An Introduction to Smooth Manifolds

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Lecture 03

Multivariable Calculus 2

Okay, so now, last time we were talking a bit about multivariable calculus. So this lecture will continue with that. And I will discuss a bit about one of the major theorems in multivariable calculus, namely the inverse function theorem. But before I get to that, let us continue with our discussion of derivatives.

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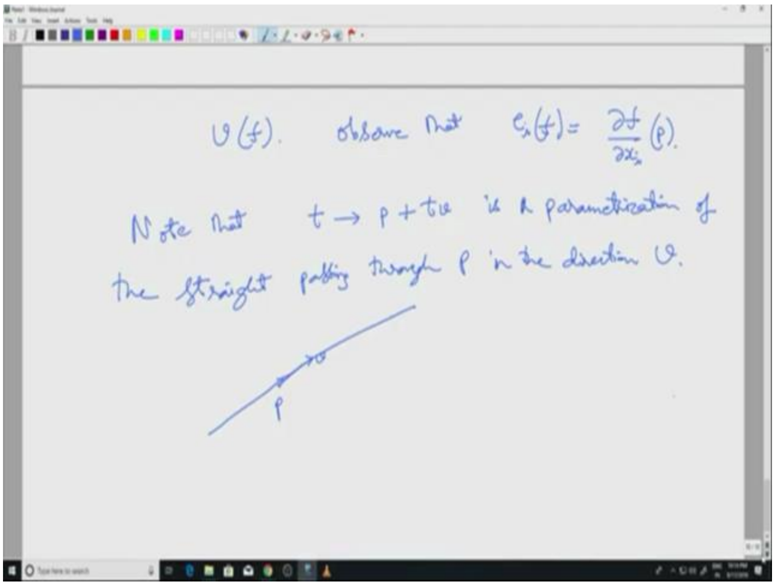

Well, so this note that t going to p plus tv is a parameterization of the straight line, straight line passing through p in the direction v.

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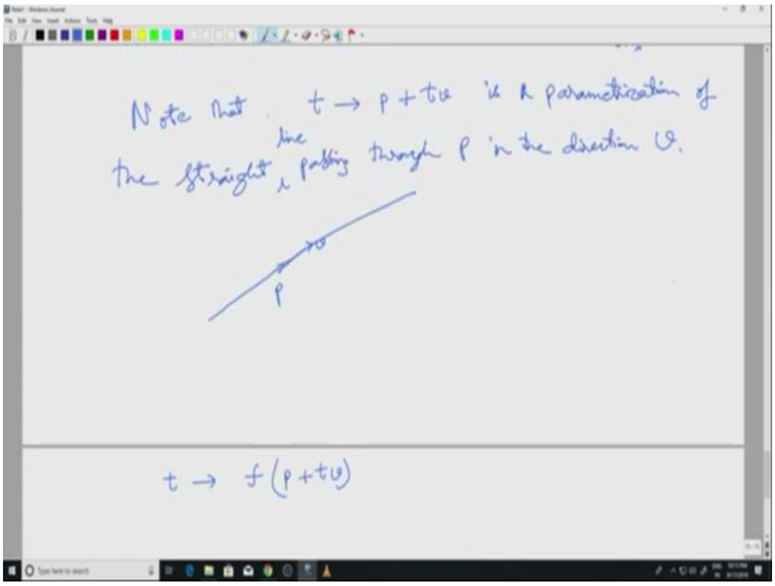

And what we are essentially doing when I look at this, what I am doing is essentially I am restricting the function f. Remember that f was defined in an open set around p, but I am just restricting it to the straight line, at least a portion of the straight line. And then calculating the what seeing what values it assumes at various points of the straight line, which is what this is. So essentially, I get a function of one variable t going to f of p plus tv.

