An Introduction to Smooth Manifolds
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Lecture-04
The Derivative Map

So, today's lecture we are going to resume our discussion of multivariable calculus. So, last time I had introduced or rather recalled the definition of derivative in higher dimensions. And so let us start with the example that I had started towards the end of last lecture.

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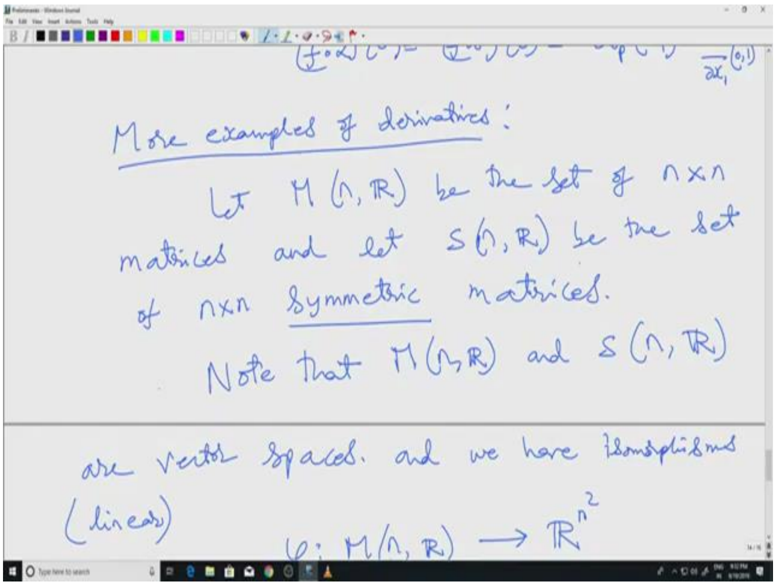

So let M n R be the set of n cross n matrices and like S n R be the set of n cross n symmetric matrices. Note that M n R and S n R are vector spaces with the usual operation of matrix addition and scalar multiplication and S n R would be then a subspace of M n R.

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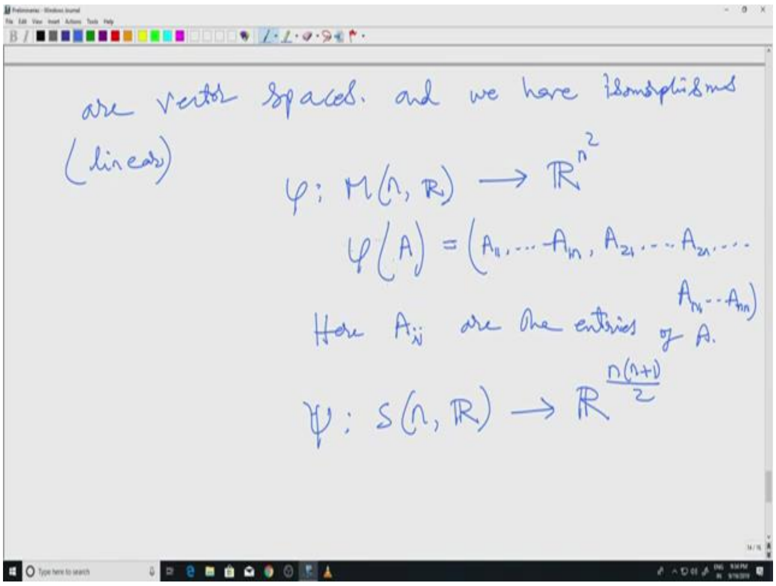

We want to regard these as Euclidian spaces. So, when I say regard these as Euclidean spaces what I means, exactly is I want to give an isomorphism, linear isomorphism from M n R to R n squared that is 1. And the other 1 is from the space of symmetric matrices to R n into n + 1 by 2 and the isomorphisms are as follows. So, for M n R I just take so I define this map phi, phi of a matrix A, here A is an element of M n R, so it is a matrix.

What I do is I just put out the entries of A row by row. So this is the first row, the second row and so on. So altogether I get n squared entries. So I get an element of r n squared. And it is easy to check that this map is linear. And 1 to 1 and on 2.

