An Introduction to Smooth Manifolds
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Lecture 32
Lie Groups 2

Hello and welcome to the 32nd lecture in the series. So, last time towards the end of the lecture I had started talking about lie groups. So, let us continue with that.

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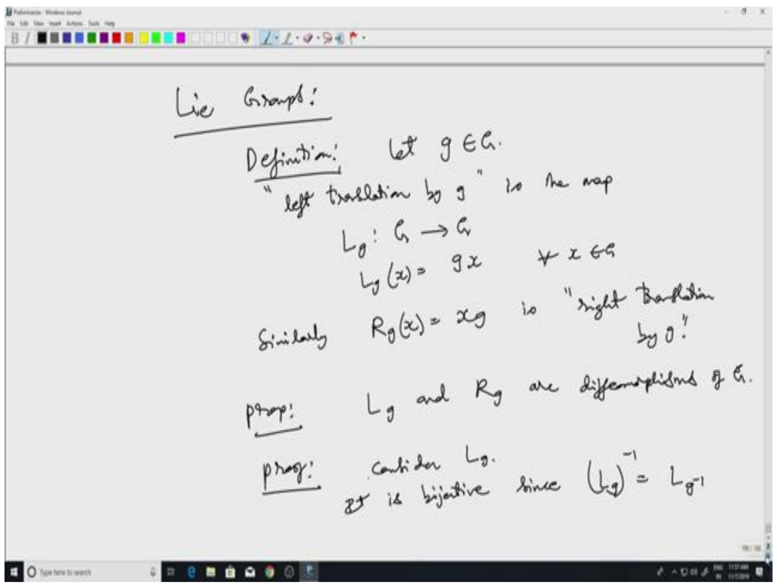

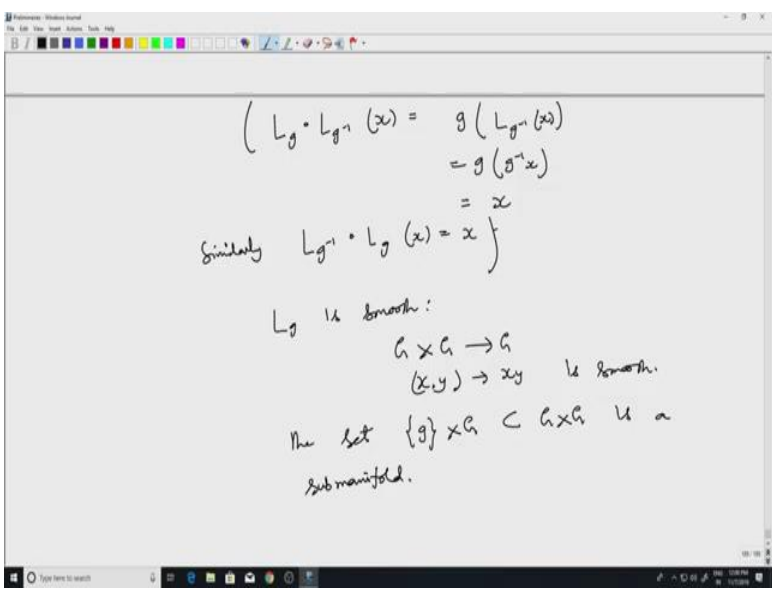

The set $\{g\} \times G$ ...

submanifold.

[More generally, in a product manifold $M \times N$, the subsets $\{m\} \times N$ and $M \times \{n\}$ are submanifolds for any $m \in M$, $n \in N$]

$\therefore$ the restriction of φ to $\{g\} \times G$ is smooth.

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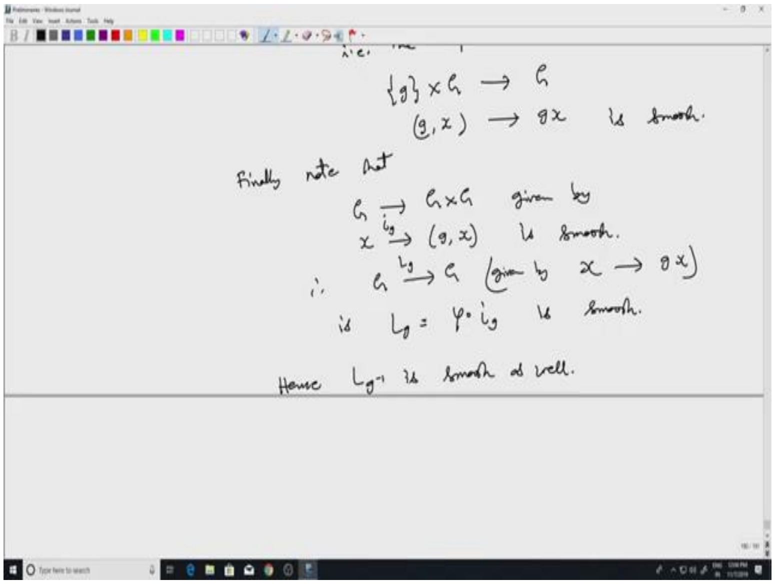
Finally note that