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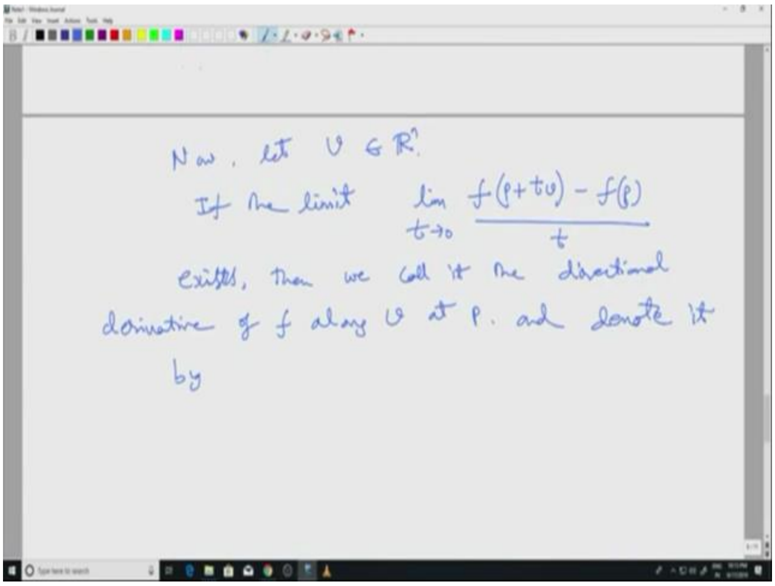

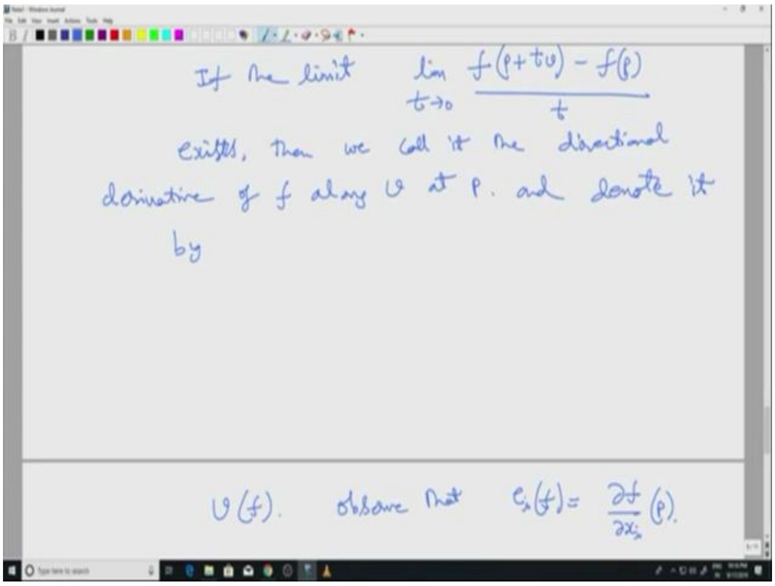

So, that is like this derivative, what I have here, this is a function of one variable. f itself is a function on an open set in Rn, but if I regard it as a function of t, it becomes a function of one variable. That is why this this is very reminiscent of the classical derivative in the one variable case. So I could easily just divide by t and take the limit.

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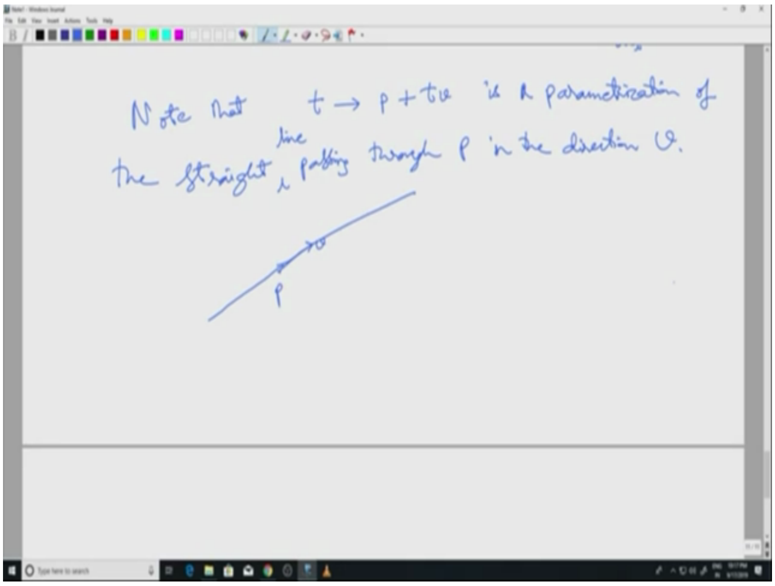

The reason why I mentioned this here is that now, there is nothing special about the straight line. So, given this point p, and the direction v, from one point of view, there is nothing special about the straight line after all, what is the feature of the straight line? One aspect is that when t is zero this p plus tv will be just equal to p. So, in other words this if I regarded it as a curve, then t as a time parameter, when t is zero, I start at p.