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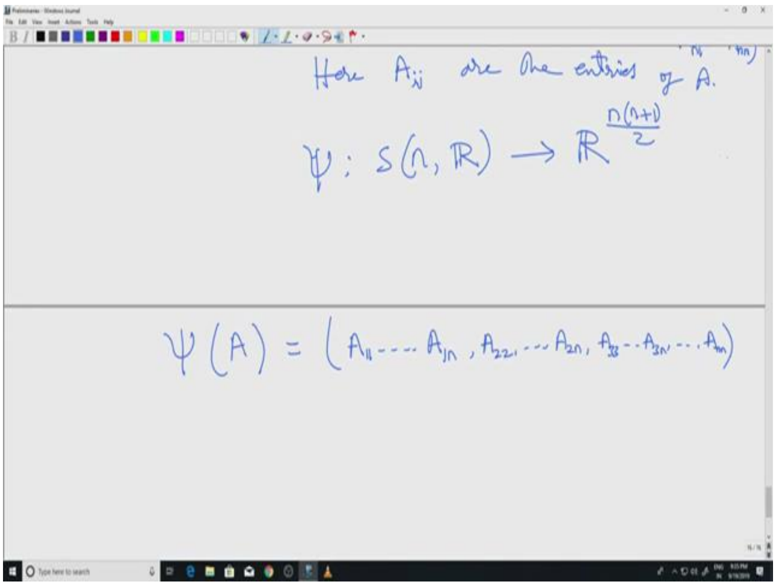

And as for the space of symmetric matrices, I do something similar. I just well, the first thing first row I again, I just put out the first row of A as for the next, I do not start with the full second row. I start with A22 go all the way up to A2n and for the third one, I would start with A33, go up to A3n and so on. So, the final entry will be Ann. Again just to verify that we have a map into this Euclidean space, notice that there are n elements here in the first block, then n - 1 and so on all the way up to 1.

So, the number of coordinates here would be n + n - 1 + n - 2 and so on + 1, which is indeed n into n + 1 by 2. And again, this map C is also a linear isomorphism from this vector space to that.

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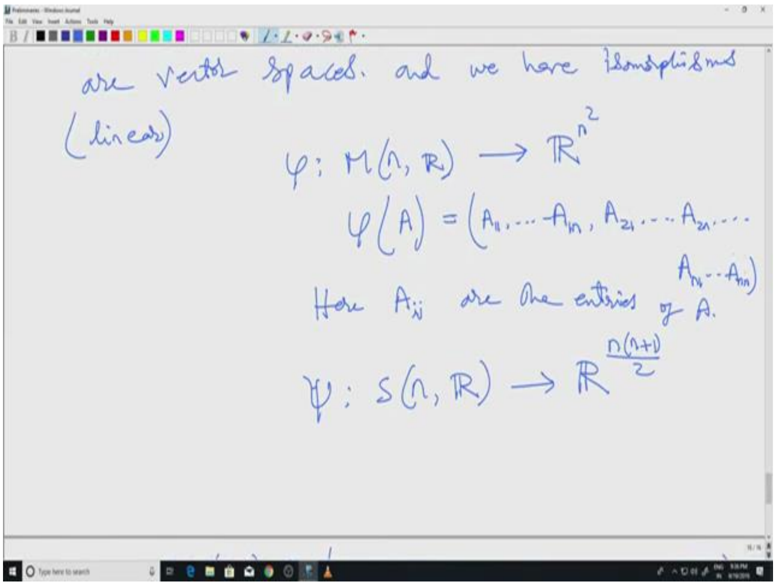

So, once I have this isomorphism, I can just basically, so the all the notions that we have defined, here, there the domain was an open subset of some Euclidean space. So because of these isomorphisms, I regard these as identified with these Euclidean spaces. So I can still talk about differentiability of functions defined on these spaces M n R and S n R.

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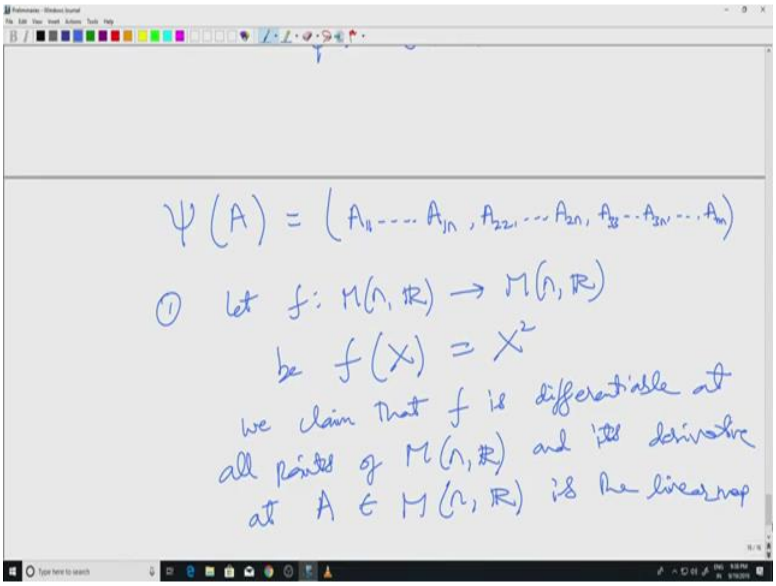

So I am going to give 2 examples. The first one is f of x equals x squared here of course, x is an element of M n R which is an n by n matrix, so and here x squared is usual matrix multiplication. So, the claim is that, we claim that f is differentiable at all points of M n R and its derivative at a point at A in M n R is the linear map.

