An Introduction to Smooth Manifolds
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Lecture 27
Tangent spaces to level sets

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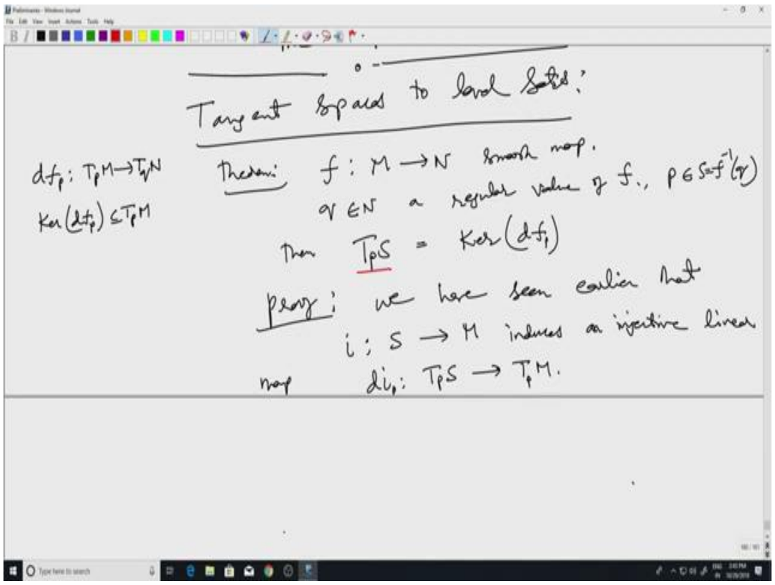

Hello and welcome to the 27th lecture in the series. So, let me having described a bunch of examples of sub manifolds arising from the regular value theorem. Now, let us talk a bit about their tangent spaces. We have already seen how the tangent space of a sub manifold is can be naturally regarded as a subspace of the tangent space of the bigger manifold. Let us use that to describe the tangent space to a regular set, regular value, level set of a regular value.

So, tangent spaces to levels sets. Well, so the setup is that I have F theorem, F from M to N is a smooth map and then q in n regular value of F and then the claim is that and of course P in S equal to F inverse q, so S is the level set, which is a sub manifold. So, then I want to describe the tangent space to S it turns out, it is exactly equal to the kernel of D F at P. So, DFP is actually a map from TPM to TQN.

And the kernel of this will be a subspace of TPM. So, we have think, we are describing the tangent space to us as a subspace of TPM. And so we have to use that earlier observation that one can regard the tangent space to S as a subspace of TPM. So, let us do that, so the thing here is that in the statement of this theorem the tangent space to S, is already regarded as a kernel of is regard as a subspace of TPM.

But recall that earlier, the way we got the tangent space to S as a subspace of this is via the inclusion map, we have seen earlier that the inclusion map I from S to M induces an injective linear map Di at the point P from TPS to TPM. So, in short where here, when I say TPS at this point I am thinking of DiP of TPS as the image of TPS under this inclusion, derivative of the inclusion map. Now, in fact, this thing, this way of looking at it will help us see why this statement is true in the first place.

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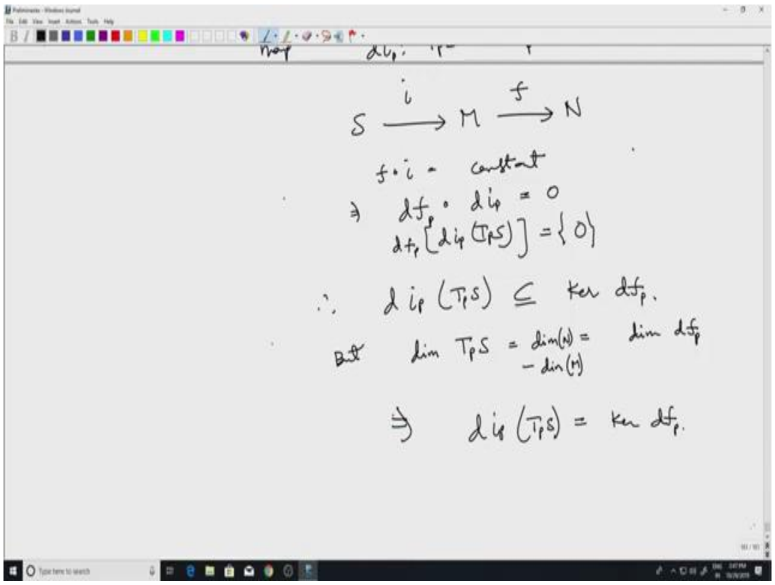

