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An Introduction to Smooth Manifolds
Lecture 35
Integral curve and flows 2

So this would be 34th right, no 35th yeah. So hello and welcome to the 35th lecture in the series.

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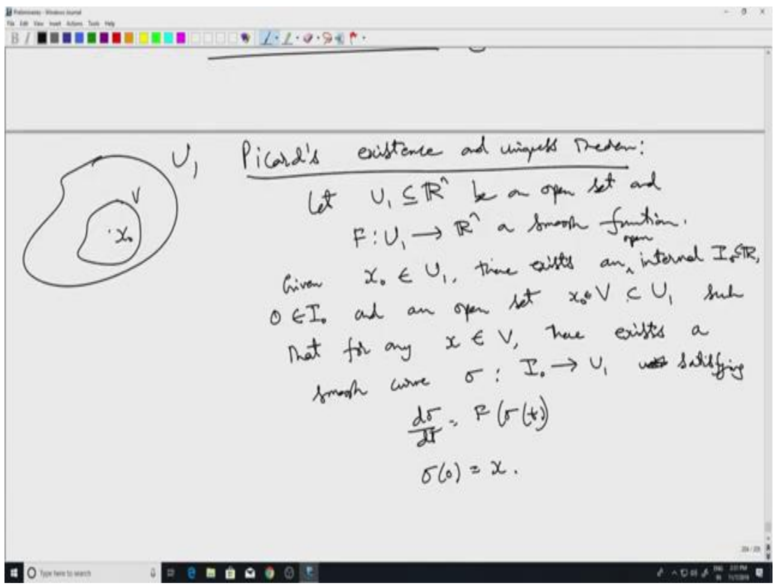

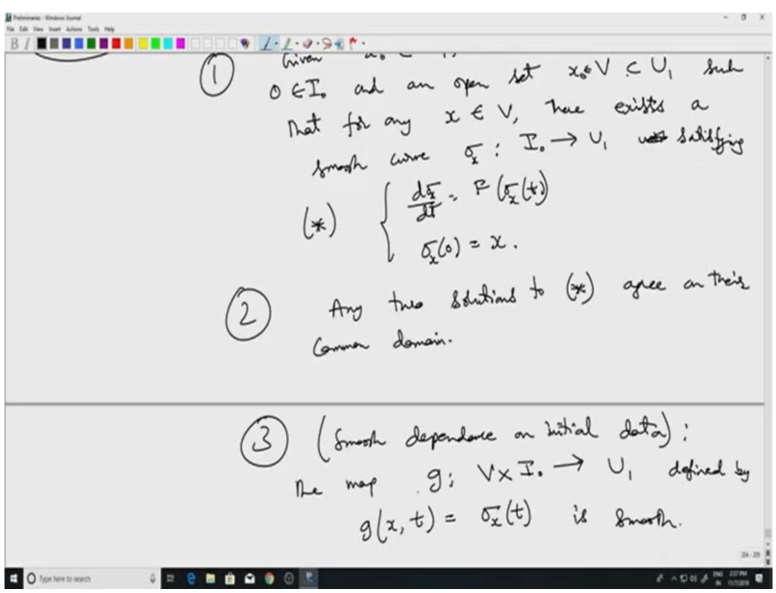

So let me resume from the last lecture we had discussed the Picard's existence and uniqueness theorem which implied the existence of integral curves on of any vector field on any manifold.

But I wanted to point out that we have something stronger than just saying that integral curves exist. So what we have is that is the following statement.

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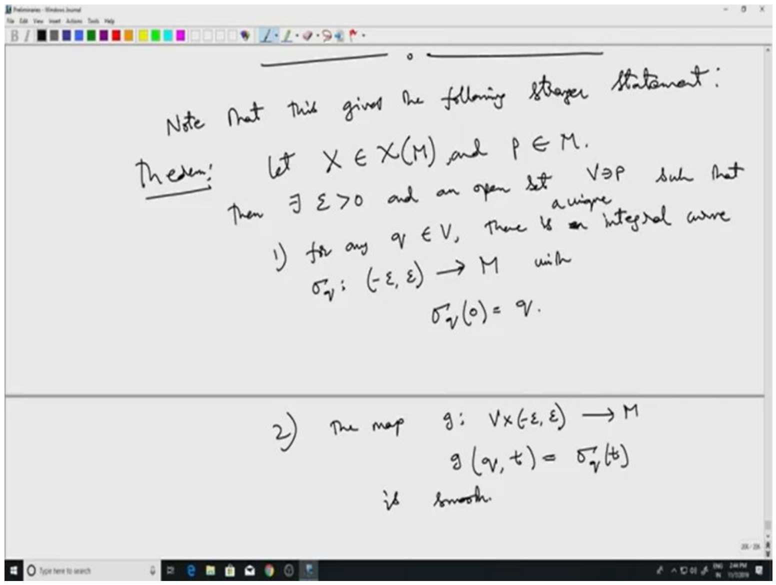

