An Introduction to Smooth Manifolds
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Lecture 24
Regular Value Theorem

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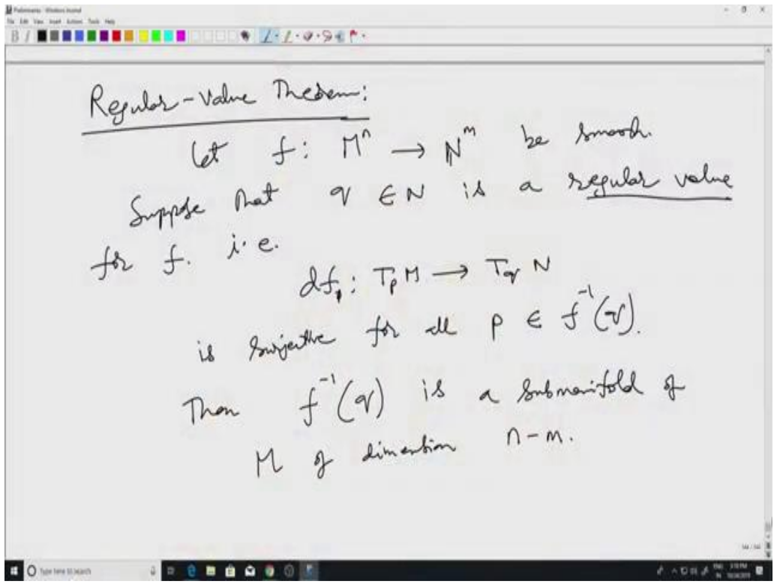

Hello and welcome to the 24th lecture in this series. At the end of the last lecture, I stated a very useful and important theorem, the Regular Value Theorem. Let me continue with that discussion of, let us discuss some examples stemming from this and we will come back to proof of this later on. I stated the theorem for maps between manifolds but in fact even the case of maps between Euclidean spaces is already very interesting.

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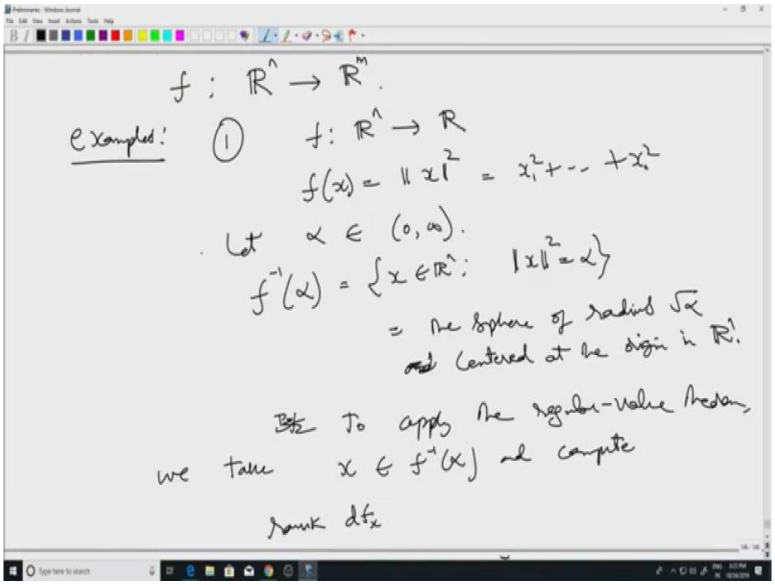

So, let us look at f from Rn to Rm see what kinds of examples we can, we will get out this, so let us start with some simple ones. So, the first thing is let us take f from Rn to just R f of X is norm X square, so it is just X 1 square plus Xn square.

Let now, I will just take any positive number, let alpha belong to 0 infinity the level set f inverse alpha is then set of all X in Rn with norm X square equal to alpha so this is the sphere, the sphere of radius square root of alpha centered at the origin in Rn. And, we know, we do know that we can, we checked it for the unit sphere but we know that the any other sphere is homomorphic to that therefore we can put as.

So, we do know that this is a manifold of dimension n minus 1 since we are in Rn by explicitly constructing charts but this theorem directly gives us, it gives a more uniform way of proofing that this is a manifold. And, all one has to do is to apply the theorem we have to check that if to apply the Regular Value Theorem we, we take X and f inverse alpha and compute rank of Df at X.

