An Introduction to Smooth Manifolds
Professor. Harish Seshadri
Department of Mathematics,
Indian Institute of Science, Bengaluru
Lecture 33
Integral curve and flows 1

Hello and welcome to the thirty third lecture in this series. So, in this lecture I will talk about vector fields in the context of differential equations, ordinary differential equations.

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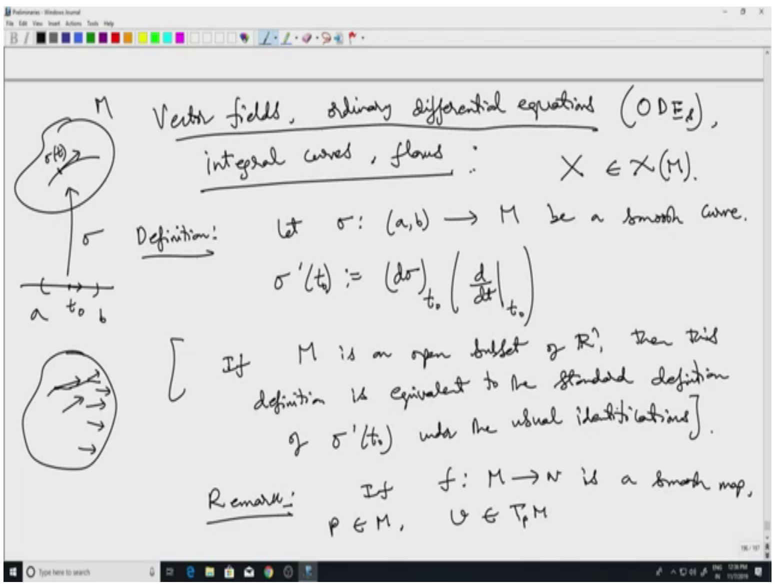

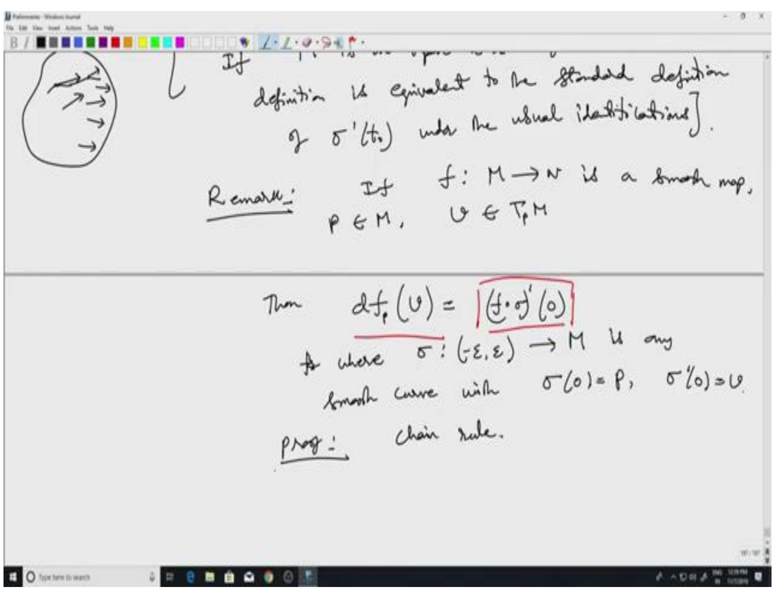

So, let us so vector fields, ordinary differential equations which elaborate ODE's integral curves and flows. Let us start with the notion of a. So, as usual X will be a vector field on a smooth manifold M I would like to define the notion, now I want to define the notion of an integral curve. So intuitively before stating a formal definition if we think of a vector field as it is an assignment of an tangent vector at each point.

An integral curve is a smooth curve in a manifold whose velocity vector whose tangent vector at each point is exactly equal to the vector field at that point. So, to state this, first, let me define the velocity vector of a smooth curve. Let sigma from an interval a b to M be a smooth curve. Well, so I want to say the, I want to talk about sigma prime t. So, what I do is I just sigma is a map from this interval a, b to this is M, so if I take a point t, this will go to sigma t.