And notice that if I differentiate this curve, I will get v. So, initial starting point is p and initial velocity is v. Actually in this straight line case the velocity is v at any t.

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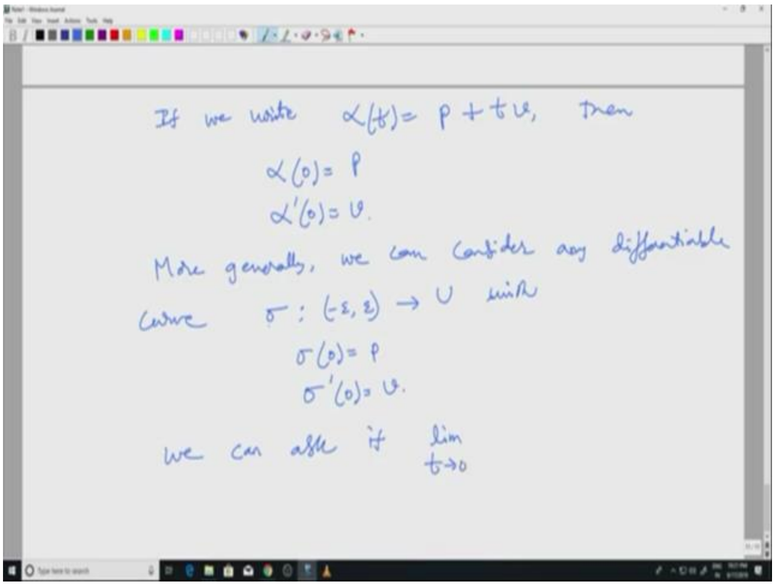

But let us just focus on two things. So, here I say that straight line note that t is a parameterization of the straight line passing through p in the direction v. If we write alpha t equals p plus tv, then alpha zero equals p, then alpha prime zero is equal to v. Well, there are lots of curves. So given a point and a direction, there are lots of curves which have these two properties which starts at p and its initial velocity should be v.

So here of course, I am using velocity in a loose sense, I just mean the initial derivative of this curve. Derivative of this curve at t equals zero. So, more generally, we can consider, more generally we can consider any differentiable curve sigma.

So, which is defined in some very small neighborhood of zero minus epsilon, epsilon. And the image should lie inside this U, the open set on which f is defined more generally we can consider any differentiable curve from here to here with sigma zero equals p exactly the same two properties, sigma prime zero equals v.

Now, what do I mean by differentiable curve. Well, curve, by definition is a map from an interval, open interval into whatever the target is. So if I say curve in U, I mean, all I mean is the domain of that function has an interval.

And by differentiable what I mean is that, of course, we already have the notion of differentiability of a function from Rn to Rm so, I can use that, but really I do not need to worry about that because this is a function of one variable, even though the target is an open set in Rn. That is not an issue. The main point is the domain is just something in R.

So this is a function of one variable. So it's, if I look at its components, sigma one, sigma two, sigma n there all just real valued functions of one real variable, so I just demand that they will be differentiable. So that would, but it's essentially the same thing as saying that this is differentiable in the fresher a sense.

You do not get anything new. So, all right, we can consider any differentiable curve. And then again we can ask, if limit t going to zero.

