An Introduction to Smooth Manifolds
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Lecture 26
Hypersurfaces

So, hello and welcome to the 26th lecture in the series. And last time I was still in process of describing various examples, important examples which arise from the regular value theorem. So, the last thing I had started on was the graph of a smooth function.

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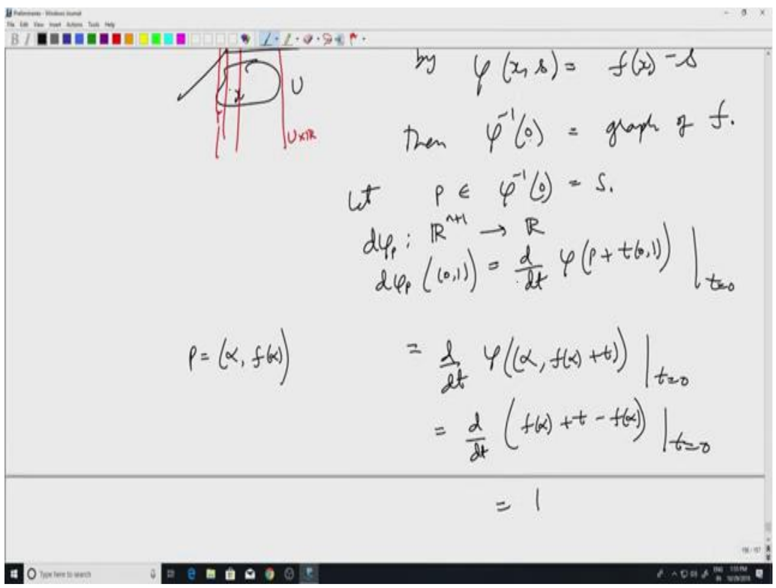

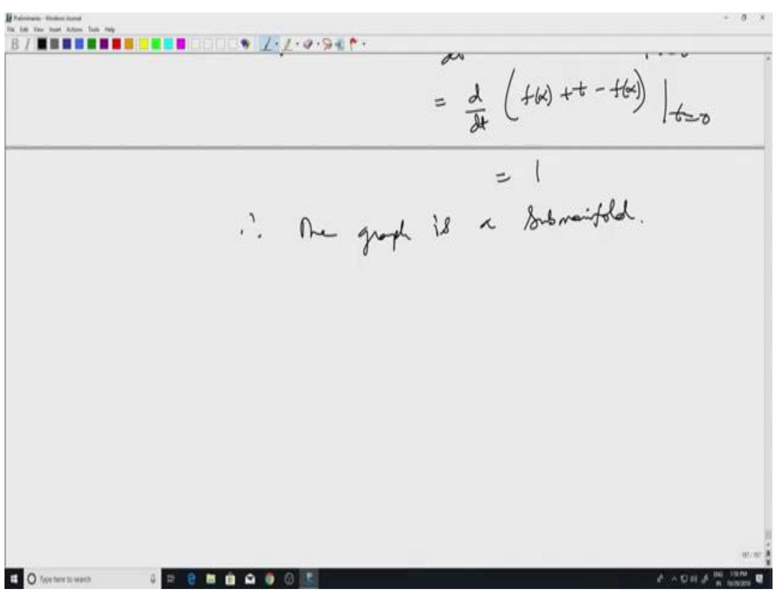

Well, the setup was I had an open set U in Rn and a smooth function f from U, graph is the set of points X F of X in Rn plus 1. Now, I would like to describe this as the level set of a function which is defined on U cross R notice that U cross R is an open subset of Rn plus 1.

So, on this open subset I put this function phi of X is equal to F of x minus S. Then phi inverse 0 is the graph of F. Now, so let us on it. So, let us now put it in the framework of the regular value theorem and prove that this is a indeed a sub manifold of Rn plus 1. So, for that, all I need to do is check that the derivative of this, which is again a map just as in the previous case of SLNR, this derivative is now a map from Rn plus 1 to R.