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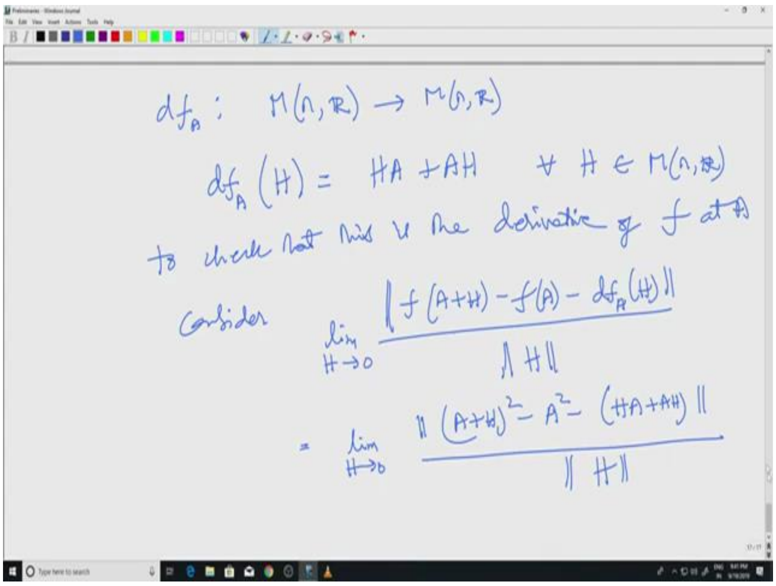

d fA, remember that if we had a map from an open subset of Rn to R m. The derivative was a linear map from Rn to Rm as well. So d fA here would be a map from M n R to M n R and the claim is that this linear map is given by its action on an element H is HA + AH for all H and M n R. Now, here I just wrote down the formula for the derivative. And let us verify that this is indeed the derivative going to the definition. In the next step, I will show how to get this in the first place.

So but to verify that this is the derivative, we just have to check, to check that this is the derivative of f at A, consider we go back to the definition. So I have to look at f of A + H - f of A - d fA of H. Norm of that divided by norm of H, limit H going to 0. Now, remember that M n R has been identified with R n squared. So when I say norm of something, norm of a matrix, we have to regard M n R as an element of R n squared and take the norm in the usual way.

So let us see. So here this is if I do the calculation, I get limit H going to 0, f of A + H is A + H squared - A squared - here this is, by definition not well, from the formula that I wrote down here, HA + AH divided by norm H.

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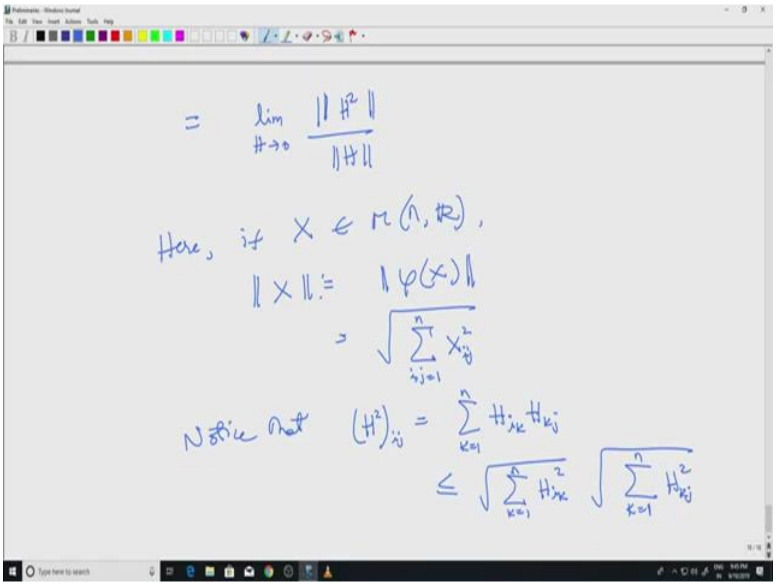

And if we expand out A+H squared, and so on, so finally, I will be left with various terms cancel out, I will be just left with limit norm H square divided by norm H. As I said, the norm here on the space of matrices comes from Euclidean space. So to proceed further, let us see what the normal for matrix is actually. Here if x belongs to M n R norm x is, so I define that isomorphism from M n R to R n squared, so I will use that isomorphism. So this is the definition of norm X, norm X is norm of phi of X.