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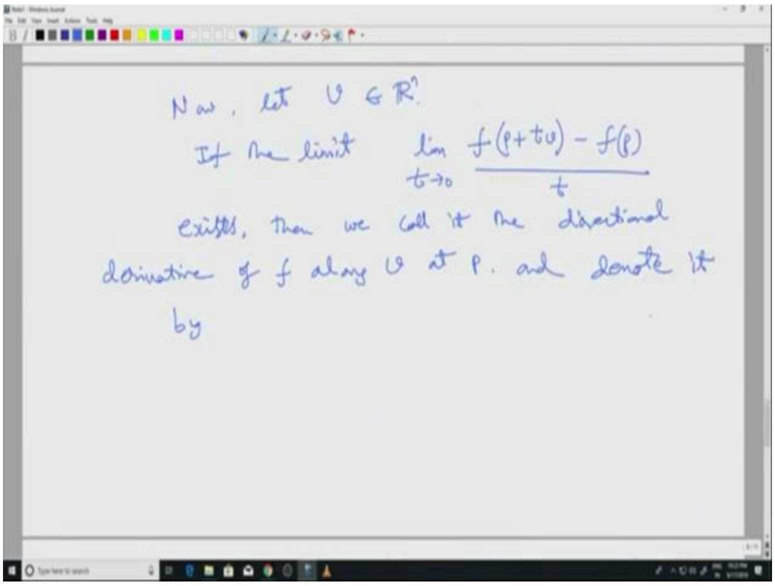

Now, instead of looking at F of p plus tv, I will just look at f of sigma t.

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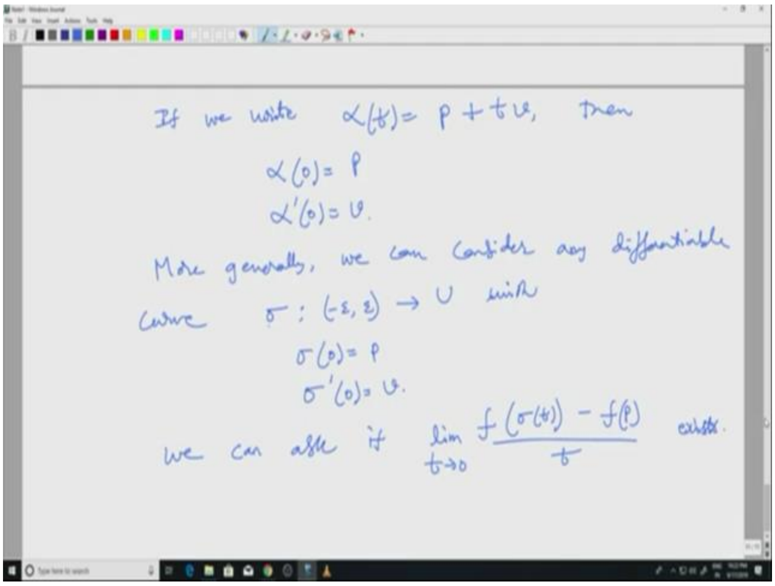

We can ask if limit f of sigma t minus f of p as before divided by t exist. And if it does exist, how is this related to the direction derivative as defined earlier. So, this is direction derivative would correspond to taking instead of sigma in arbitrary sigma, I would end up taking this specific sigma which is alpha t equals p plus tv.

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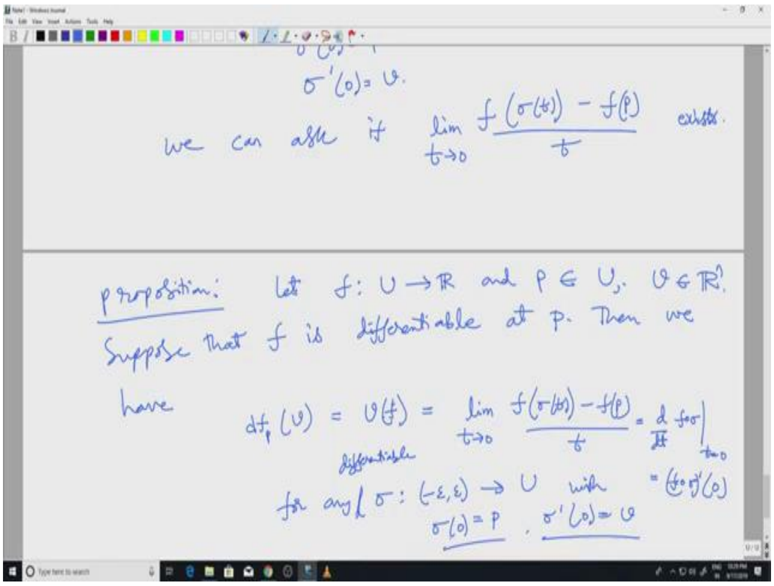