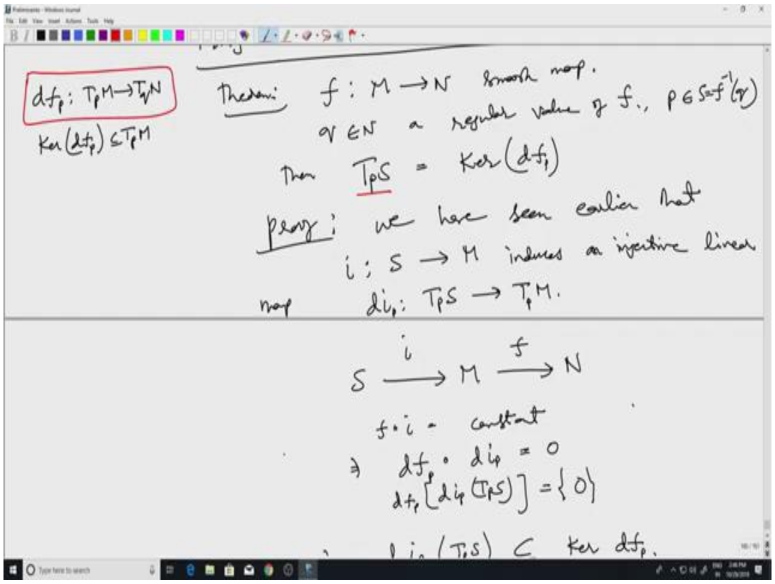

So, I have a map, and so let me use black. So, I have a map, the inclusion map i is from here to M. Then I have the map F to N, F compose with i is the constant map. In fact, the whole thing S is being taken to Q. So, therefore, since it is a constant map, its derivative is 0.

DF now I takes P again so DF at P of compose with DiP is the 0 map in particular DFP of if I will cut the image of this TPS this is 0, the 0 subspace. So, this immediately shows that therefore, DiP of TPS is contained in the kernel of DFP. On the other hand, the dimension of this, we already know that the dimension of this is TPS is n minus 1, which is also the dimension of DFP.

Well, not quite, no here I am not dealing with hypersurfaces so it is not minus 1 rather it is dimension of, so this is the mention of the target n minus the dimension of the this the first equality is because we are applying the regular value theorem. And therefore, we get the dimension like this. The second one is just that again, simple linear algebra. The DFP, this DFP here is a surjective linear map.

Therefore, the rank nullity theorem tells us that the kernel the machine of the kernel is dimension of the image minus dimension of the mean so both are the same. Therefore, this implies that DiP of TPS is actually equal to the kernel of this map. So, this is a very useful thing, as it turns out.

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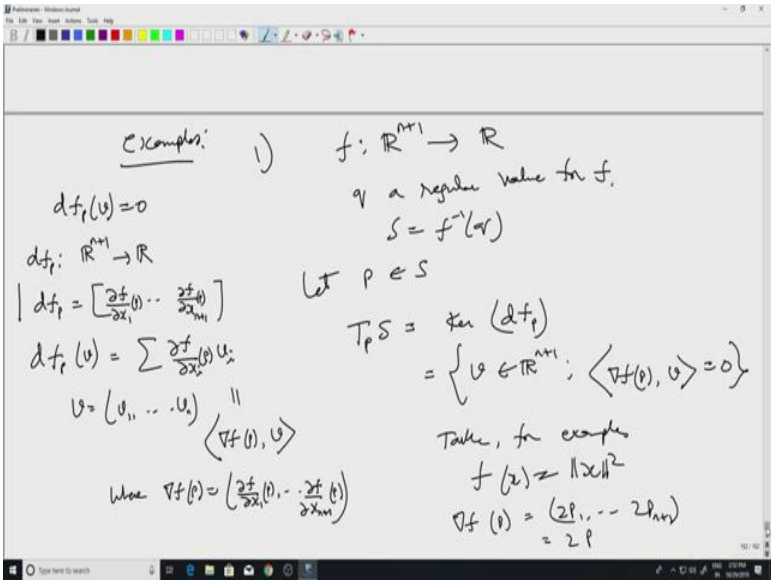