The point being that this phi of x is an element of R n squared. And this norm is the usual Euclidean norm. And so and remember phi of X was just the entries of x laid out row by row. So, if one takes the Euclidean norm of this, one get square root of sigma x i j squared, i j equal to 1 to n. So, all we are doing is we are just saying that, that this norm is just amount to saying that you square all the entries of x, add them up and then take the square root of the sum.

So, with this definition of norm notice that, now, I want to go back to this norm of H squared. First, notice that if I look at the matrix H squared and take the ij th entry, this is the same thing as Hik, Hkj, k equals 1 to n, this is from the usual definition of matrix multiplication. So, these are the entries of the matrix H squared. Now, when I take norm, I have to square each entry and add them all up and then take the square root.

So, but before I do that, let me see how big can this be, so, this is by this Cauchy Schwarz inequality, this is less than or equal to summation square root Hik squared, k equals 1 to n square root, K equals 1 to n Hkj squared.

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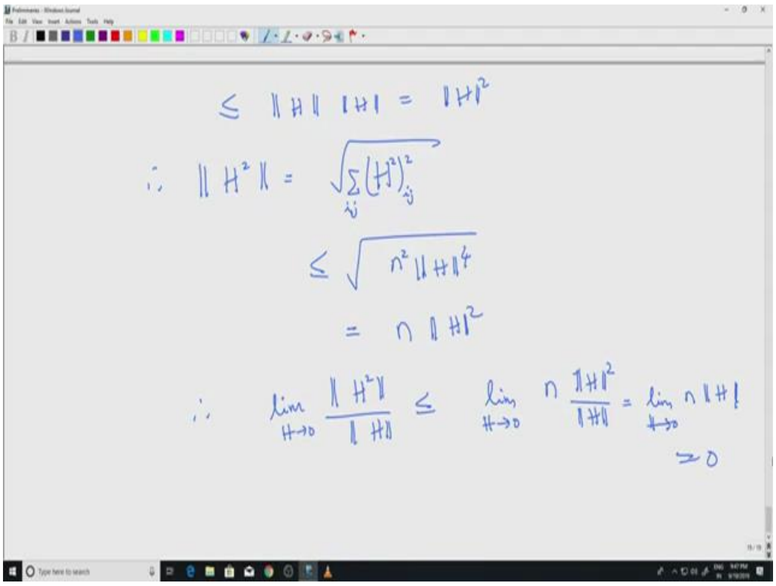

Well, each one of these will be less than or equal to norm H, norm H times norm H multiply equal to norm H squared. So the point is that every entry of the H squared matrix is less than or equal to norm H squared. So let us use that. Therefore, if I look at the norm of this H squared matrix, that is equal to, by definition, I look at its entries and sum over i and j. Now, each entry, I have to square this and then sum double. So now each entry is less than or equal to H squared. And but I am squaring them, so I will get H to the 4 here.

So this is less than or equal to square root, and there are n squared terms. So n squared times norm H to the 4. So this is equal to n times norm H square and that is good enough for us. So I get this estimate. Therefore, now I will go back to the limit, limit this divided by this H going to 0 is less than or equal to because of this what I just proved, limit H going to 0. So I get n H squared divided by H is equal to limit H is going to 0 n times norm H, which is 0.

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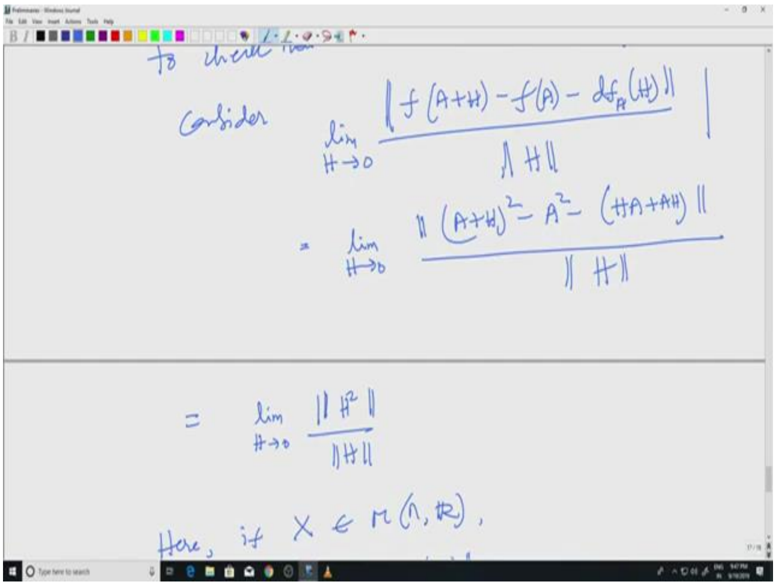

So, the upshot is that I started with this. I wanted to claim that d fA H satisfies the condition that this limit should be 0, so and we checked that it does. But this process does not, this is just a verification of an expected, we already guessed ahead and guessed for the derivative and then we verified, but suppose we want to find the formula for the derivative.