So, I just have to check that it is a non-zero map. So, I have to find one vector where it is non-zero. So, let us see what the derivative actually looks like, let P belong to phi inverse 0 equal to S. So, I just have to check d phi at P from Rn plus 1 to R. I have to look at the rank of this linear transformation. And since the target is 1 dimensional I just, it is enough to show if I can find a single vector in Rn plus 1 whose image is non-zero.

Then I know that this map is surjective. So, in fact let us just take this d phi P of the vector 0 1. So, first n coordinates are 0 and the last coordinate is 1. Now, this in order to calculate this, I will take a curve passing through the point P in the direction 0, 1 compose it with phi and calculate the derivative. So, this is D by Dt phi of P plus t times 0, comma 1 evaluated as t equals 0.

And that is the same as d by dt of phi of well, so here now let me take this point P in S has two coordinates, one,let us call it alpha, f of alpha, then P plus t time 0, comma 1 will be alpha, comma f of alpha plus t at t equal to 0. And that is the same as going by the definition of phi it is just the second coordinate minus F of the first coordinate. F of alpha plus t minus F of alpha at t equals 0. Now, I am just left with t, derivative of t, which is 1.

So, you immediately see that the 0 is a regular value of this map phi of this function phi. And the graph is a level set of this corresponding to this regular value. So, therefore, it is a sub manifold. Therefore, the graph is a sub manifold.

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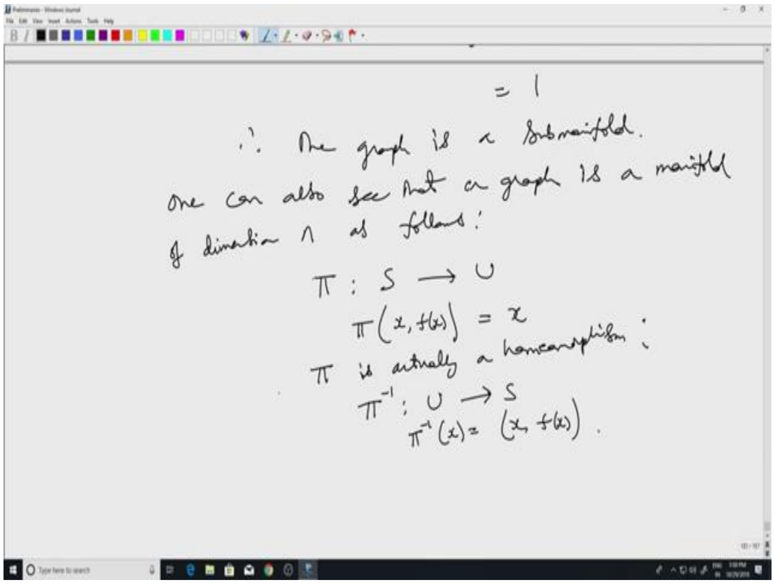

Now, if one is just interested in checking that, the graph is actually a manifold. There is a more direct way of doing it. One can also see that a graph is a manifold of dimension n as follows. So, in fact, the graph is just diffeomorphic to the open set U that we started with. So, I will define a map, the projection map from the graph S to U in a very simple way which is the graph consists of points of the form x, comma f of x and I just project it to you x itself.

So, now pi is actually a homeomorphism. In other words, Pi is obviously continuous and pi is bijective as well. It is clearly one to one and it is clearly surjective as well. And pi inverse, the inverse map is also easy to describe from U to S is just pi inverse of X. I will just put it at height f of x that is it. So, pi is actually, so pi is a homeomorphism and therefore pi provides a single chart. So, S is covered by singles chart and the topology on S is coming from Rn plus 1. If that topology this S itself well, so S is covered by single chart and that is given by pi.