So, example, let us revisit all those examples of sub manifolds. The first one was the case of F from Rn plus 1 to R and Q regular value for F S equal to F inverse Q. The level set of corresponding to Q which is hypersurface in Rn plus 1.

Now, in this case, let P belong to S according to what we just discussed the tangent space to S is now the kernel of DFP. So, the kernel of DFP would be set of all v such that DFP v is 0. Now, in this particular case, DFP is a map linear map from Rn plus 1 to R. So, in other words, it is as a matrix. It is just a, well, it depends on, it is a 1 by n. So, del F by del x1 at P, del F by del xn at P.

The matrix is just a row vector with o n plus 1 actually, not n. Now, this is DFP, DFP acting on v would be just summation del F by del xi at P vi where v equals v1, v2, vn. And of course, when we think in matrix notation this would be a column vector rather than a interpolate like this, what we get is, this is equal to, so we have this.

Now, the point is this can also be written as the gradient of F at P in a product with V where gradient of F at P is again the same thing what I had here, except that I do not think of it as a linear transformation now as a matrix, but as a vector. I think of it as a vector with the same entries, xn plus 1 at P. So, in short, with this discussion, this is set of all V in Rn plus 1, is that the gradient of F at P in a product with v equal to 0.

So, one reason for writing it like this is let us take the case of the sphere. So, take for example, F of x is nom x square, as we have been considering and well f of x is nom x square in this case, grad F at P would be, 2P1 2Pn plus 1 which is the same as 2 times P. So, the gradient of F at P is just P itself up to a constant multiple of 2.

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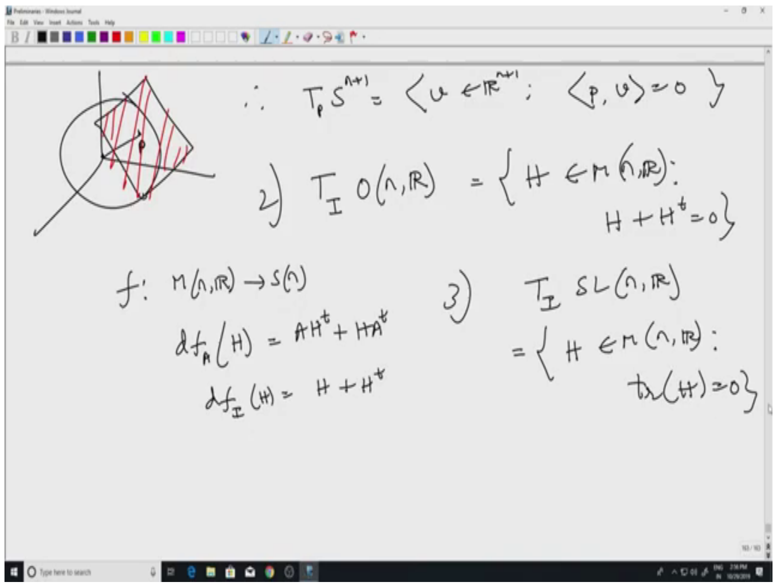

So, therefore, the tangent space to the Sn plus 1 if I take the regular value Q equals 1 so I will get S equal to Sn plus 1, the tangent space would then be set of all v in Rn plus 1 such that I can forget the 2 which is equal to 0 theta P in a product v equal to 0. This is the tangent space at P and which is which can be kind of expected. If you have a point on the sphere the setup for, the tangent space is precisely all those vectors which are, if you think of the vectors as actually you move the tangent space.