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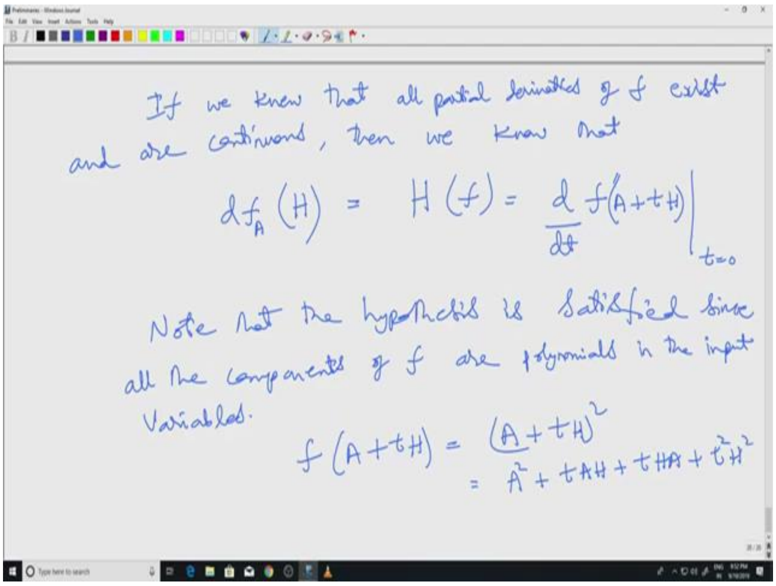

Then I go back to this directional derivative interpretation. Suppose, but that involves an assumption if we knew that which is satisfied in this case, if we knew that all partial derivatives of f exist and are continuous then we know that d fA acting on H is nothing but the directional derivative of f along the direction H at the point A, which I denoted by H of f. And which was in fact, this is by definition, H of f is d by dt, f of A + t times H evaluated at t equals 0. This was the definition of directional derivative.

And last time I had remarked that the directional derivative is related to the full derivative, the fresh A derivative by this formula, well they are literally equal when the full derivative is evaluated along that specific direction. This is under the assumption that if we know that all partial derivatives of f exist and are continuous. So, now let me just say that this hypothesis, all partial derivatives of f exist and are continuous is satisfied in our case, simply because all the component functions.

Note that the hypothesis is satisfied, since all the components of f are, well the function f of x was just x squared, where x is a matrix and f of x, all the components of f are actually just polynomials in the input variables, polynomials in the input variables. Therefore the partial, all partial derivatives exist and they are polynomials as well. But that is all we need to say at this for this hypothesis to be true, but if we actually want to know what the derivative is, we use this formula.

So we already know that d fA is A H + H A according to the previous calculation that I did. But now let us use the right hand side. Now, f of A + t H is by definition A + t H squared which is A squared + t AH + t HA + t squared H squared. Yeah, right. So this is this.

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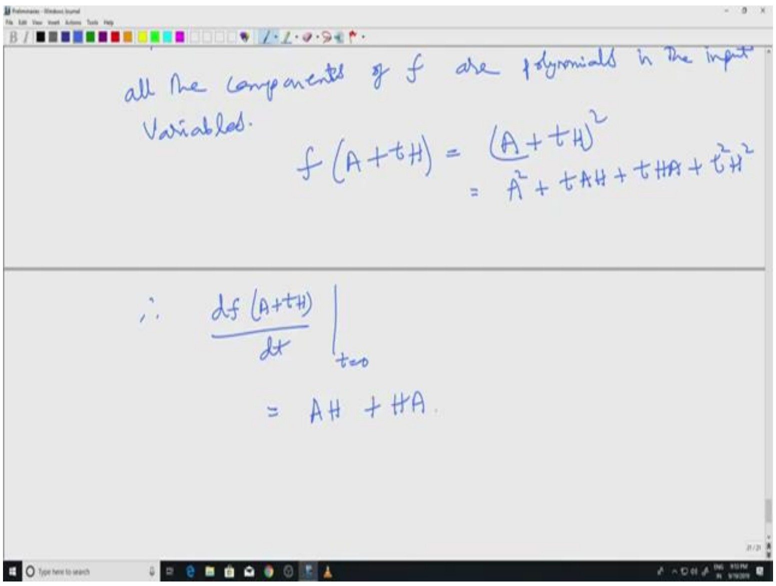

Therefore, d f of A + t H by dt at t equals 0, I just have to differentiate with respect to t in the usual sense here. And then, basically I will get 2t here in the last term and this first term disappears because there is no dependence on t. And in the second 2 terms, the t becomes 1 after differentiating. So I will just, so when I put t equals 0, finally, I will get AH + HA, which is the formula that we had earlier. But the point here is that we are able to derive the formula if we use this directional derivative interpretation of the derivative acting on as a linear map.