Let, so let X be a vector field on a smooth manifold and P belong to M, then there exists epsilon greater than 0 such that epsilon greater than 0. And a neighborhood of an open set V containing P such that first for any q in V. There is an integral curve sigma q defined from minus epsilon to epsilon to M with sigma q0 equals q, again the point is that the same epsilon will work for all q inside this neighborhood.

The interval of definition of the integral curve is not shrinking. The second thing is that the smooth dependence on initial data. So the map let us again continue to call it g, g from this V cross minus epsilon epsilon M defined by g of q comma t is sigma q t. So here I should say that there is a, there is a unique integral curve with this property. So in fact that is how I am able to say this moment you know q the sigma is predetermined of course given X.

So the map defined by like this is smooth, was a smooth dependence on initial data. And this completely sort of this the best one can say in a without assuming anything further about the manifold. For instance, the, it turns out that if the manifold is compact then every integral curve can be defined on, the domain can be taken to be all of our not just some small interval minus epsilon epsilon, but in general without any further assumptions on M this is the best one can do.

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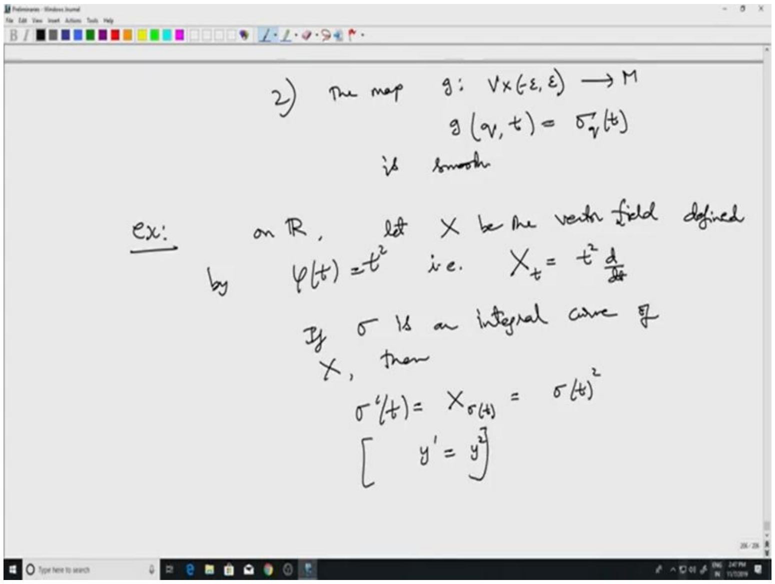

Now, since I have talked a bit about this interval of definition. Let me quickly give an example to illustrate why we need to even mention this, so on R, let X be the vector field so remember that on Euclidean space a vector field is just defined by a function from the open set back to the same Euclidean space. So on R a vector field would be given by a smooth map from R to R to field defined by phi of t is T square. So i.e. X at any point t if one thinks in terms of derivations this is t square times d by dt derivative at that point or in terms of functions ,this is just phi of t is t Square.

So, now let us look at integral curves of this vector field. So the defining property if sigma is an integral curve, of integral curve of X then sigma prime t would have to be equal to X at sigma t. In other words this is thinking of it as in this Euclidean terms. So this is just sigma t square. So in more familiar notation if we think of a function, y is a function of X or t this we are looking at the equation y square, y prime equals y square with some prescribed initial data.

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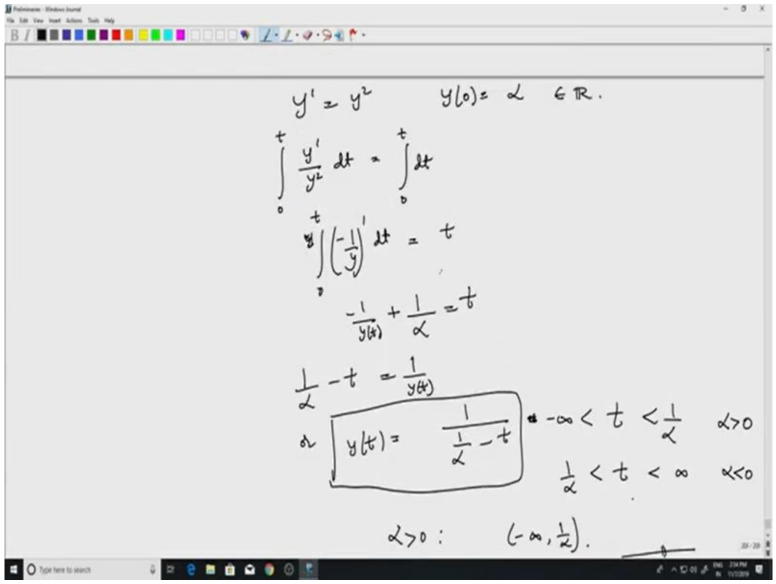