If you wanted to be a vector space, we would have to think of it as passing through the origin, in which case you will get all vectors which are perpendicular to this given vector P. Which is what I have here. But in fact, this is more generally this the same thing holds for any hypersurface given by a smooth function like this. Now, let me want to the tangent space to the Orthogonal group TA o n R.

Again so this time the map was a call that we had a map from MNR to what I call Sn the group of symmetric matrices, the group of symmetric matrices and the derivative at the, at A turned out to be AH transpose plus HA transpose. Now, in fact, in the context of this the orthogonal group and the special N A it turns out to the important thing is that tangent space at identity. And so let us look at that.

Identity belongs to O N R and I want to look at the tangent space and the expression for the derivative here is just there is no A anymore. H plus H transpose. So, this is set of all H in MNR such that H plus H transpose is equal to 0. In other words, this is exactly the space of skew symmetric matrices. So, the tangent space to the orthogonal group consists of skew symmetric matrices.

Similarly, if I look at the tangent space to the special linear group this will turn out to be so the derivative in this case at identity turned out to be just the trace function. So, this is set of all H in MNR with trace H equal to 0. So, one has very concrete descriptions of tangent space or when we start with a regular level set of regular value.

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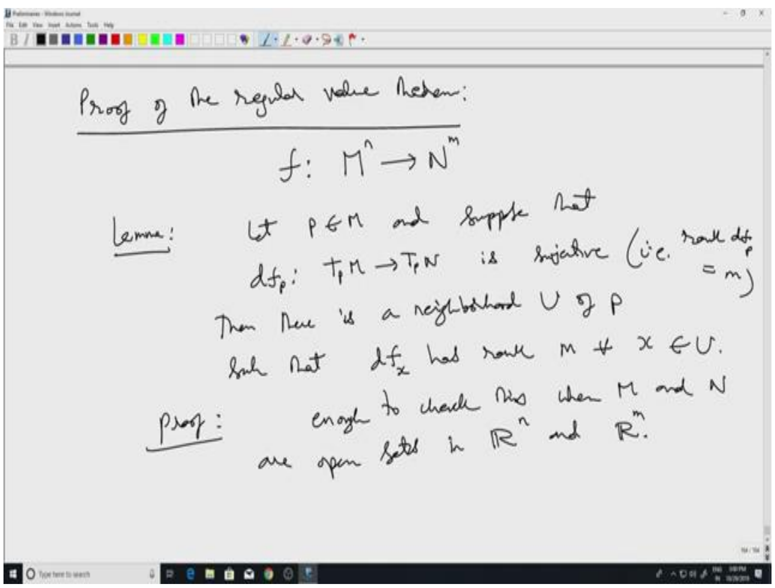

Now, let me quickly describe the proof of the regular value theorem. So, what we will need a small lemma for this. So, here before a state the lemma, let us recall that the setup was I had a map from here to here and I took a regular value q and proceeded to do and proceeded to look at the levels corresponding to q. So, lemma let P belong to M and suppose that DFP from TPM to TPN is surjective i.e. rank DFP equal to M the target. Suppose, then there is a neighborhood U of P such that DFP has rank M for all.

Well, let me use a different. It is not DFP anymore. DFX as rank M for all X in U so in short, if the differential is on to a single point, then it will be on to a neighborhood of that point. And the proof now, as usual, when one talks about the derivative the statement is purely local. So, in short, one can work with the coordinate charts transfer the map F to the coordinate charts and then prove the statement there.

So, enough to check, check this when M and N are open sets MRN and RM respectively. I did not add respectively, since I did not give any notation for this. Open sets no it is ok. Open sets, well, open sets in this.

