemma that the left-invariant vector field is just given by multiplying the A with p, that is one thing.

The second thing is that, with that in hand, now we can proceed to show that, we can proceed with our calculations and show that, use this to simplify the computation of the Lie bracket. So, I will have to stop at this point since I am out of time. I will resume with the calculation next time. So, thank you. We will resume at this point in the next lecture.