Now on this interval I have the standard basis, if I take the point t for the tangent space at this point, the usual basis of which just consists of a single vector. So, this vector regarded as a derivation is just d by dt in Rn we have always talked about del by del X1, Del by Del X2, etc. but if we have just one variable, we can still use the del by del t notation but the convention is that when it is just one variable, we call it d by dt.

So, I have this d by dt then I can take the derivative of sigma act it on this basis vector whatever I get I call sigma prime t sigma prime t, as by definition d sigma at the point t of actually let me call it t naught this to sort of d sigma at the point t naught of this basis vector which is d by dt evaluated at t naught. This is my definition of velocity vector, and it is of course, it is when we have the manifold is just another open subset of Rn.

This definition just coincides with. So, if M is an open subset of Rn, then this definition is equivalent to the standard definition of sigma prime t naught, the standard definition of signal prime t naught is just differentiate all the components of sigma and put them together as a element of Rn again is equivalent to the standard under the usual identifications, of course, the usual identification amounts to what we have been doing all along.

The tangent space to Rn on the one hand it can be identified with Rn and this on the other hand it is the space of derivations. And so in effect, the derivations at a point on Rn can be naturally identified with Rn. And we have seen that the reason is that there is a frame on Rn, in other words, there is a basis natural basis for the tangent space at each point, namely the partial derivatives using this one can go back and forth between derivations and the tangent space as Rn itself.

With that in hand, then this is the same as they usually one. Then, in this connection having introduced this I should also mention that remark something which I should have done earlier. Recall that when we started the course, we heard a nice way of computing the derivative as a linear map. Suppose the derivative is acting on a vector v, we saw that it could be computed by just by seeing the action of the map on any smooth curve tangent to v.

And then taking one more derivative. So same thing holds here remark, if f from M to N is a smooth map P is a point in M, v is a tangent vector at t. Then the derivative of f at p acting on v is now the with this notion of sigma prime f composed with sigma prime at 0 for where sigma from some small interval to M is any smooth curve with sigma 0 equals p and Sigma prime 0 equals v sigma prime, I have used this sigma prime the velocity vector of a curve in two places f compose with sigma prime and sigma prime itself.

And based on this definition that I made here and this is the proof of that, the derivative can be written like this is immediate, it follows from chain rule. While the proof is quite trivial, it is still a very useful fact, because the point here is that the left hand side of this equation here makes no reference to any smooth curve.

All that matters is the velocity vector v and the smooth map f, the right hand side seems to involve a smooth curve, but actually in reality it does not matter which curve you take. You end up getting the same answer dfp of v and depending on the smooth map f one can choose some nice curve sigma so that it is easy to compute the right hand side f compose depending on the depending on f sometimes one can choose sigma so that f compose with sigma prime is easy to compute.

So, that is the usefulness of this, so I will just remark. So, here proof is chain rule.

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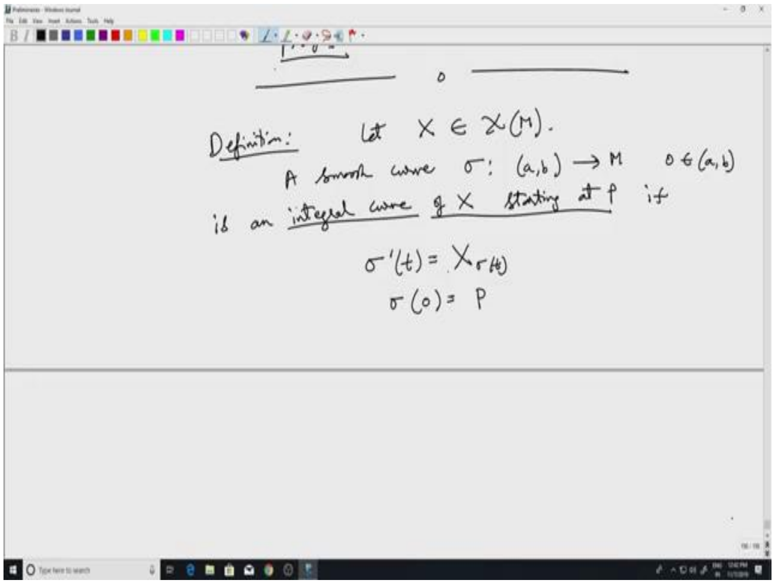

Well, so now let us return back to now the other main definition. So, this is something very general. It has nothing to do with what I defined here and the subsequent discussion has nothing to do with the vector fields. Now, let us come back to vector fields, let X be a vector field on M. So, I would like to yeah as I said, the notion of a integral curve is that I would like to find a curve whose velocity vector in the sense defined here is exactly the arrow which has been prescribed.