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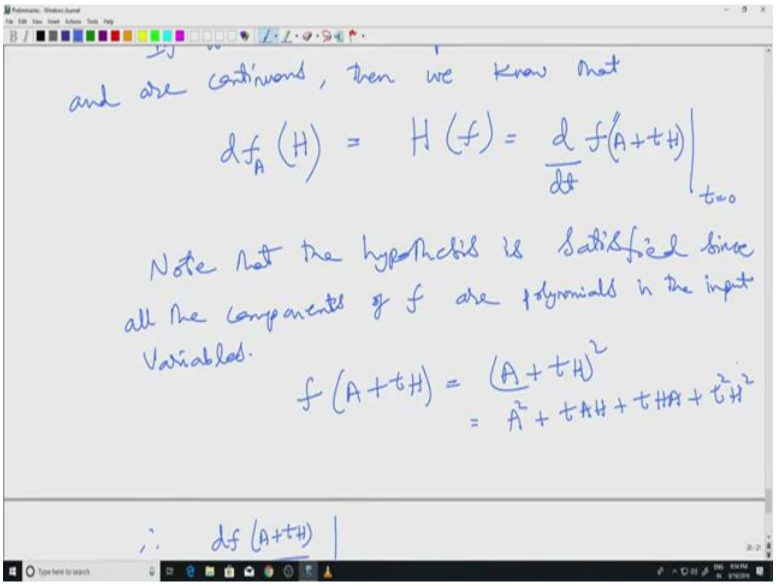

So that is one of the main uses of this, it actually helps us to compute derivatives. Because essentially, as you can see here, the problem gets reduced to a problem of one variable calculus, there is only one variable here. Since I have restricted f to the straight line A + t H, the only variable here is t. So I can do the calculation quite easily.

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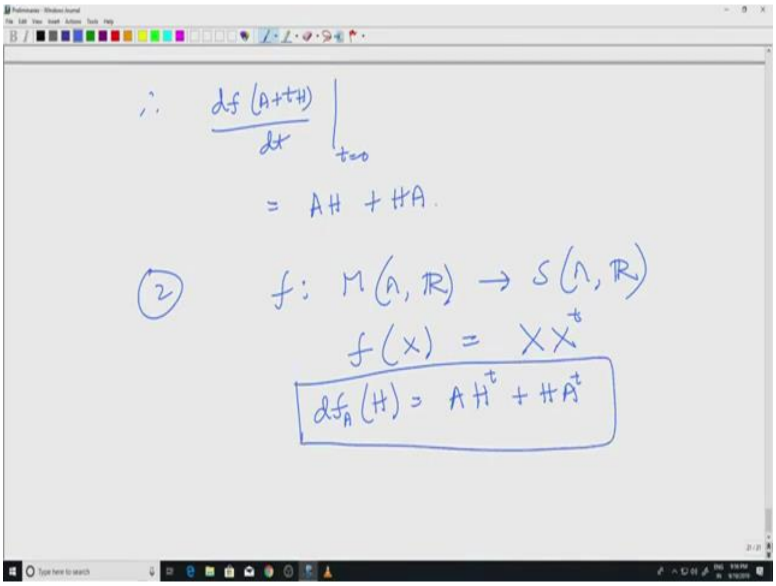

Now the second example is something similar. So this time, I will take f from M n R to S n R symmetric matrices, and again, the input is x, and the output is instead of x squared, I will take x times x transpose. The point is that whatever x is, x times x transpose is always going to be symmetric. So I actually get an element of S n R, not just any other matrix. And again, 1 can do the calculation, directional derivative interpretation here as well.

Because the hypothesis is again, satisfied all partial derivatives of exist, of f exist and are continuous because again the component functions, components of F are just polynomials in the input variables x ij. So, we can use them to and this time we find that d fA H, now it turns out to be A times H transpose, A times H transpose + H times A transpose. This formula I will use later on in the course. So, this is the final thing. So, I have been saying that one has to be a bit careful when dealing with these different concepts of derivatives.