$$G \to G \times G \quad \text{given by}$$
$$x \xrightarrow{i_g} (g, x) \quad \text{is smooth.}$$

$$\therefore \quad G \to G$$



So, lie groups are essentially manifolds, which have been group structure such that the group operations group multiplication and inversion are smooth maps. So and we saw that GL n R is a lie group. And because these the orthogonal group and SL n R are sub manifolds of GL n R. The fact that multiplication on GL n R is smooth will immediately imply that there is a corresponding operations on these groups are smooth as well.

So, these are also lie groups. Now in the context of a vector fields, there is a special class of a vector fields on a lie group. So, let us define that, to begin with let us consider, so definition. We take a group element G small g corresponding to this I define left translation by g is the map L g, from G to G given by L g of x is just g x for all x and G. Similarly, R g the right translation, right translation by g. The first thing to note is that L g and R g are diffeomorphisms of G. And this is clear because L g for instance. First, let us observe that, consider it is the proof for R g is the same as proof for a L g.

So, consider just left translations. Well, we have to check to say that it is a diffeomorphism we have to check that L g is smooth bijective and the inverse smooth as well. First of all, it is immediate that L g is bijective. Since we can immediately write down an inverse for this L g inverse is just L g inverse. So, in another words, if I do L g composed with L g inverse acting on x, I get by definition g of L g inverse x which is the same as g or g multiple not g of g multiplied by a g inverse multiplied by g, which is the same as this and which is x.

Similarly, the other composition is also x. So, this the fact that we are able to explicitly construct a inverse map shows that it is bijective. Now, as for smoothness, we will proceed as follows. We know that G Cross G to G the map x, y go into x y this is smooth. By the definition of a lie group. So, the set g cross G contained in G cross G is a sub manifold.

This is a more general fact about product manifolds, we had seen it earlier. In any case, let me just mention it again. More generally, in a product manifold M cross N the subsets m cross N and M cross n are sub manifolds. These are what we call slices for any m in M little n in capital N. So, the point is that it is quite straightforward to write a slice chart for this, these two sets insight m cross n.

Now, yeah with this in hand, since this is a sub manifold therefore the restriction of well, restriction of this, so let us call give this a name, this map as phi the multiplication map. The restriction of phi to g cross G is smooth, but this restriction is i.e. the map, the first coordinate has been fixed to be g. So, the map is g cross G to G and this map has g, x going to g times x is smooth.

Then all we have to do is observe that, finally actually we are interested in just G, not this small g cross G, but that is the same as G. Finally, note that G to G cross G given by g going to not g, let us take x, x going to g cross x smooth this we had seen earlier. This just embeds or rather sort of places G in G cross G as a slice, so given by this smooth.

So, let us call this ig. Therefore, G to G given by x go into g x, so this is what this is our left translation map. As L g equals well, first I start with G. I put it inside G Cross G by ig. Then I do this phi both of these are smooth. So, therefore this a smooth as well. So, in short this map is smooth and that well. So, we were trying to check that L g is a diffeomorphism bijective it was immediate because we could construct an inverse and I just shown the smoothness of L g. And this is true for any g and so I might as well start with g inverse, L g inverse is smooth as well. So, the inverse map is also smooth. So, and the same considerations apply to R g.