So, in other words, the velocity vector is the value of the vector field at that point. So, let us write that as an equation a curve, a smooth curve sigma from a, b to M is an integral curve of X starting at p if sigma prime t should be equal to the value of the vector field at that point. Now sigma prime t as we have defined it is actually a tangent vector at the point sigma t. So, I want something sigma t and that thing is X at sigma t. This is the defining property and then the starting at p amounts to saying that sigma. So, smooth curve, so instead of saying, that iss okay let me retain that, a smooth curve so let me add that zero is in a, b the interval contain zero.

So, I want sigma 0 to be p. So, one thinks of this t the parameter t as time. So, at initially the curve starts at P and then subsequently it moves in such a way so that its velocity vector is dictated by the vector field. Actually, it is actually literally equal to the vector field.

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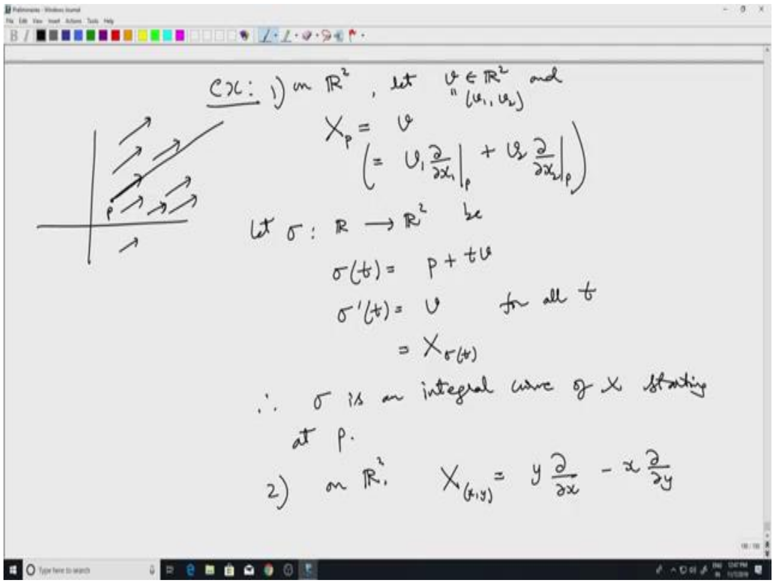