So, therefore, well I have the following proposition. Let suppose that f is differentiable again when I just say differentiable, I mean fresher differentiable. So which is the existence of a linear map etcetera suppose that f is differentiable at p then we have df at p so here I also should take v belongs to Rn.

So f is differentiable then we have dfp. The derivative as a linear map acting on this vector v is the same thing as the directional derivative of f along the direction v. So same as v of f and which is also the same thing as this thing here, which is limit t going to zero f of sigma t minus f of p divided by t for any sigma from minus epsilon, epsilon to U with Sigma zero, zero sigma prime zero equals v any differentiable curve, for any differentiable sigma with this property.

Notice that so this the last thing here is just, it is the same thing as derivative in the usual one variable sense d by dt of f of f composed with sigma at t equals zero. All right it is same thing as f composed with sigma prime at zero. So in short what this says this is a very useful (())(11:57) this going to be very helpful later on what we are saying here is that if f is differentiable in the fresher sense at p, then in order to calculate, there are many ways of looking at this equation here, one thing is that in order to calculate the derivative as a linear map, so what the derivative does to a vector v, I can use this that dfp of v. One way is that it is actually equal to the directional derivative along v.

But actually this is not so, what is more helpful is that, this last one here, namely, I can take any curve, not just the straight line, I can take any curve with just has to satisfy sigma zero equals zero sigma prime zero equals v. The moment I have these two conditions all I have to do is look at F composed with sigma, this is a function real valued function of a real variable, and then calculate the usual derivative.

So sometimes one is able to find, depending on the function f, one can find an appropriate curve sigma, where it becomes quite easy to see what f composed with sigma is. So, that is the value of this proposition that in order to calculate the fresher derivative, I can use this f compose with sigma prime zero.

But there is also something else here namely, the second, the middle quantity equals the last quantity here. So remember that v of f was obtained by restricting f to the straight line through, here a small changes needed here sigma zero is not zero. Sigma zero is p right? so v of f was obtained by restricting f to the straight line through p in the direction v, and then calculating the derivative.

What this is saying is that it really does not matter if it is a straight line or not. All you need is some curve which passes through p and those initial direction is p. So immediately it may change directions. But as long as it starts there, it is one is in good shape.

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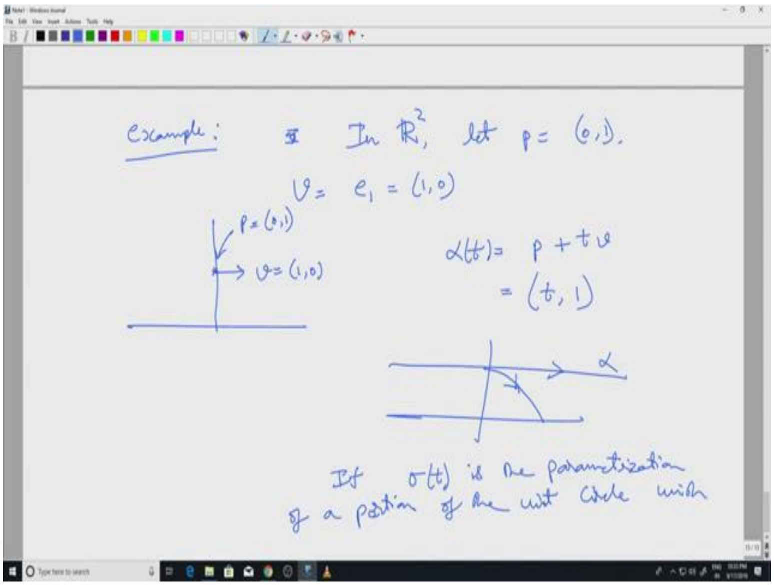

So for instance, as an example, if example so let us take p so in R two let p equals zero comma one. v equals, let us say p one so I am looking at this picture. So this point p is zero comma one and perhaps I should write it on the other side or rather below. So this is, so this point p is zero comma one and this v is one, zero.

