An Introduction to Smooth Manifolds
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Lecture 31
Lie Groups 1

So hello and welcome to the 31st lecture in the series. And I was giving some examples of vector fields last time.

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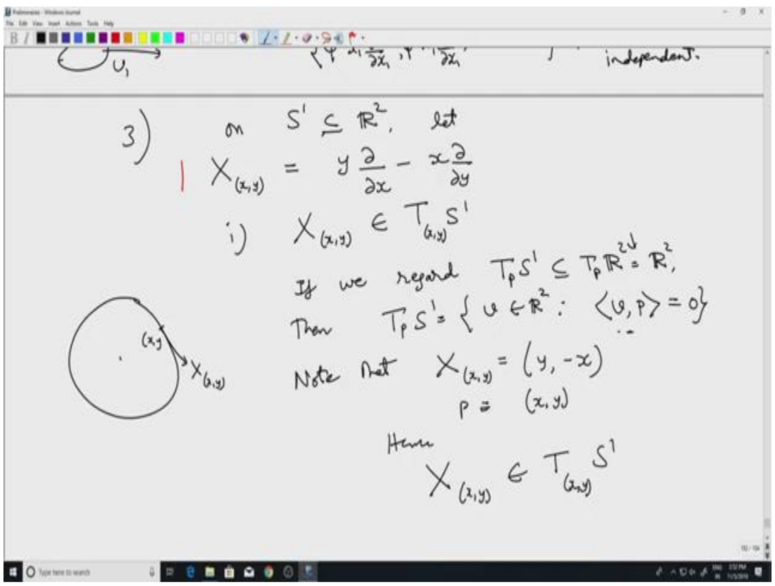