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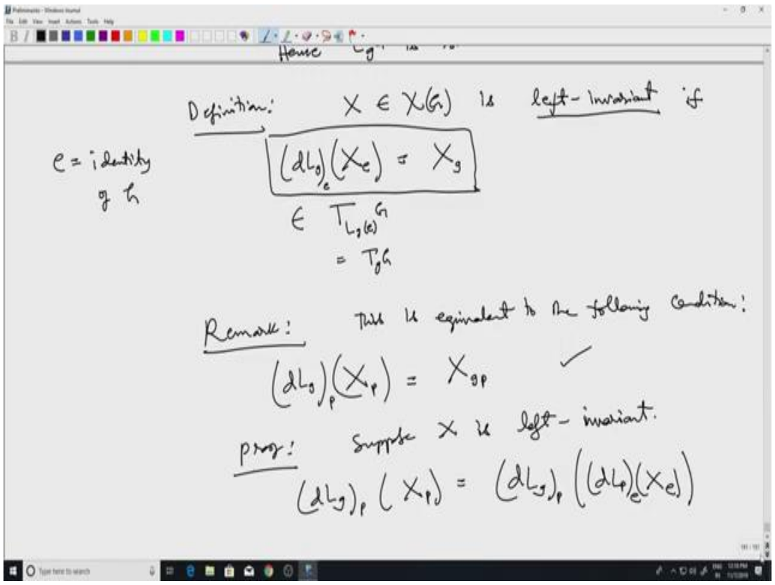

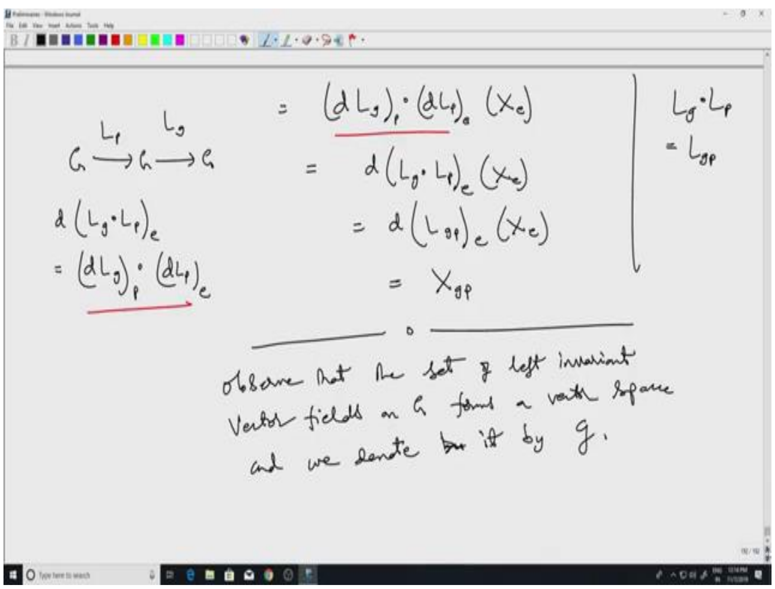

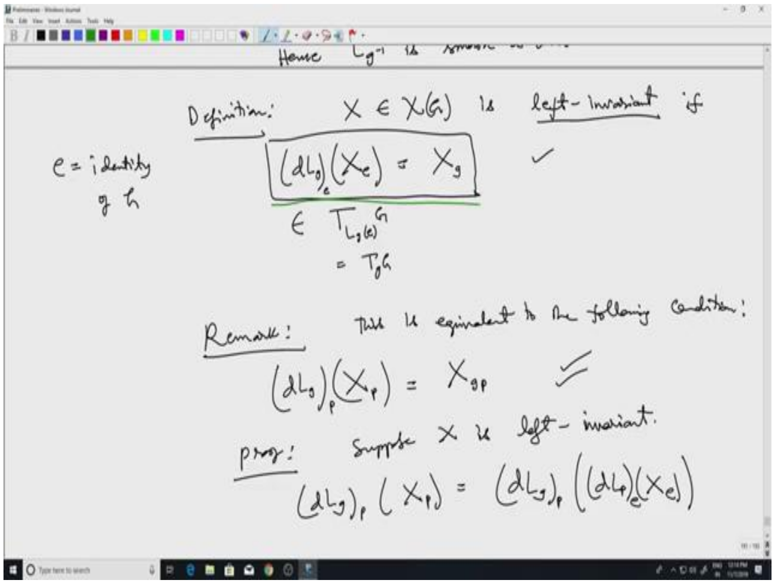

Now, let us say that, so the second definition I need is. If I start with a vector field, smooth vector field on G, I should change this more to look like Chi, Chi x a vector field x on G is said to be left invariant. If, what I can do is I can start with the value of the vector field at the identity element. So, here e equal to identity of G. So, I start with the identity element, then I look at the derivative of the left translation L g at identity. Now, this is in general, this will give me a map, a vector so this is a tangent vector at identity X e is a tangent vector at a identity. So, when I do this it will give me a tangent vector at g times L g acting on e but L g acting on e is just G.