So, through this point and in this direction, of course, the straight line is what I call alpha t, alpha T is the parameterized curve P plus tv, which is t comma 1. So I will just get so it is this, this is alpha t, this horizontal line. Now, of course, I have another natural curve, which passes through this point and a tangent to v namely if I, I can take portion of a circle here. If sigma t, so I need, if sigma t is the parameterization of a portion of the unit circle with…

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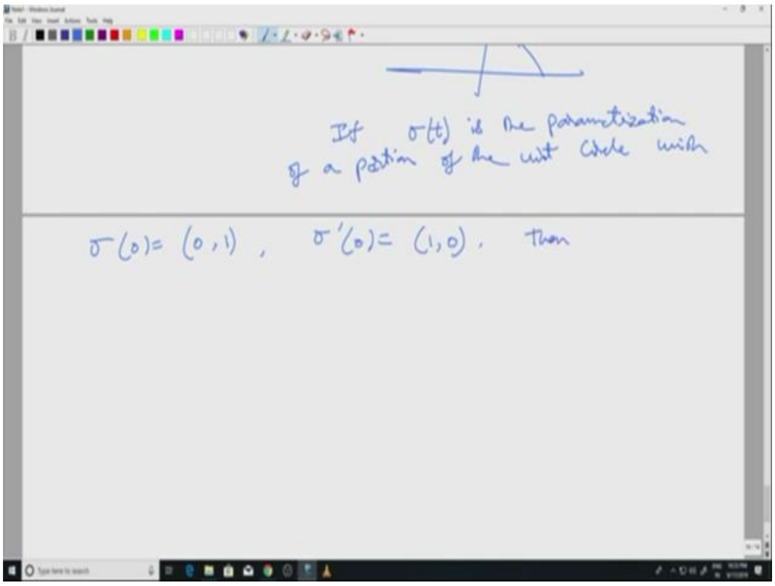

Unit circle with sigma zero equals zero comma one, sigma prime zero equals one comma zero then, so basically these two curve…

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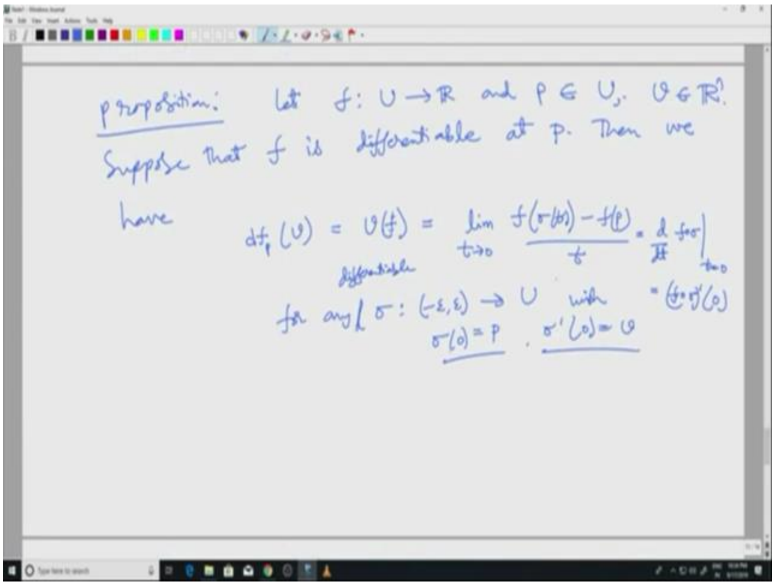

Well this sigma satisfies the hypothesis of this small proposition here.