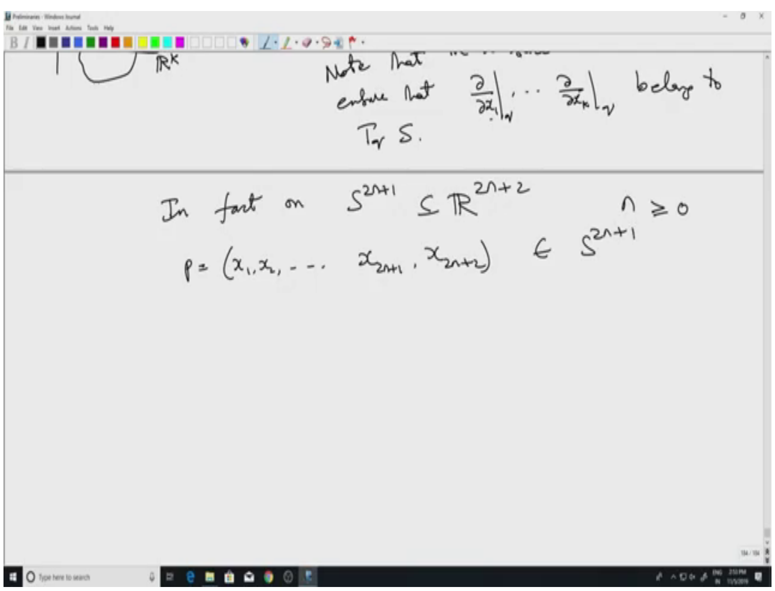

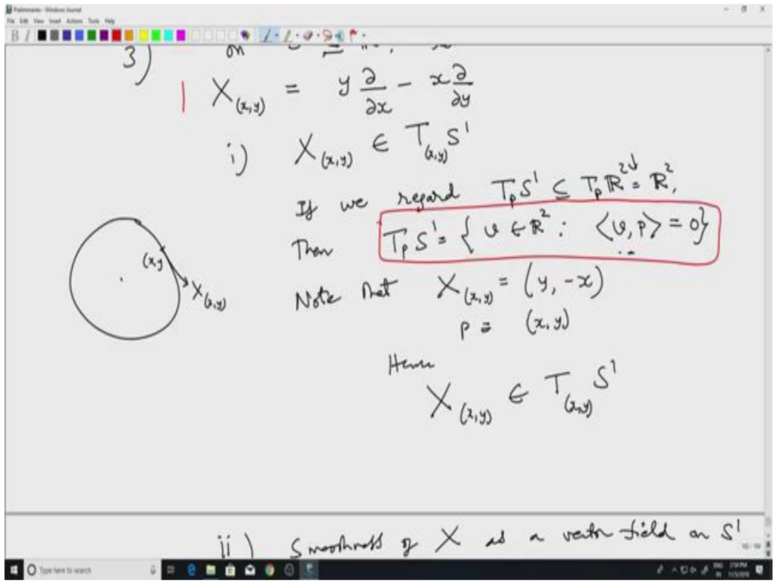

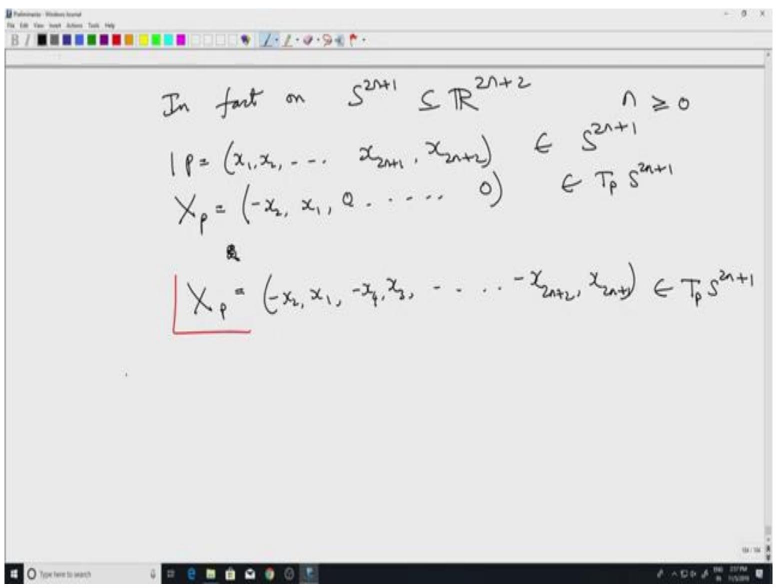

So, in particular I had this I gave an example of a vector field on s1. The point X Y, I am looking at X, x y. Now notice that, yeah in fact this it can be, this example can be generalized to all odd dimensional spheres. In fact on s 2n plus 1 contained in R 2n plus 2 for any n bigger then or equal to 0.

I can write down a similar vector field. So, let us start with a point x1 P equals x1 x2 x 2n plus 1 x 2n plus 2. So, let us start with this point in the sphere, 2n plus 1 dimensional sphere. Again, the tangent space to the sphere in any dimension we know is given by the regular value theorem. The tangent space is still given by the right hand side. It looks at the point P it consists of all vectors v such that v in a product with P is 0 and the subspace of the ambient Euclidean space.

So in this case too, that is the case, so I just to come up with a vector field, I just use Xp well I just take minus x2 x1. And I just put 0's as well. This will give me a smooth vector field on the 2n plus 1 dimension sphere and in fact, I can play around with this and come up the other ones as well. Yes, that is or another example would be Xp, I do this on each pair of variables minus x2 x1 minus x3 rather x 4x 3 and so on minus x 2n plus 2 x 2n plus 1.

All I have to do is ensure that Xp is perpendicular to P which is this, P is here, so the way I have written it is that is perpendicular to this. So, each of this, this or this gives rise to a smooth vector field on two dimensional sphere. So, this is in TP S 2n plus 1 and so is this and this smoothness again. These are all even though I have defined it only for P raised to 2n plus 1 if I use the same formula this is a vector field on defined on the whole of Rn, whole of R2 in plus 2.