So let us solve this y prime equals y square y of 0 equals some let us say alpha in R. We know that the solution always exist like Picard's theorem. Actually here in this case we can explicitly solve, anyway, so if I solve this. We will solve it in the usual way by prime by y square equal to 1 assuming y is not 0 then I integrate between 0 and so integrate this between so put a dt. So what I get as using substitution and so on what I end up getting is y of T.

So here this is the derivative of actually minus 1 over y prime dt 0 to t, actually I can just use the fundamental theorem of calculus. So this is minus 1 over yt plus 1 over y0, y0 is alpha. So as t and then I play around with that so I take it to the other side and so on, 1 minus alpha minus t equals 1 over yt or yt equals 1 divided by 1 by alpha minus t equals.

Now, let me just leave it like this. Now at t equals 0. Of course here when I did this, these calculations were formal calculations without paying any regard to whether y is 0 or not but once I have this formula what I have here? I can disregard these calculations and then just check directly that this is indeed a solution to this y prime equals y square. And y of 0 is actually just is exactly equal to alpha.

So this is a perfectly legitimate solution and we know that this is the unique solution beginning at t equals at the point alpha. Now the point is that let us see what is the interval of definition of this. So this makes perfect sense until t equals 1 over alpha, so as long as t is less than 1 over alpha. And there is no other constraint on t, t less than 1 over alpha 1 is in good shape.

If alpha is negative then it is, even then so we just want to ensure that we want to have 0 in our domain of definition. So this is, here if alpha is positive if alpha is negative then I would want t greater than 1 over alpha and no other constraint yeah this. These two cases arise because I want to impose the condition that 0 is inside the domain of the solution.

So now the point is that as alpha changes so the domain is either so for the moment let us fix alpha greater than 0. In this case the domain is minus infinity to 1 over alpha. So as alpha changes the domain of definition does change so for instance as alpha goes to 0, as alpha goes to 0 then this becomes larger and larger. On the other hand if alpha is actually equal to, if alpha goes towards infinity then this, the domain 0 should always be in the domain.

So but it gets closer and closer to 0, it just goes slightly past 0 on the right side. So the point is that even though we started with a very simple case of R and perfectly smooth, in fact, it is a polynomial vector field still the domain of definition of the solution is not so. It is not that it is always going to be all of R.

And also another thing one can notice here is that let us take let us note that the vector field is actually equal to 0 when T is 0. So if I had taken alpha to be actually 0 then this were, this will not work anymore. So all along I had assumed that so these calculations make breakdown if I alpha turns out to be 0. This has to be treated slightly differently.

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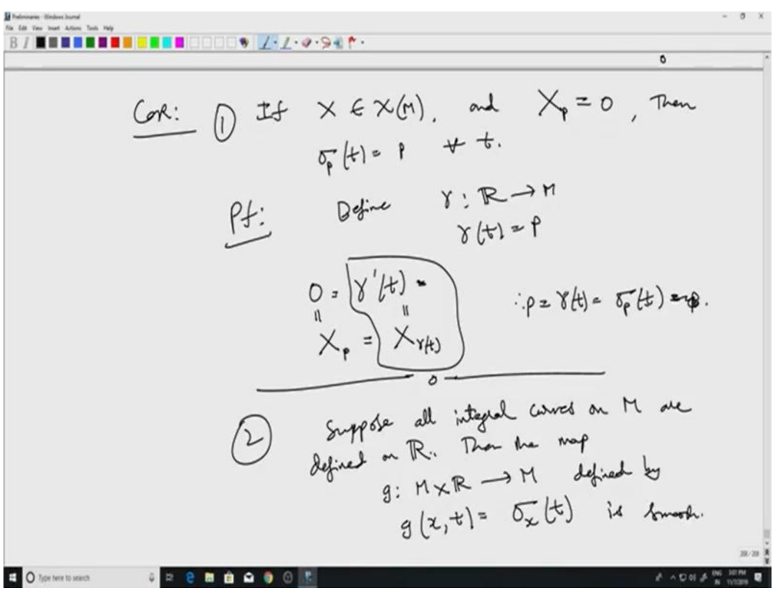

And in fact here is a small corollary of the uniqueness part if X is a vector field on a manifold and XP is the 0 vector. Then the sigma P t equals P for all t. So the integral curve it starts at the point P is the constant integral curve it just stays there for all time. So even in this case of Y prime equals y squared the solution when alpha is equal to 0 is just Y of t is 0. The 0 solution is the only one for all t.