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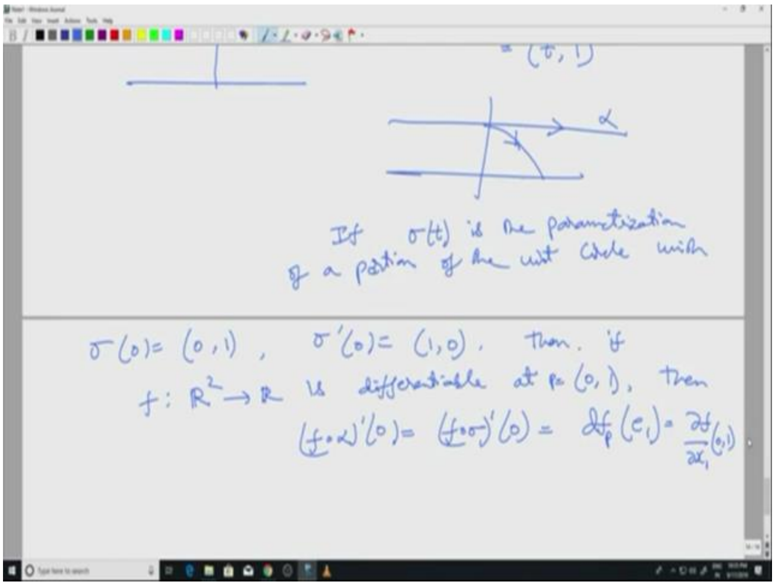

So if we have a function which is differentiable then if f from R2 to R is differentiable at, its differentiable at zero comma one then f composed with alpha prime zero equals f composed with Sigma prime zero and both of them are equal to df at p. So this is P, so dfp acting on e1, right and this is also equal to df by the del x1, del f by del x1 at zero comma one, right.

So, as a small exercise one can write down the explicit equations for this sigma t. So just the normal parameterization of circle would be something like sigma t is cos t sine t. However, that it would go in the wrong direction whatever at t equals zero it will start at this point, the normal parameterization will start here rather than here. So, one has to modify the cost t sine t a bit and then one gets an equation for sigma t, alright. So, that takes care of this.

Now, let me do something which is slightly more interesting than interesting more interesting examples.

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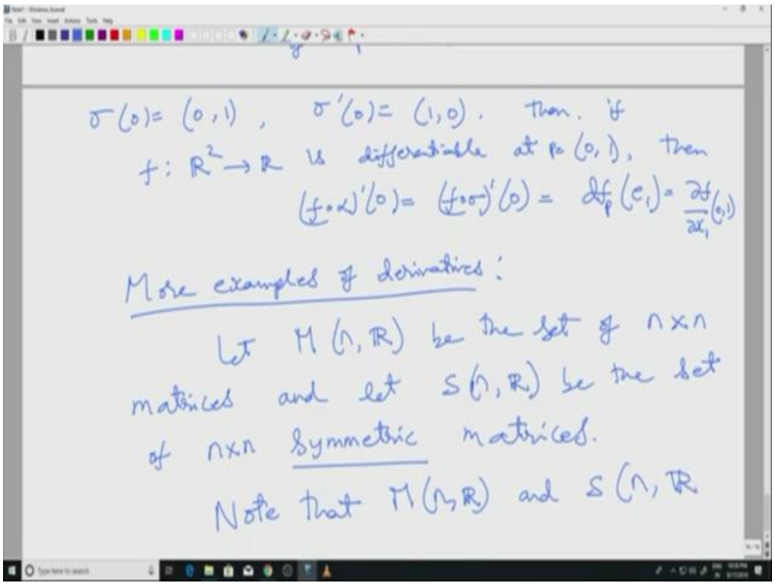

So, more examples of derivatives. So, to talk about these examples, there is a couple of things I have to introduce first is let us look at the set. Let MnR, let MnR be the set of n cross n matrices. And let SnR be the set of n cross n symmetric matrices. Note that both of these…