So, but it is just so we again use this small proposition here that if I have a vector field on the whole manifold, such which is tangent to the sub manifold at each point, then the restricted vector thing is also sooth. So these are all smooth vector fields. The last one, this is especially interesting because it gives rise to a vector field on the sphere, which is not zero at any point. It cannot happen that this, you get the 0 vector because if this is 0, then the coordinate vector would also be 0.

P would also be 0 which is not the case since P belongs to S 2n plus 1. So the existence of a vector field which is not 0 at any point is related to some questions in topology, but we are not going to go into that. Instead, let us talk about this is the right place to talk about lie groups. So, this was example number three. Now, I am moving on to example number 4.

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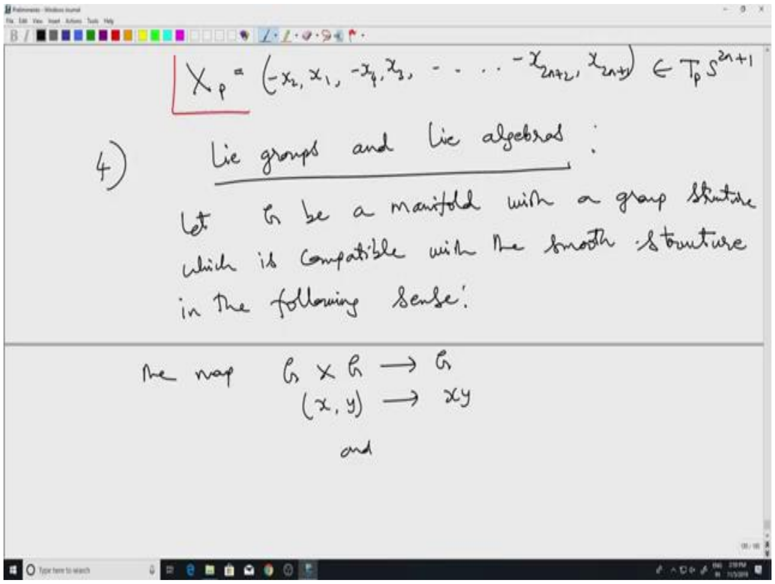