And this is true by the uniqueness part, so let us just check this, define gamma from R to gamma t is the constant curve. Just take the constant curve then gamma prime t it is a constant curve and whatever definition of derivative however one likes to calculate it is easy to see that you get 0, so the 0 vector and this should be equal to, so this is 0, on the other hand the 0 is also X at by assumption this is Xp which is also equal to X at gamma T.

Since gamma t is the constant curve so what we end up concluding is that? The integral curve equation holds for gamma therefore gamma t. And we know that integral curves a unique so gamma t equal to sigma V t. And so when I make this statement about uniqueness and so on one should to be careful one should always mention the domains. So here for instance when I say sigma P T the normal thing to do is let you specify sigma P to be the integral curve defined on the maximal domain on definition around starting at P.

So in other words extend start with some integral actually see how far you can extend it. Take the largest possible interval and work with that otherwise you can always restrict to a sub-interval and still it could be a solution and there will be ambiguity but that is more of a technical thing. So with that understanding this gamma t is equal to sigma Pt and one is done, so this is equals to P, rather the left hand side equal to P.

So when a vector field has a 0 at a point the integral curve is just that point itself. Now we also talked about the smooth dependence on initial data that actually gives rise to a very instruct, interesting construction on in the setting of a manifold. So let us call this corollary one and corollary two is for simplicity. Let me take the case, so actually one need not restrict to this case but let me take the case when the integral, suppose all integral curves on M are defined on R.

So one must be careful that this is a somewhat I mean, this is a non-trivial condition as we saw even the simple quadratic polynomial vector field on R does not, this condition is not satisfied, the intervals of definition are of this form minus infinity to 1 over alpha, 1 over alpha to infinity.

So in particular they are not all of R. But there are lots of other examples where this condition does work, all integral curves on M are defined on R, then the map G from M cross R to M defined by g x t is so.

Just look at the integral curve starting at X at time t. So this map is smooth so this is, now this follows from what I said earlier that regards the last part of Picard. This statement here smooth dependence now 1 this in the in the Picard's theorem. I had to restrict this V a smaller open set V on which I could make this statement that this smooth. And so on but one can check that in fact and smaller open set and.

So one can check that actually even in this case so I will not go into too much of details here that on the, for one thing smoothness is always a local issue. So, for example, here I can restrict X if I want to check smoothness at a point X naught D naught, I just have to restrict to an open neighborhood of X naught and a neighborhood of t naught. And then consider this so I do not have to work with all of M and R. So in fact this does follow from the Picard statement, so one gets this map and not only that if I fix. Now, let us turn it around and in fact what we will do is?

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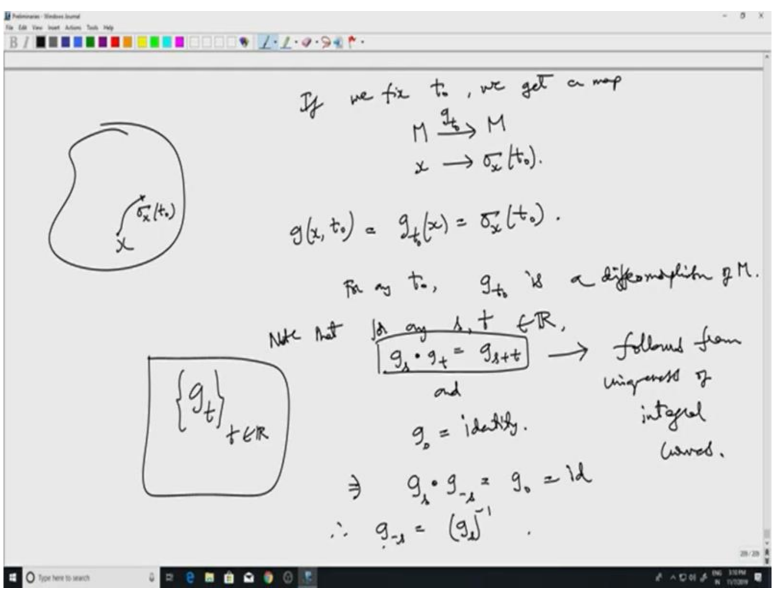

Let us fix this T and regard it as a function for, if you fix for, if you fix t naught we get a map M to M which is X going to sigma X at t naught. Let us call it g subscript t, gt X naught g t naught x is my x at t naught if it is just g of x t naught. So I regard it as a function of one variable so all we are doing is we are taking any point X that what the map does is take any point X? Look at the integral curve starting at x travel along that for time t naught.