So, one can put a smooth manifold structure on S in a very trivial way via a single chart and it is a small exercise to check that the single chart structure and the smooth structure given by the regular value theorem are the same. So, now let us I would like to say a few words. In many of the examples, for example, the case of the graph or the special linear group SLNR it turned out that the sub manifold that we get is one dimension less than the dimension of the ambient manifold.

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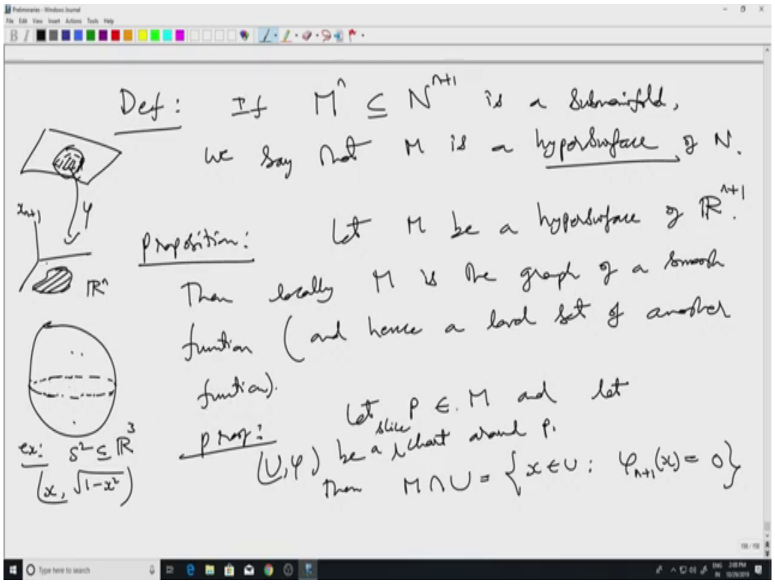

In that case, we say that the sub manifold is a hypersurface. If Mn contained in N plus 1 is a sub manifold. We say that M is a hypersurface of N. Now, I would like to describe so, yeah, so now that I have this going back to the case of the graph, given a smooth function as able to get a hypersurface in Rn plus 1. Here is a converse to that, proposition. Let M be a hypersurface of Rn plus 1. Then locally M is locally M is the graph of the smooth function.

Once and we already seen that the graph of a smooth function can be realized as the level set of some other function and so level set of another function. So, the proof is quite simple let, so when I say locally M is the graph of a smooth function, what I mean is that given a point P in N I would like to find an open set around P and that portion of that so for instance, actually the case of the sphere is good enough.

When I have a sphere. I know that locally I can so for instance graph of a smooth function. Well, I know that every point on the sphere lies in a the if I am in the upper hemisphere, then this is in the graph of the certain functions similarly in the lower hemisphere, in the graph of a certain function, and that description continues to hold even when this I am on the equator. So, for instance, so as an example, let us look at the case of S2 inside R3.

If I am in the upper (hem), closure of the upper hemisphere so the including points on the equator, then any point can be written as X, comma X is in R2 square root of 1 minus X square. So, this function, square root of 1 minus X square will serve the function for which the graph is what we have here. Now, one wants a similar statement here. So, take it. So, let us start with let P belong to M. And let U phi be a chart for chart around P.

So, here actually what I need is not just a chart, I need a slice chart. The slice chart around P. So, then M intersection U now, a slice start, by definition has the property that this the portion of U which intersects with M is taken to in the case of a hypersurface, phi will take M intersection U to Rn. So, in other words, the last coordinate becomes 0.

So, it is a set of all X in U such that phi n plus 1 X equal to 0 because phi is actually a by diffeomorphism which, so the picture is that this is P. I have a bigger open set and this portion is taken to something in Rn. So, the, this is the N plus first coordinate X and plus 1. And so this is phi. So, phi inverse of if I look at where phi n plus 1 become 0, that precisely corresponds to M intersection U.