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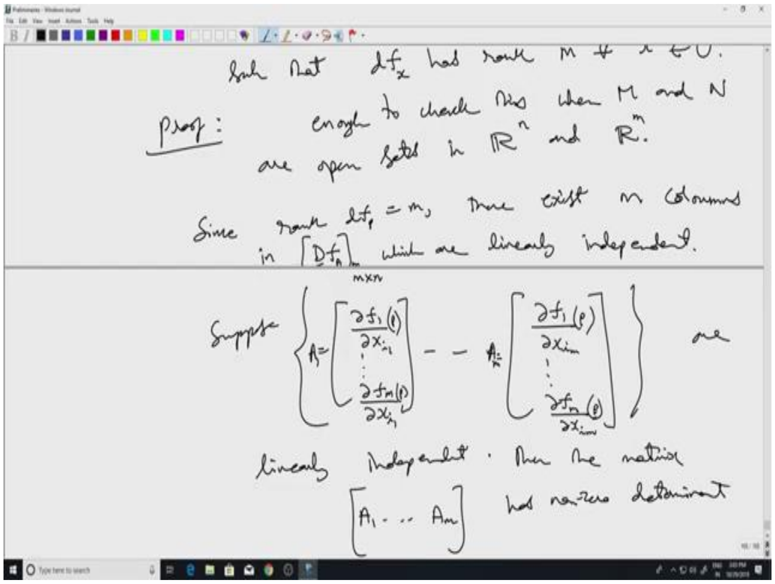

Now, let us look at since the good thing about moving to Euclidean space is that now the derivative can be thought of in the classical sense. And in particular, I have a matrix, the Jacobean matrix associated to DFP.

Since rank DFP equal to M there exist M columns. In the derivative, which I use, the classical derivative and use capital DFP which are linearly independent. Remember that this is a matrix of size M cross, M cross N, it has N columns and M rows. So, these are there are some linearly independent columns.

Suppose, let us take some columns del F let us pick out their linearly independent column. Del F M divided by del xi1. This is one column dot, dot, dot del F1 by del Xim. So, these columns are linearly dependent. Here everything is evaluated at the point P. So, the fact that these are linearly independent would mean that then the matrix, if I just put these columns together, the full Jacobean matrix, DFP had more columns. Now, I just pick out and what these and put them together. So, let us call these columns as A1 Am. So, here A1 Am this is this matrix has non-zero determinant.

So, if I now here P was already fixed to get these columns, but I can vary P and look at the same matrix and look at the determinant function. So, I will get a function on I will get a function defined on that open set and if it is by continuity of that function, after all, we are assuming everything is psi infinity so all the partial derivatives are continuous determinant is continuous by continuity of that function.

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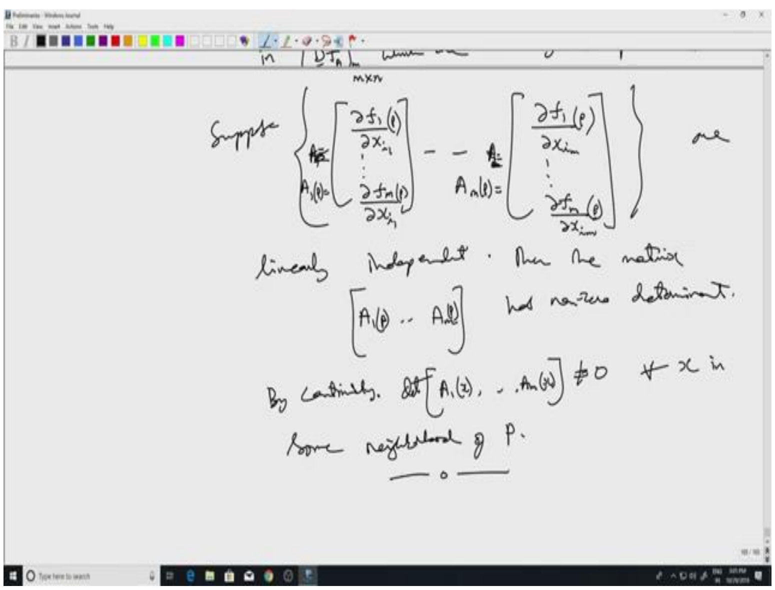

By continuity. So, let us put A1 instead. In fact, let us call it A1 of P. And here AM of P. By continuity so this is A1 of P and A1 of P by continuity even if I change it to some X AM of X. In other words, I just evaluate the derivatives at some other point X by continuity that of this is not 0 for all X and some neighborhood of P. Now, let us go back to the full Jacobean matrix.