So we will end up with sigma x t naught that is your new point so every point on M have associated another point of M that is this map from M to M like this. And the claim is for any t naught, g t naught is a diffeomorphism of M. This so this family of maps gt is called the flow associated to the vector field. So the moment you have a vector field, you have integral curves. Using integral curves you get this flow and each, the flow consists of diffeomorphism of M. Now how does one see that this is a now how does one see that?

This is a deofformism of the smoothness of well you have to check that gt naught is smooth and bijective and its inverse is smooth, the smoothness of g t naught just follows from the smoothness of g. Since all I have done is a fixed one variable regarded as a function of the other variable. So this is smooth, now to see that g is bijective. There is a neat way of seeing, this namely just notice that if I now, note that for any s and t in R gs composed with gt is g of s plus t.

If we have this then it is immediately clear that and also the other thing is when I take t naught equal to 0. And g 0 that is, well if I put t naught 0 I will be looking at sigma x at 0. So the integral curve that starts at x. I want to see where it is at time 0 by definition it is at x. So this is just the identity map. So gs composed with gt is gs plus t and gs subscribe g naught is identity.

This implies that gs composed with g minus s g naught equal to identity. And so therefore g has an inverse which is, gs has an inverse which is g minus s, g minus s is gs inverse. So this proves two things one is that gs is bijective, the other thing is the inverse of gs is also of the form g of something therefore it is also smooth. So it immediately follows that, it is a diffeomorphism. Now this statement the gs composed with gt is just.

I will not do this but I will just write the basic reason follows from uniqueness of integral curves. 1So in short if all integral curves are defined for all time, then one has a nice family of diffeomorphism of the manifold which depend on a single parameter t not only that this family of diffeomorphism of gt t belongs to are actually a group in the sense that and a composition, if you compose two of them you get one more thing like that and identity is inside that because g 0 is identity and inverse is there because of what I just saw. So if you have a vector field with integral velocity find for all time then and as I said there is one very important case where integral curves are defined for all time.

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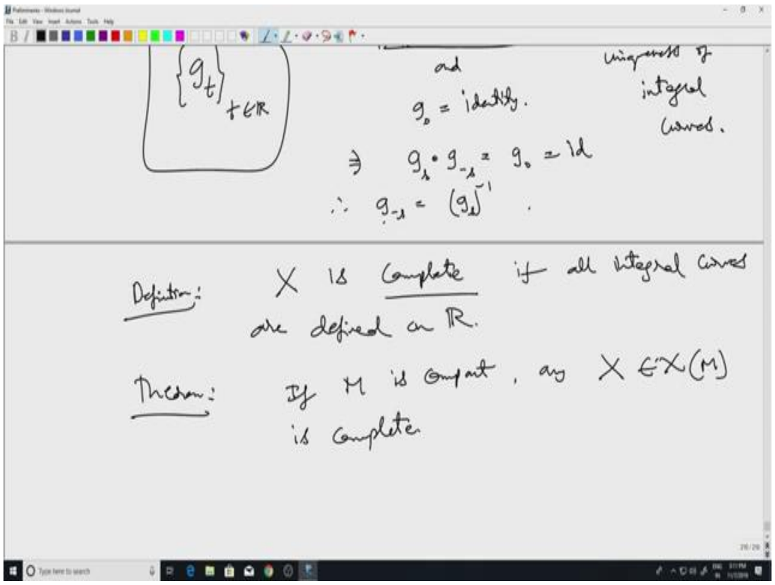

So this is a theorem in ODE's actually which I will not prove if so first let us say that let us give it a name. When integral curves are defined for all time X is complete all integral curves are defined R. Then the theorem is that if M is compact any X and any vector field is complete, so in the case of a compact manifold one has this nice flow and so on, but even if it is not compact. If the vector field is complete one has a such a flow so all right. So we will stop here and then I will see you in the next lecture. Thank you.