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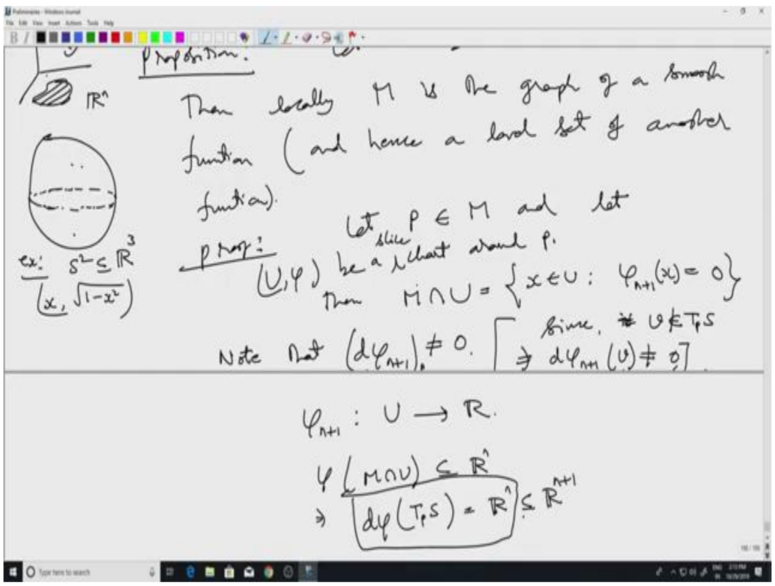

Now, this immediately tells us that also note that d phi n plus 1. If I look at this d phi n plus 1 at the point P is not equal to 0. Not the 0 map. So, here I am thinking of phi n plus 1 as a map from U to R. And then the claim is that this derivative is not 0. So, this can see that this is not 0. Since d phi n plus 1 acting on V is so let me write it slightly different way, so this is actually yeah, the point is that phi takes U in M intersection U to Rn.

So, the point is that because of this property, if I take any vector V, which is not in the tangent space to, So this will imply here d phi, the tangent space to S is actually equal to Rn itself phi is a diffeomorphism and this is contained in Rn plus 1. So, the tangent space to S is taken to the special subspace Rn by under phi. So, that means that if I take a vector V since V not in the tangent space to S then D phi plus 1, the last coordinate cannot be 0.

If it had been 0, then remember that d phi is supposed to be an isomorphism then it would from this equation, if this part had been 0, then it would mean that d phi of V would actually be inside this Rn by definition of the last coordinates. So, but that would mean that V would actually have to come from TPS. So, which is not the case since I took v not to, so this is not 0. So, in short, what I have is that here I have already shown that this is actually M intersection U realized as the levels set. Once I have yeah so now I have a level set.

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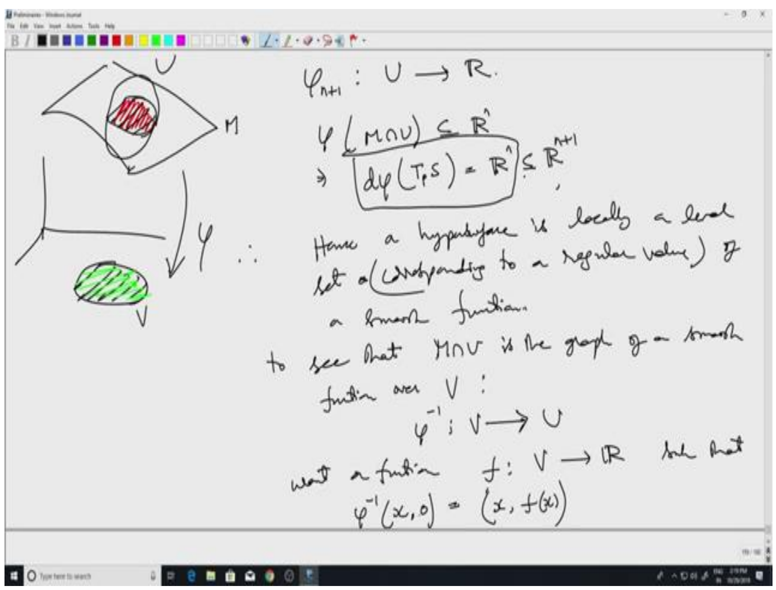