So, this will give me this in general, this would be a tangent, an element of the tangent place at L g of e of G and L g of e is just g itself. So, this will give me a tangent vector in at the point G. Now, the condition that it is the vector field is left invariant is that this tangent vector is precisely the value of the vector field at G. So, this is what this is the defining property of a left invariant vector field.

And as a remark, this is equivalent to the following condition. I can start with instead of starting at the tangent vector at identity, rather the value of the vector field at identity I can start with at any point. So, let us see let us take X at some point p, let us apply left translation by g. So, the derivative at the point p and what we want to say is that the condition is that this should be gp. Well, this is what I wrote is looks slightly more general than the defining property of left invariants.

Because if I take this p here to be the identity element, then I have the original definition of left invariant, but here I am demanding it for any p but it is quite easy to see that this more general thing follows from this. I want I will just quickly sketch this. So, as I said, this condition is more general than the original condition. So, I if I want to show that this is equivalent. Let me start with the original condition and show this more general condition.

So, suppose X is left invariant as in the original definition, I want to check this thing here. So, let me just (())(15:41) X p. Now, according to the original condition, X at any point is given by this the left hand side here. So let me use that, so, this would be dLgp and then instead of X p I plug in this dLg rather d, not L g, L p at e acting on X e. And well, now so it is essentially a composition of two maps two derivatives dLgp composed with dLp e of acting on X e.

Now, just let us just apply chain rule. According to the chain rule, so I have two maps here, the first map is L p, the second map is Lg is translation by G. So, according to the chain rule and evaluating the derivative of the if I evaluate the derivative of the composition the first thing well L g composed with L p at identity chain rule would tell me that this is the same as dLg of at the point L p of e, which is just p composed with dLp at e which is exactly what I have here.

So, what I have here and what I have here are the same, so I can just write this as dLg composed with L p identity acting on X e. And finally, you just note that L g left translation by p, then you compose with left translation by g is the same thing as. So here, I will make L g composed with L p is the same as Lgp. So, dLgp at e of X e, again going back to the definition, original definition of left translation. So if I had a g here, it would be g would occur here, but here I have gp. So, what I get is Xgp which is what we wanted. So, I get this equation here starting with this one.

So, now so this is the definition of a left invariant vector field. Then we also observe that the set of left invariant vector fields on G forms a vector space over R and we denoted by this gothic g. Now, the fact that it is a vector space, if we add two left invariant vector fields. So it is quite it is immediate just because left translate the derivative of any smooth map is a linear map. So, if I start with two left invariant vector fields, if I add them up it is immediately clear that this equation here is will continue to hold for the sum.

Similarly, if I take a multiplied by a fixed real number vector field left invariant vector field by a fixed real number, the new vector field will also satisfy this. So, we get a vector space now, even though, as I remarked earlier in the last lecture, I remarked that the set of all vector fields on this smooth manifold is not a finite dimensional vector space. However, and the same thing holds true on a lie group as well, that if one looks at the set of all vector fields, it is certainly not finite dimensional.

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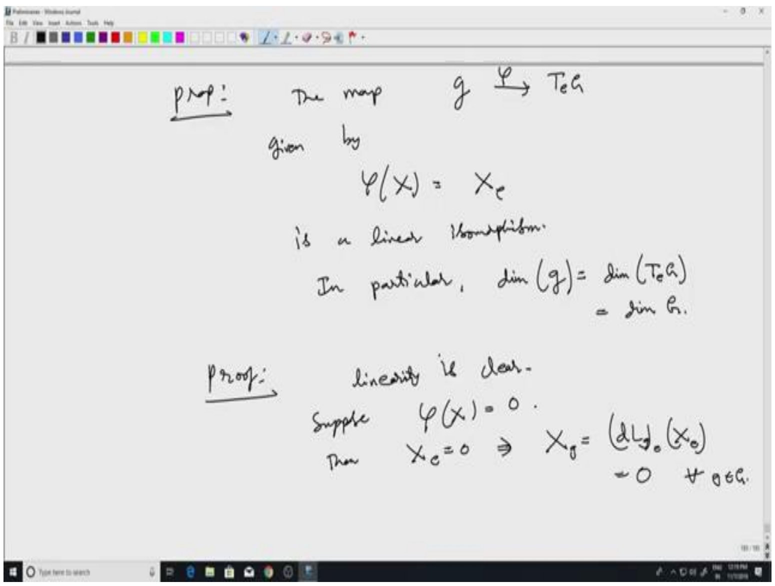