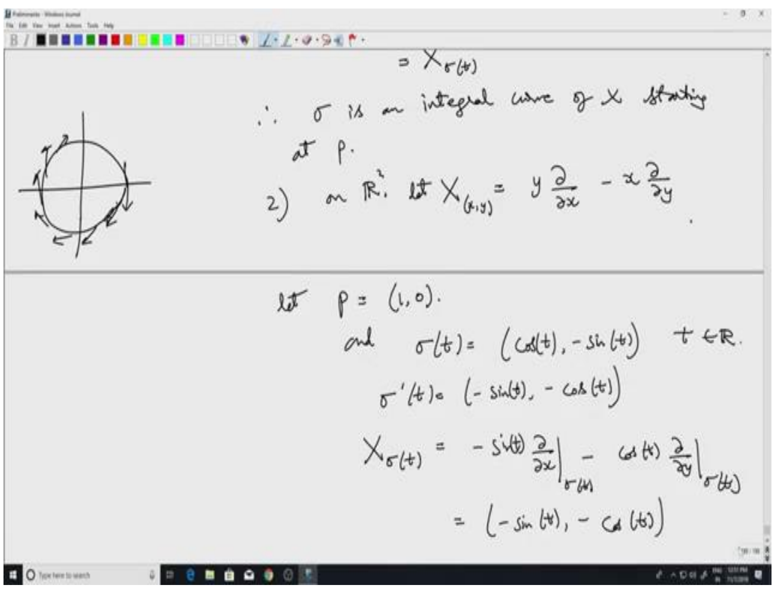

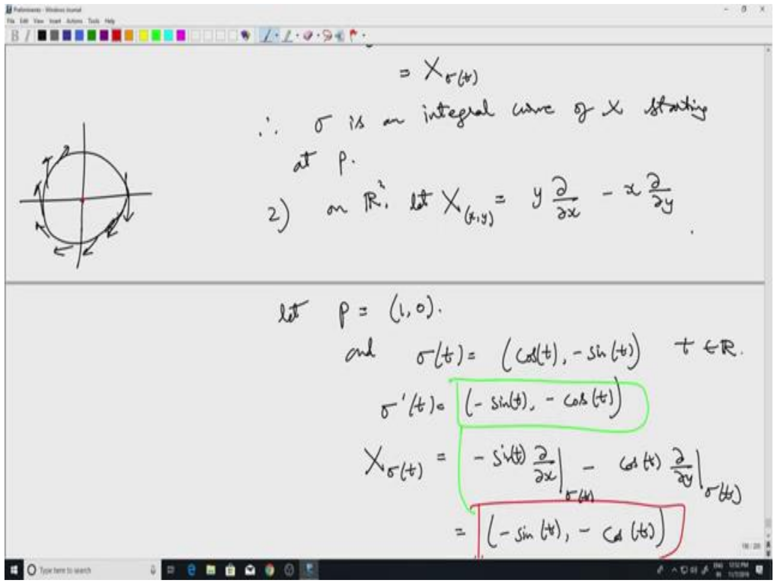

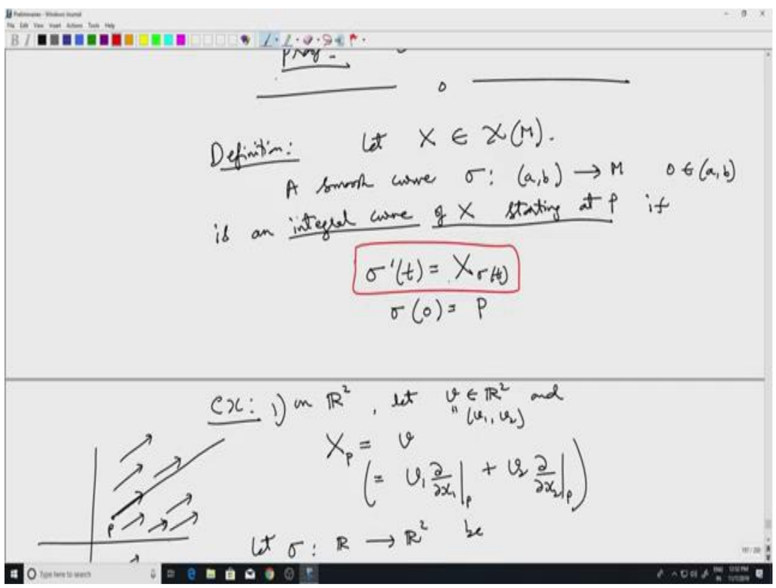

So, it is a let us look at an example on R2. Let us take a trivial example to begin with, let the vector field X at any point p is just a fixed vector v. So, I start with v in R2 and X at p is equals to v. So in terms of so if v is v1, v2 then in terms of this when I write Xp equals v I am thinking of v as a element of the vector space R2 but has a derivation we know that this is the same as v1, del by del x1 and the point p plus v2 del by del x2 to p. So, we should in the context of Rn one always goes back and forth between these two viewpoints. A tangent vector has a derivation and a tangent vector is just an element of Rn.

So, now here pictorially all one is doing is just this one has a fixed vector v and one imagines that at every point one is putting the same vector all over R2. So, it is clear what an integral curve has to be in this case. So, if I start at a point p, I just move along the straight line in the direction of v. So, in fact let us check that, the claim is that if you take any point p look at the curve this time, I can define the domain of the integral curve, which I said as a, b here I can just take to be the whole real line.

Let be just the straight line, which starts at p and moves in the direction of v. In other words, p plus t times v. Now, if I compute sigma prime t as I remarked earlier, this notion of the velocity vector of a curve. So, in this parenthesis, I have remarked that this definition is equal to the standard definition of sigma. So, I can just use the standard definition here to say that sigma prime t is just v. For all t, but which is exactly X at sigma t, therefore, sigma is an integral curve of X starting at p.