Since in that full Jacobean matrix, this A1, AM are all certain columns. So, the fact that this determinant is non-zero will mean that these columns are linearly independent. Therefore, the full Jacobean matrix at the point X will have rank M as well. So, this completes the proof of the lemma.

Now, once we have that, that is a main point in this proof, actually that is the other thing is the actual well that is not the main point that is just small observation that the actual thing which we are using is the constant rank theorem.

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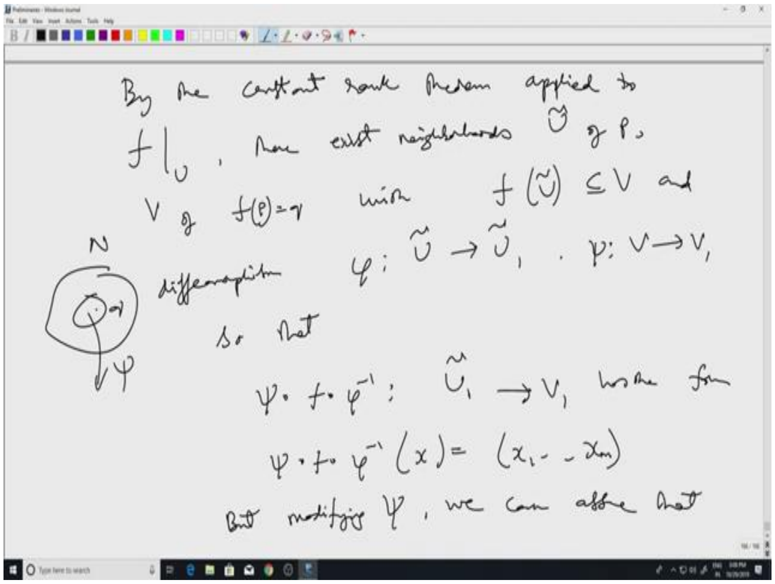

So, by the constant rank, now that we know that rank is constant by the constant rank theorem applied to F restricted to U.

We recall that this U was the open set around P on which DF at constant rank and there exist neighborhood's U tilda of P, V of F of P equal to Q with as usual, F of U tilda contained in V and we have diffeomorphism as well, phi from u tilda to U tilda 1, psi from V to V1. Notice that this we are all even though we are already in Euclidean space, the constant rank theorem still gives us some diffeomorphism only after changing by these diffeomorphism we will have the required form for F.

Diffeomorphism such that so that psi composed with F composed with phi inverse from u1 tilda to V1 as the form, phi inverse of X equal to X1 up to Xm and this psi in this picture Q was here in N and the psi was here. Now, by modifying psi, we can assume that there is nothing I mean, instead of considering psi, I can consider psi minus psi of X minus psi of Q.

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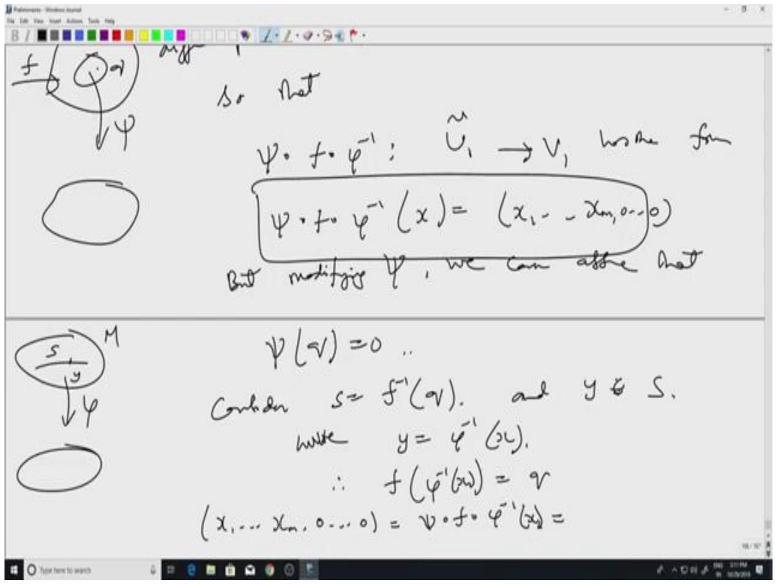