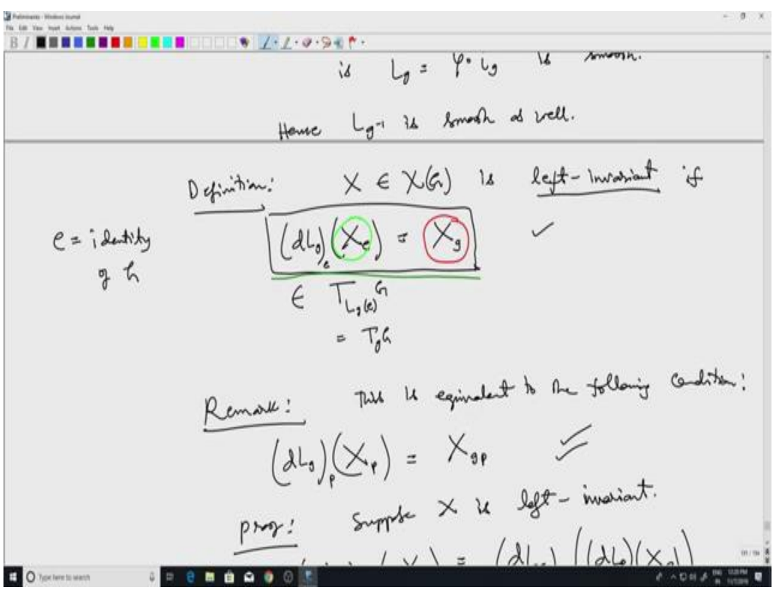

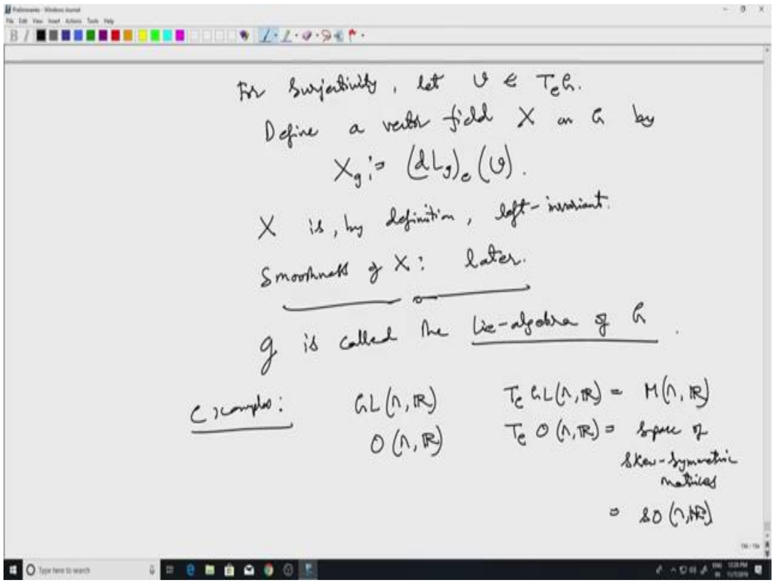

But this left invariant vector fields form a finite dimensional vector space. And in fact, our claim is that the map from the tangent space at identity to g, let us call this map again as some phi given by actually, I wanted the other way, from g to the tangent space at a identity given by phi of X start. So, this g on the left side consists of left invariant vector field. So, I start the input as a left invariant vector field and I just evaluate it at the identity that is this map.

So, this map is a linear isomorphism. In particular the dimension of the set of left invariant vector fields space of left invariant vector field is the same as the dimension of the tangent space at identity, which is the same as the dimension of G. So, let us prove this, so first thing is it is I said that it is a linear isomorphism. So, the first thing is that linearity is clear, since if I add if I consider X1 plus X2, I will be looking at the value of X1 plus X2 at a entity, which is the same as the value of X1 at a identity plus the value of X2 at identity.

Linearity is clear and what does. So, what we have to do is check that the map is bijective, so since it is linear we injectivity for we just have check that the kernel is just the zero element. So, suppose fee x is equal to zero, then X e would be zero. Since phi x is X e, since X is left invariant X at any point g is dLg of X e this would also be zero for all G.