Now, the second example is let us look at something slightly more interesting, again on R2 I look at this vector field X at the point x, y is y del by del x minus x del by del y. The partial derivatives are evaluated at that point x, y, let X be this. Now let us take lLet us take the point let p equals the point 1, 0. So, let us say I want to find the integral curve on this vector field starting at this point. So, I guess first actually, in order to guess the answer, one has to so let us see what does that add up at p itself this would be the X coordinate is one y coordinate is 0 so what I get here is minus del by del y, so, it will be like this.

And as I move along one can check that the vector field looks like this at various points of the circle. So, in other words, the circle is automatically tangent to this vector field. The unit circle with centre, the origin is tangent to this vector field. Therefore, we would expect that the circle, the nice parameterization of the circle should give interval of curve. So, indeed that is the case. So, let us take let and sigma t equals cos t.

And then now I want to move in the in this direction. So, I will take the minus sin t and again t I can take to be any real number. So, if we compute sigma prime t, sigma prime t is minus sin t, minus cos t. Now, let us see what if we suspect that this is the integral curve. This is an integral curve I should know what X at sigma t is.

So, X at sigma t is, well sigma t is this. So, X at sigma t would be so minus sin t del by del x at sigma t minus cos t del by del y at sigma t. And this is the derivation interpretation of a tangent vector if we go back to the Euclidean interpretation, then this would be minus sin t, minus cos t.

So therefore, what I get here and what I get here, these two are the same. The one concludes that therefore sigma is an integral curve of X starting at this point p. Now notice that this vector field that we had here at the point 0, 0 at the origin, the vector field is just the 0 vector, 0 tangent vector. In that case, suppose I wanted to find an integral curve which starts at the origin. Well, since the defining equation so my sigma prime 0 would have to be x at sigma 0 which would be 0.

So, the curve would have to have 0 initial velocity. And in fact it turns out that it is easy to check that after I discuss a bit connection with ODE's that the constant if curve it is the only integral curve if the vector field is 0 at that point.

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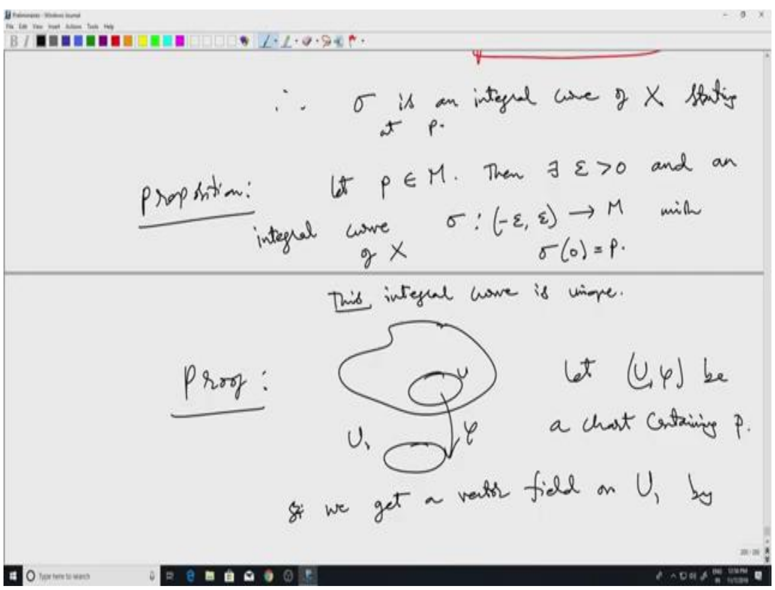

This integral curve is unique.

Proof:

Let (U, φ) be a chart containing p.

So we get a vector field Y on U_1 by

$$Y_y = d\varphi_{\varphi^{-1}(y)}\left(X_{\varphi^{-1}(y)}\right).$$

Note: If $M \xrightarrow{f} N$ is smooth and $X \in \mathfrak{X}(M)$, we cannot use df to get a vector field on N.