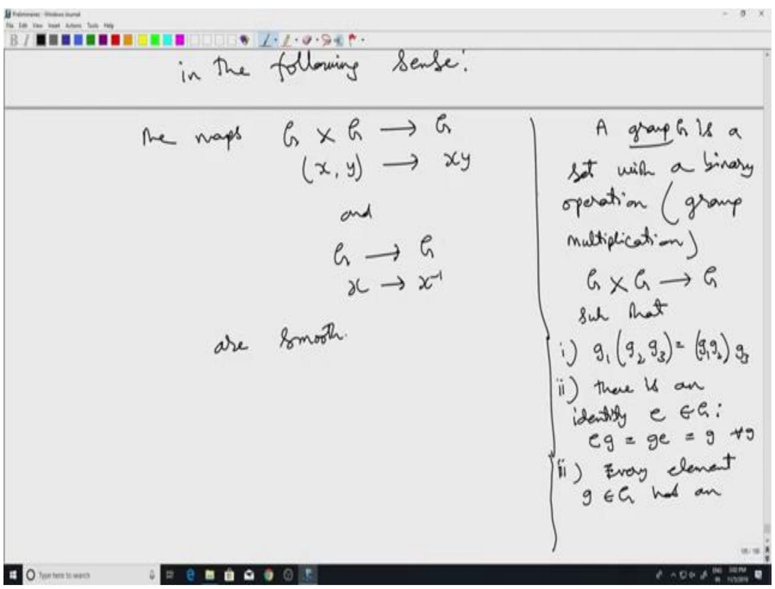

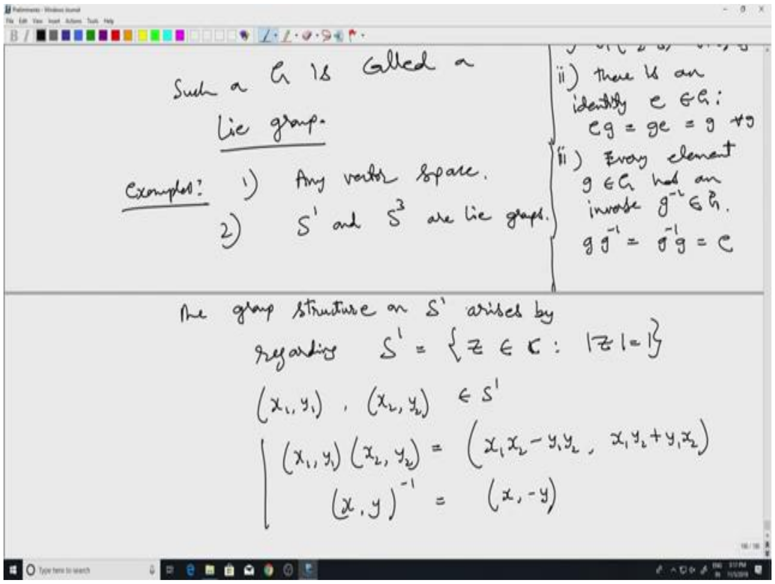

Lie groups and lie algebras. So let G be a manifold with a group structure. I will quickly recall what I mean a group structure is but let us assume for the moment, let us proceed with a group structure which is compatible with the smooth structure in the following sense. I just demand that group multiplication which is a map from G Cross G to G X Y going to X times Y, the map and G going to G, X going to X inverse. I demand that these two maps so the maps, these two maps are smooth.

So well, so in order for this to make sense, we need a notion of a group, so groups is a set, set with a binary operation. Also core group multiplication and what a binary operation just means that if you start with two the same thing that I have written here G cross G to G. So binary operation is just a map from G cross G to G, but satisfying such that one G operation is associated. So in other words, G1 G2 G3 is G1 G2 G3.

Secondly, there is an identity each and the definition of an identity is that e g equals g e equals G for all G. And every element g in G, as an inverse and this definition of inverse is that g g inverse equals to g inverse g equal to identity so that is a group. So now the thing is that but here normally when one first sees this notion of a group, this g will be a typically a finite set, say the group of permutations, etc.

Now here it is definitely not going to be a finite set. Third dimension of M and dimension is greater than 1, dimension is at least one, so let g, here we are having a manifold and which has a group structure as well which is compatible with the smooth structure. It is not that the group structure and the manifold structure should be completely independent. They should be related in the following sense that the group multiplication and inversion operations are smooth maps.

Here G cross G is regarded as a product manifold but one of the good thing is that one has lots of natural examples of this, so, such g I called a lie group. So one can think of it as a manifold, which is a group or a group which is also a manifold, so examples any vector space is trivially a lie group. Now, the group operation here, the binary operation here is just addition of vectors. Even though here I said the binary operation is called group multiplication.

When we are in the context of vector space, multiplication means actually addition here, addition of vectors. So, any vector space with vector addition is a group and we know that vector spaces are also manifolds and we know that the addition operation is smooth. To say the addition operation and the inversion operation is just X going to minus X in this case and those are smooth simply because ultimately the manifold structure comes from identifying n dimensional vector space with Rn.

And when we do that the, this addition in the vector space just corresponds to a usual addition of vectors in Rn and so it is certainly smooth. Now, more interesting examples come from looking at matrix groups, which I will talk about shortly, but before that, it turns out that there are even spheres, some of them are grouped so let S1 for example is a lie group and in fact it turns out that S1 and S3 are lie groups.

Let me just take the case of S1. Now, the group operation on S1 is comes from regarding S1 as the set of complex numbers with modulus one. And if you think of it S1 like that way, then of course we can multiply two complex numbers. And if both have modulus 1, then so does the product. The groups structure on S1 arise by regarding S1 as set of all Z and C with one set equal to 1.