So, this proves that the only the zero vector field, which is also a trivial left invariant is the only one which gets map to zero, the zero tangent vector under this map. Now, as for surjectivity, so this proves that Hence, phi is injective for surjectivity. Let V be an arbitrary element of TeG. So, I would like to realize v as the value of a left invariant vector field at a identity.

So, I do the expected obvious thing, which is let us go back to the definition of a left invariant vector field. What infected saying is that, if I know the value of the vector field at identity I know it at any other point X g. So, if I know this, I know X g and they are related by this equation here.

So, now we can turn this around, and suppose we are given some tangent vector at identity, I use this equation to define the vector field at any point G. So, define vector field X on G by X at g, when I want to define a vector field, I just have to specify what its value is at any point. So in other words, what tangent vector I get at any point. So, here I am saying that X g is by definition dLg e of this vector V that we started with.

So, now this is by definition left invariant. So, this X by definition left invariant, but what is missing is I have to when I said that I said define a vector field X according to our definition of a vector field, it should be smooth an assignment, which is smooth.

Here, I just given an assignment which is left invariant. Smoothness of X it is a, it is not immediately clear because the problem is that we are varying its smoothness with respect to the point G as G varies. This left translation maps also change, and of course, the derivatives changes and so on. So, but we have what in effect what we are doing is we are considering the action of the family of maps, family of derivatives on a fixed vector V. So it requires some work to prove this.

So I will come back to this later on. For now, let us assume that the smoothness of X will prove it, return to it later. So given the smoothness, the proof is complete that I have a bijection from the set of linear isomorphism of from the set of this to the tangent space identity. So, the significance of left invariant vector fields at this stage is not so clear, but as it turns out the algebraic properties of the group are entirely reflected in a entirely captured by the lie algebra of by the way, yes I forgot to mention that this g is called the lie algebra of G.

Now, so far g is just a vector space, but it has a crucial additional algebraic operation on it called the lie bracket, which I will talk about shortly. Then, the statement is that this lie algebra with this additional lie brackets structure entirely captures the algebraic properties of G. And this statement can be made precise, though I will not be able to prove anything. I will be able to state the precise relation between the algebraic properties of this G and g and big G.

So, now as examples, of course for GL n, R GL n, R is an open subset of Rn m, n R which is a vector space of dimension n square and therefore, the tangent space at identity, for that matter, the tangent space at any point can be identified with a ambient vector space, which is a M n, R the when we come to O n, R, we have seen that the tangent space at identity is set of space of skews symmetric matrices.

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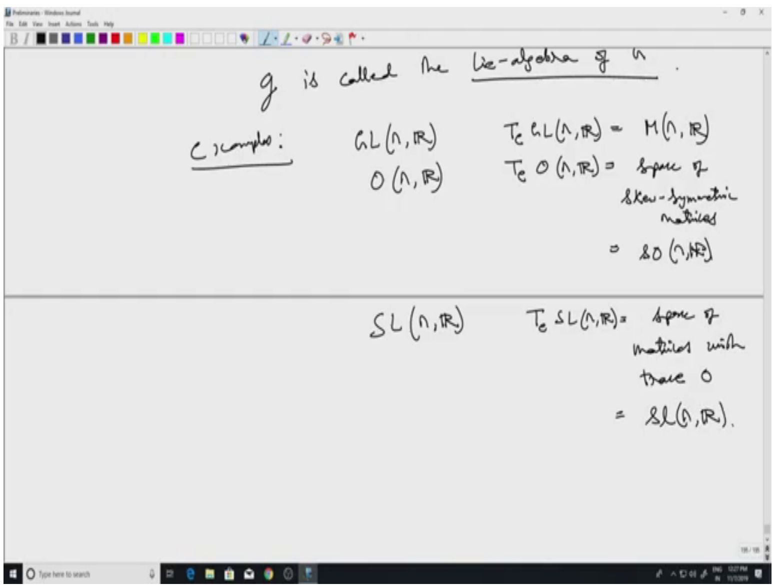

And this is usually denoted by, yeah so little as so n, R and the final thing that I mentioned is we have also seen that the tangent space of SL n, R at identity as the space of matrices with trace 0. And this is denoted by sl n, R. So we will stop here in the next lecture I will talk a bit about the relation of vector field ordinary differential equations integral curves and flows. So thank you.