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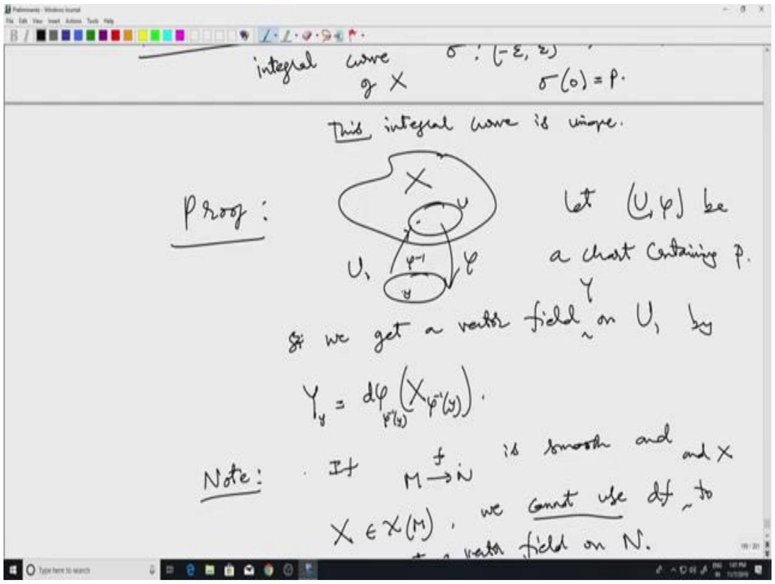

So, here is an important in fact, let me state it as a proposition. So, whatever manifold one starts with and whatever vector field on has take any point, let p belong to M. Then there exists epsilon greater than 0. And an integral curve sigma, which is defined on this integral minus epsilon, epsilon integral curve of X it start with the point p, that is one thing. The second thing is this integral curve is unique, this meaning that this it is starts at p.

So, there is only one integral curve and there is exactly one integral curve which starts at the point p is the content of this (())(26:27)proposition. And how does one see this? So, this is where ordinary differential equations enter into the picture. So for this, let us use a coordinate chart to transfer the situation to Euclidean space and see what happens there. So I start with a chart U, phi let U phe be a chart containing p we get a vector field on, so, this is the open set U, what we have been calling U1 and this is in Rn.

So, we get a vector field on U1 by, so here I want to define a vector field on U1 so I start with a point let us say X here. So, I would like to specify, so I already had a X here so I want to define a new so instead in fact, let me call it just to be clear. Let us call it label this as small y I want to define a vector field y on U1.

So, Y at the value little y, well I just use this since phi is a dimorphism that derivative is a isomorphism, so I can take the derivative of d phi inverse phi inverse is go in this direction, sorry rather so, this point is y I use phi inverse to get to phi inverse y here in the manifold or in the open set U. They have the original vector field x value and then use the derivative of phi to come back here.

So, in other words I start, I look at X at phi inverse y then use the derivative of phi d phi at the same point phi inverse y and this is what I defined to be y. Now, so in other words given a vector field here I have been able to get a vector field on this U1. I should remark that, so at this stage it is important to observe that, note if you have a smooth map between two manifolds and X is a vector field on M, it is not possible to get a vector field on N by using the derivative of f. If I had a single tangent vector, if I had a tangent vector on M then I know that the derivative will give me a tangent vector on N.

But I cannot define a vector field, we cannot use df to get a. We cannot use df and X to get a vector field on N. In any natural way, I mean cannot use a has to be interpreted suitably, but in any natural way we cannot get a vector field on N. And the reason is that if I start with a point here, so this is M this is N if I start with a point here, if I want to get a and I already have a bunch of arrows here the vector field here.

So, if I start with a point here, I would like to use one of these arrows and push it forward by the derivative to get something here, but the problem is, if F is not bijective, then first of all, this y may not even be in the image of f in which case it is there is no inverse image, so I do not know which point to take.

And even if y is in the image, the map may not be injective. So, there might be several points in f inverse of y. So, it is not clear which tangent vector I take to push it forward. So, that is the reason, but this this situation changes for a diffeomorphism if I had a diffeomorphism, I can push forward a vector field like this. And of the way I do it. So notice that I have already used the fact that the map phi has an inverse.

So, I will stop here in the next class I will resume at this point. We will quickly check that this map, we have already seen some version of this. We will check that this y is smooth. And essentially we can transfer the situation to an open subset of Rn and find an integral curve and an open subset of Rn, use that to get back an integral curve on the manifold. So, let us stop here. Thank you.