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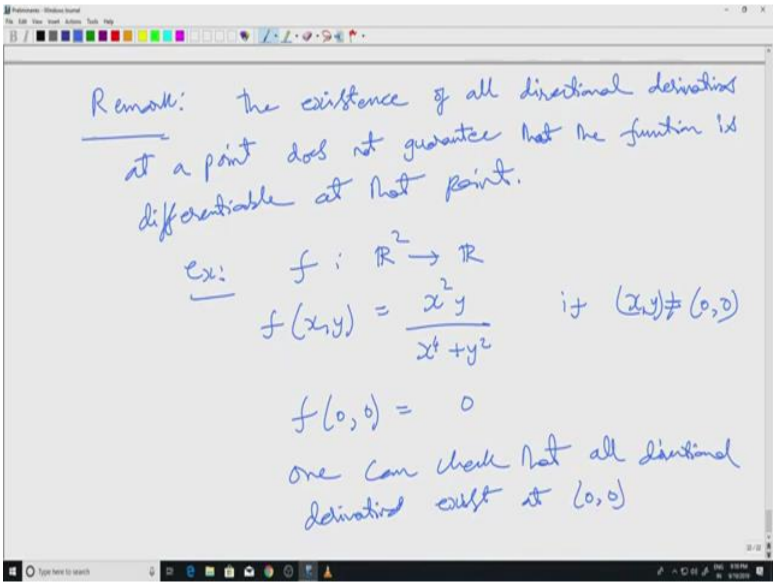

So, let me just make a remark, remark, the existence of all directional derivative derivatives at a point, here, all refers to all possible directions. So, existence of all directional derivatives at a point does not guarantee that the function is differentiable at that point. So the standard example is an example f from, I will define a function from R2 to R by f of x y is x squared y, x to the power 4 + y squared, if x y is not equal to 0, 0 and f of 0, 0 I just define it to be 0. One can check that at the point 0, 0, all directions derivatives of f exist but f is not differentiable, one can check that all directional derivatives exist at 0, 0. And the way one is that one just goes, the definition of a directional derivative.

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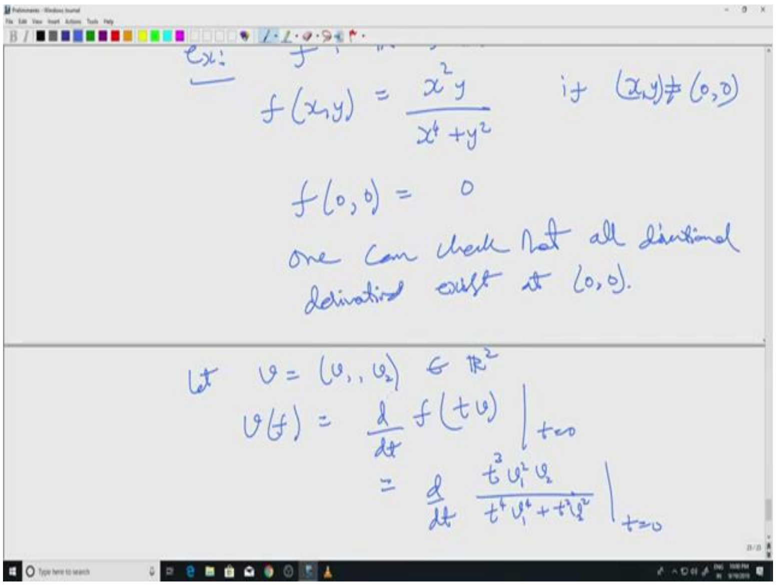

So once you start with let V equals v1, v2 in R 2, then V of f is d by dt, f of, now the point is 0, 0, so the straight line was p + t V, P is, the origin, so all that I get is p of t V at t equals 0, and this would be the derivative f of t V would be, well, t v2, tv2. So here I will get t cubed v 1 squared v2. And here I will get t to the 4, the V1 to the power 4 + t squared v2 squared, right evaluated at t equals 0.

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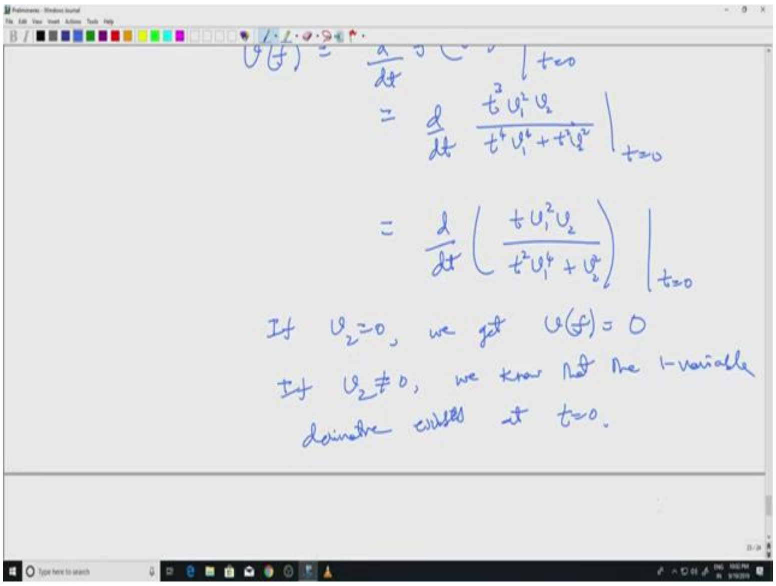