So if you think of it like that, then Z1 Z2, we just do usual complex multiplication, of course this is how one gets a structure, but ultimately one does not have to mention complex numbers at all. So, one just uses this complex numbers as a motivation and then define. So, if I just start with x1 y1 x2 y2 in S1, then I will define the product x1 y1 times x2 y2 to be well I just do complex multiplication of x1 plus I y1 x2 plus I y2 that will give the x1 x2 minus y1 y2 and then x1 y2 plus y2, not y1 x2.

This if I write it like this, it is not entirely clear how we came up with the right hand side but since I have already said that structure, so here I need structure on this one arises right. So I have already said that one can think of S1 as complex numbers with unit modulus and then one can get distracted. So, this is how, so this is the group operation and we know that S1 is indeed a group. The identity element is literally the complex number 1, which would correspond to the point 1 common 0 in R2.

And we know that the inverse of a, an element with modulus 1 is also another element with modulus 1. So, this is the operation and in fact the inverse I can even write down the inverse. So inverse of x comma y is just, it says of the complex conjugate of this. So it is x minus y. So and so this since we have explicit formulas for group multiplication and the inversion operations, it is immediately clear that again, we are using the fact that these operations are actually well defined maps on all of R2, so then we restricted to S1, we will continue to get smooth maps.

So therefore the group compatibility condition which I talked about here this stuff is satisfied for S1 with this operation, so s1 becomes a lie group. The group structure on S3 is well again, one can think of S3 as a subset of R S3 is a subset of R4 which is C cross C. However, it is the C cross C with all those elements with norm equal to 1, however the group structure on S3 is not immediately, it is not clear.

In fact the S1 the group operation on S1 has since it is coming from C complex multiplication, it is just an commutative a billion group. It does not matter whether I do x1 y1 whether I put this one first or x2 y2 first I get the same answer. But the group operation S3 is actually it is a non a billion group operation and it is not it immediately clear how to how one sees that lie group, but it is not that difficult either.

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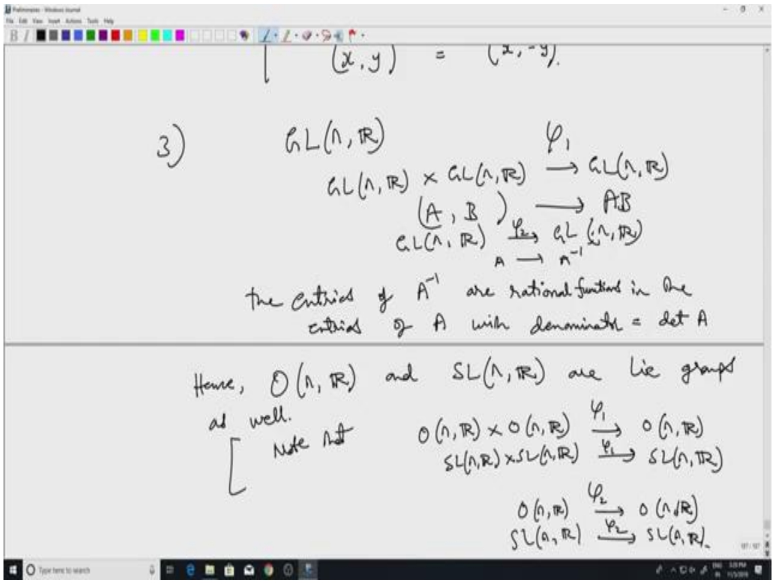