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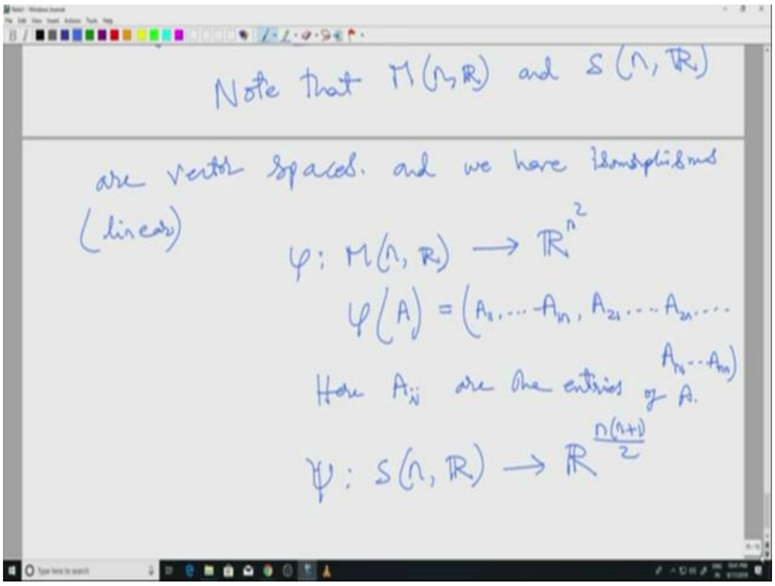

Both of these are vectors spaces with the usual operation of matrix addition and multiplication by a scalar. Well, and in fact, since we know that and any finite dimensional vector space is, (()) (23:15) any two finite dimensional vector spaces isomorphic as soon as they have the same dimension. So, this we can easily see is isomorphic to some familiar Euclidean spaces.

So, our vector spaces and we have isomorphisms, linear isomorphisms. So, first is from phi from MnR to R n square and what this do, well if I take a matrix A I am supposed to get a point in Rn square. So, I should have n squared coordinates. Well, I just put the rows of A side by side. So, A one one, A one n and then I start with A two one, A two n and so it goes on till I have the last row A n one, Ann this is the map from I, given a matrix, I just put the rows of A side by side, and then get, get an element of R n squared.

Here, of course, here Aij are the entries of A. So this, it is easy to check that this map from here to here is actually by ejective map and it's linear. So I have a linear isomorphism from here to here. So this is one map and similarly from SnR. I have a map to well, so this time the Euclidean space dimension is going to be less than this Rn square. So it is in fact it is n into n plus one divided by two.

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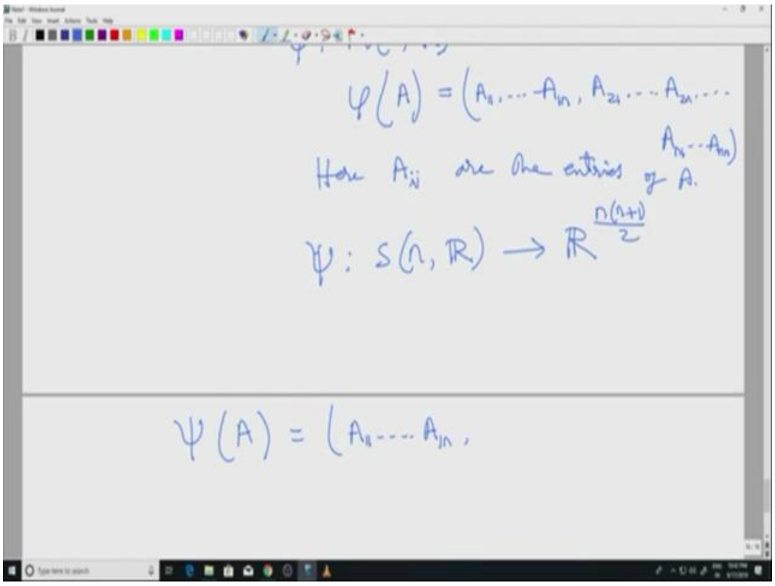

And this map again, I start with the matrix, this time the matrix is going to be symmetric. I just instead of laying out all the rules of A side by side, I just use the recalling that symmetric just means that the terms above the diagonal are basically reflect the terms about the diagonal along the diagonal, you get the terms below the diagonal as well. So what I do is I will use the, write the first row.

Now, as for the second one, then so at this point, we will stop and I will resume with this example and move on to other topics and multivariable calculus in my next lecture. So…