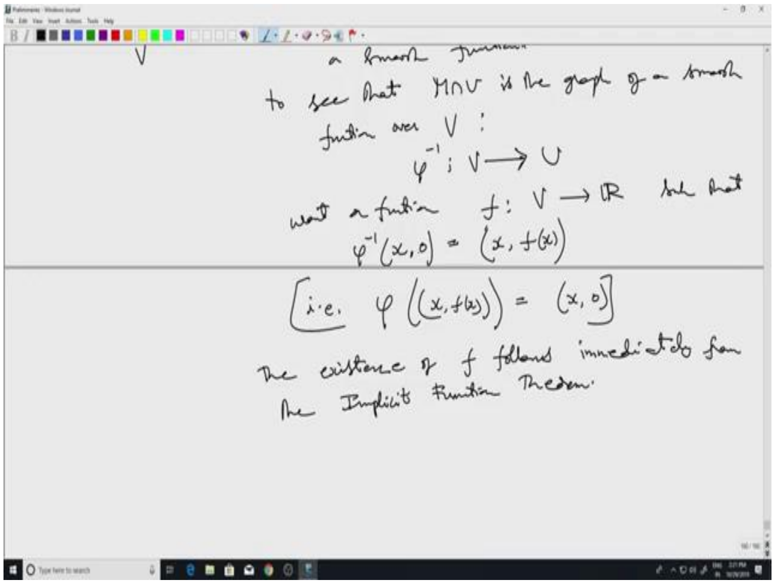

So, actually, yeah so what I have shown is that this if I start with a hypersurface, I have I can write it locally as a level set as a hypersurface is locally a level set corresponding to a regular value of some smooth, of a smooth function and that smooth function just came from the looking at the last coordinate of a slice chart. Well, one can pursue this further then and then one would like to write it as actually the graph of a and that part is also clear how to write it as a graph.

Again, to see that this is M intersection U again, the same portion. So, I have a manifold I have hypersurface M and I am looking at a small portion of it and my slice chart is pushing it down to this. So, this the slice chart is actually defined on an open set in Rn plus 1 not just an open set in N but it takes this to this. And so to think of this upper the red portion, I would like to think of the red portion as a graph or the green portion.

So, I have already given it some names. So, this the open set, big open set we had call U. So, let us call this as let us say V. So, I would like to it is natural to try to use phi inverse as a to regard it as a graph. To see that M intersection U is the graph of a smooth function over V. Now, what we would like to do is so phi inverse already. First, phi inverse is a map from V to U which takes this to this green portion, to the red portion.

However, what I want to do is I want to write want a function F from V to R such that, so essentially every point of inverse phi inverse of X should be notice that the inverse of X is actually a point in Rn plus 1, so maybe phi, I would like to think that it in so like this phi inverse

X, comma 0 so then X would be a point in so X would have n coordinates and n plus first coordinate is 0. So, that is my V.

So, phi inverse of V, phi inverse of this point should be written as I would like to write it as X, comma F of X. So, this is what I want, I want such a function i.e. phi of X F of X equal to X, comma 0 which is the same thing. So, now this is a sort of a functional equation for F, so which functional equation for F in terms of phi. Now, this is precisely the set up for another consequence of the inverse function theorem which I did not talk about much which is called the implicit function theorem.

So, let me not go into it at this point. I will just briefly say that the existence of F follows immediately from the implicit function theorem which is actually, again, a very quick consequence of the inverse function theorem. But since I will not be using this graphical representation on that much, I do not want to pursue this so alright so we will stop here and resume with the description of tangent spaces in my next lecture. Thank you.