We can so, if we do that then we can assume that psi of Q is actually 0. So, in that case, so therefore, we have that consider now consider S equal to F inverse Q the level set. And Y in S, so in other words when F of Y equal to Q. Well so this was F was landing here. This was psi so what I do is, I can write Y equal to Y in S. So, Y was actually I need another picture, this is M and the set S was here, Y was some point here and phi was going here. So, I look at I can write Y equal to phi inverse X therefore, F of phi inverse X equal to Q.

And now if you look at X1 up to XM 0 0 0, this is psi composed with F composed with phi inverse X. This is just because the general form that I have here, actually I should put zeros here as well zeros. Let me leave the bracket. Yeah, put lots of zeros there. So, because of this general form I have this. But on the other hand F of phi inverse X equal to Q so therefore, this is psi of Q and I have assume that psi of Q is 0. Here by small by changing Q and by changing psi if needed.

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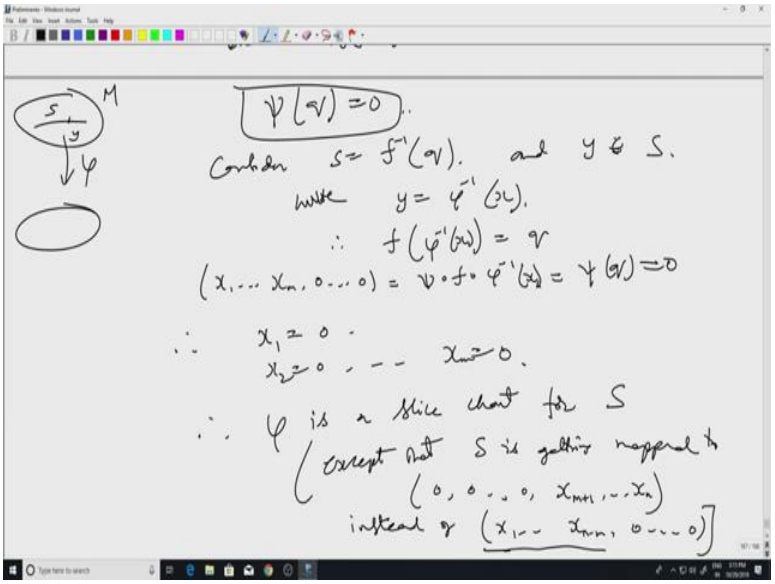

So, the upshot is that so this is equal to 0. Therefore, x1 equal to 0, X2 equal to 0, etc. Xn equal to 0. So, what has all this accomplished is just that therefore phi is a slice chart for phi is a slice chart for S. And except that the only difference is that when we say a slice chart, we wanted the sub manifold coordinates to appear in the first few places and zeros to follow that.

Here we are getting the opposite. We getting the sub manifold is getting taken to points in Rn plus 1 rather Rn where the first certain number of coordinates are 0 and the last ones are variables. So, that again one can change phi, modify the diffeomorphism phi so that this last M minus N coordinates N minus M coordinates will appear in the first. So, in other words, except that this S is getting mapped to because of what we said here, 0 0 0 Xn plus 1 Xn.

See, this equation that we have here shows us the first n coordinates are 0. So, under this map phi, S is getting mapped to this where rather what we want is instead of we want Xn minus M followed by zeros.

Now, if we want S to get map to this standard form like this, all we do is we change the slice chart phi by a suitable linear transformation of Rn which will take the last M minus N coordinates to the first one like this. The sort of permutation matrix will accomplish the job. So, let us stop here. Next time I will make a few remarks about smooth maps into sub manifolds and then start a discussion of vector fields on Manifolds. Thank you.