Of course, this t squared cancels out and I am left with d by dt. So I will get t v 1 squared, v 2 t squared, v1 to the 4 + v 2 squared, at t equals 0. And the point is that this function, so, 2 cases will arise if v2 0 or v2, when v2 is 0 and v 2. If v 2 is 0, we get v of f, if v2 is 0, well, independent of what v1 is, the numerator itself becomes 0. So the directional derivative would be 0. If v2 is not equal to 0, one can check that what the specific value does not, then the denominator does not vanish when t is 0. So we know that by elementary calculus that this derivative does exist.

If v2 is not 0, we know that the one variable derivative exists at t equals 0, simply because the denominator does not vanish at t equals 0.

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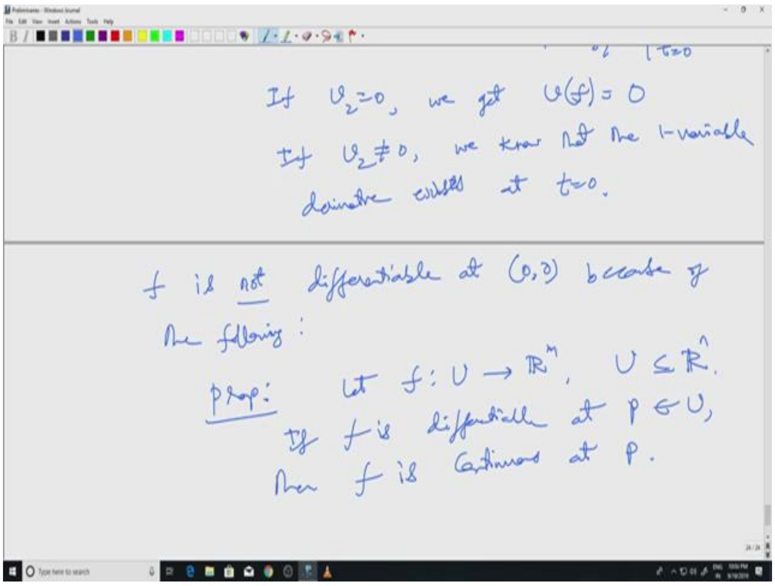

Now, why is f not differentiable at it, f is not differentiable in the fresh A sense at 0, 0 because of the following. So, just like in the one variable case we have let f from U to Rm, U is an open set in Rn, if f is differentiable at p in U, then f is continuous at… So differentiable in the fresh A sense will guarantee continuity at that point.

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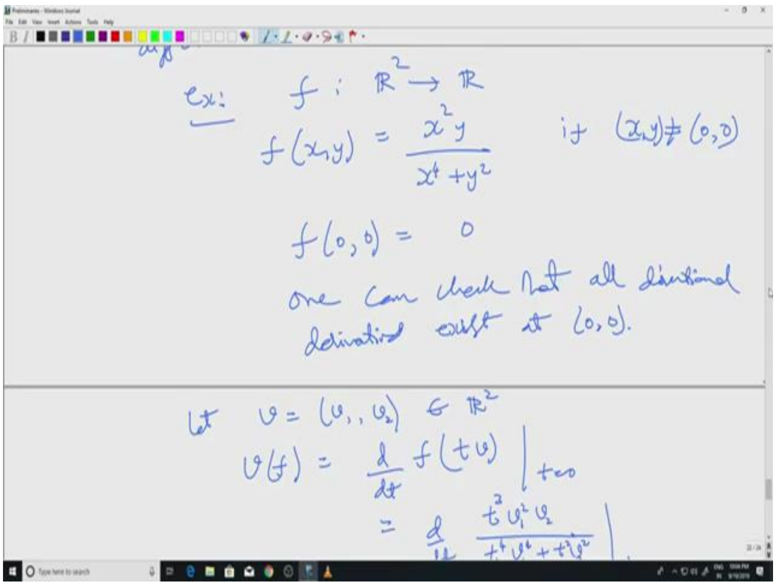

And one can check that this function, what I have here is not continuous at the origin. So that is the reason why it is not differentiable. So, in particular, the existence of directional derivatives will not imply. Right so in the next class, I will talk about one of the main theorems in multivariable calculus, namely the inverse function theorem. So let us, we will stop here.