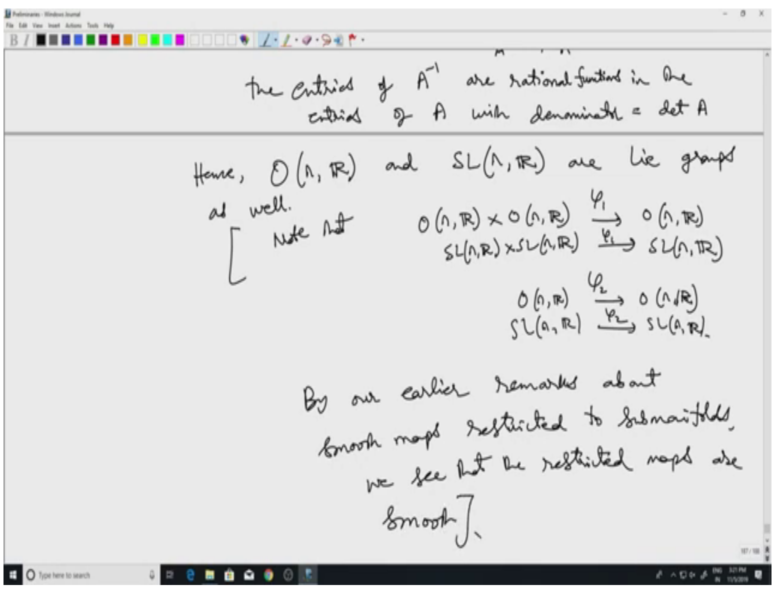

But however, I will not go into that instead I will talk a bit about matrix groups, which are the most important classes of lie groups. So first, let us start with GLnR. Now here it is the operation is just matrix multiplication group operation is matrix multiplication. And if so, one has to check that GLnR cross GLnR, GLnR, A B A comma B going to a times b. Whether this map is smooth or not, well we do know that first of all GLnR we have seen is just an open subset of MnR of the set of all M cross N matrices, which is Rn square.

So, when talking about smoothness, we can just use the fact that there is an open subset. And so the issues regarding smoothness are just, we can think of this as maps of corresponding the Euclidean space Rn square cross Rn square to Rn square and matrix multiplication is just corresponds, it will give rise to a polynomial map. These variables input variables, so therefore this is smooth.

And similarly, inversion GLnR with GLnR the map is a going to a inverse. Here it is still smooth but it is certainly not going to be a polynomial operation. The entries of a inverse or not polynomials and the entries of A but the next best thing, their rational functions. So entries of a inverse or polynomials in the entries of a, oh not polynomials rational functions, the ratio of two polynomials.

Well, rational functions can cause problems if the denominators becomes 0, but here the denominator we know is denominator equals to just the determinant of A. And since A belongs to this opens set GLnR, it is not going to be 0 anywhere on this set. So it is a rational entries of A inverse rational functions with non-zero denominator. Therefore, it will also be they will be smooth. So both of these are smooth therefore GLnR is a lie group.

Once we have this, we can immediately see that. Hence, we do not have to repeat the calculation here. Hence OnR the other two matrix group consider it R, these two are lie groups as well. Well, why? Yeah. So I read this word hence, so this is supposed to follow from whatever this discussion of GLnR and that is again to do with the fact that these are sub manifold of GLnR. Note that, these operation matrix multiplication, so let us call this one as phi 1 and this is phi 2.

Matrix multiplication and inversion phi 1, when I do OnR times OnR or not, we know that the product of two orthogonal matrices is after all this is a subgroup, so matrix multiplication preserves. So the same phi 1 that I had here when I restricted to OnR cross OnR, it will go back to OnR and similar thing for SLnR as well. And I take and I restrict the product operation. This is the very definition of a subgroup.

So I had phi 1, similarly phi 2 will take OnR and SLnR to SLnR, so since we have already established that phi 1 and phi 2 are smooth on the big manifold GLnR. We can use our earlier remarks about smoothness of maps, then districted to sub manifolds to conclude that this phi 1 and phi 2 when regarded as maps from OnR cross OnR to OnR and OnR to OnR, similar for SLnR, they are also smooth.

So we conclude that by our earlier remarks about smooth maps, restricted to sub manifolds, both in the domain and image, we see that the restricted maps are smooth. Therefore, we get that these are lie groups as well. Now, the reason for mentioning for bringing up this notion of a lie group in the context of vector fields after all we talking about vector fields is not clear from what I have said so far.

So, in my next lecture, I will talk briefly about that. That vector fields on lie groups see the role that they have to play. So okay, so we will end the lecture today at this stage. Thank you.