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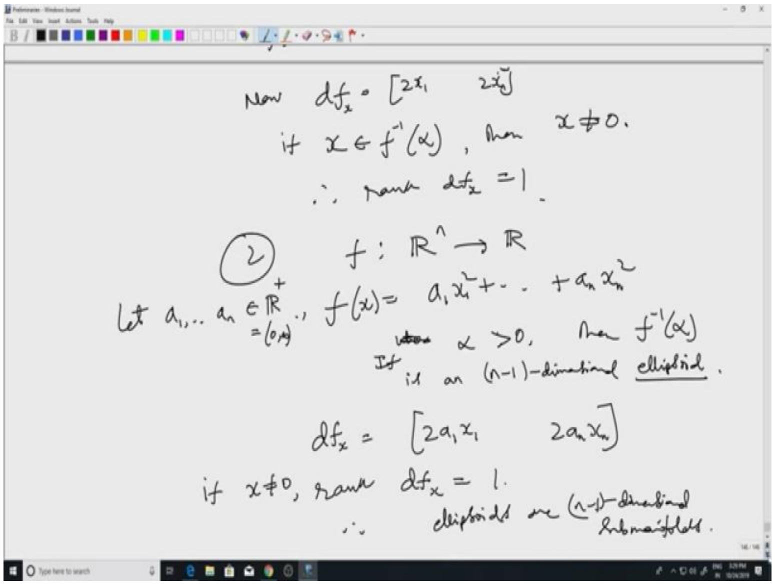

Now, Df at any point X is just the formula for f of X is X1 square plus etcetera, so the derivative, matrix of the derivative is this 1 by n matrix given 2X1 2Xn, now this is equal to this. So 1, and we have to check that the rank is to say that something is a regular value I need this derivative to be a surjective map, so the target is just R. So, I have to check that the rank is 1. Well, this is a just a row vector its rank will be 1 unless it is the 0 vector, so the only time its rank is not 1 is when all the coordinates are 0.

So, if X belongs to, but we are actually interested in the rank only for points in the level set, so if f, X belongs to f inverse alpha then X is cannot be 0. Since its norm square is equal to alpha which is not 0. Therefore, rank dfX equal to 1, so what we have shown is that for any alpha any alpha which is positive strictly positive is the regular value.

Therefore, all the level sets are sub manifolds, so that proofs, gives another proof that this sphere is a manifold well it proof something more. Here it is, we are saying that it is a sub manifold so in other words we have this, not only do we have charts on this f inverse q we have slice charts coming from charts on the bigger space. Now, the power of this Regular value theorem is that we can change this function a bit and still we can get a similar thing. For example, we can proof that Ellipsoids are, so what is an Ellipsoid?

Let us look at f again from Rn to R this time what we will do is we will look at f of X is A1 X1 square plus AnXn square where, so we have to fix this before. Let A1 A2 An belong to R plus which I denoted by, R plus is the same thing as 0 to infinity open network. So, fix some positive constants and define a function like this.

The level set if alpha greater than 0 again, then we can say f inverse alpha is n minus 1 dimensional Ellipsoid. In fact, what I have, we can take this as the definition of an Ellipsoid but if one actually starts with for example n equals 3 we do get our usual notion of Ellipsoids in R3. Or if start N equals 2 you get an ellipse in R2, but this works in all dimensions.

Now, as I was saying the power of this regular value theorem is that the proof that this a manifold or a submanifold is virtually the same as that for the sphere. So, we again compute dfX this time it will be 2A1X1 and then 2AnXn, and this will have rank again it will have rank 1 unless all the xi are 0. Rank dfX if X is not equal to 0, rank dfX equal to 1. So, therefore, everything, every positive number is a regular value and so the level sets are all submanifolds n minus 1 dimensional.

Therefore, Ellipsoids are n minus 1 dimensional submanifolds. The thing is that we do not have to really worry about the geometrical features of this level set namely an Ellipsoid and try to construct special charts. It is just that at one go all we have to do is just compute the derivative, check its rank we know that it is submanifold.

The disadvantage is that if we just use the theorem without going through the proof we still do, we will get that, we will get submanifolds but we will not know what exactly the charts are, what exactly the slice charts are? And so on. So, but off course if one goes through the proof one will see how to get hold of the charts as well.

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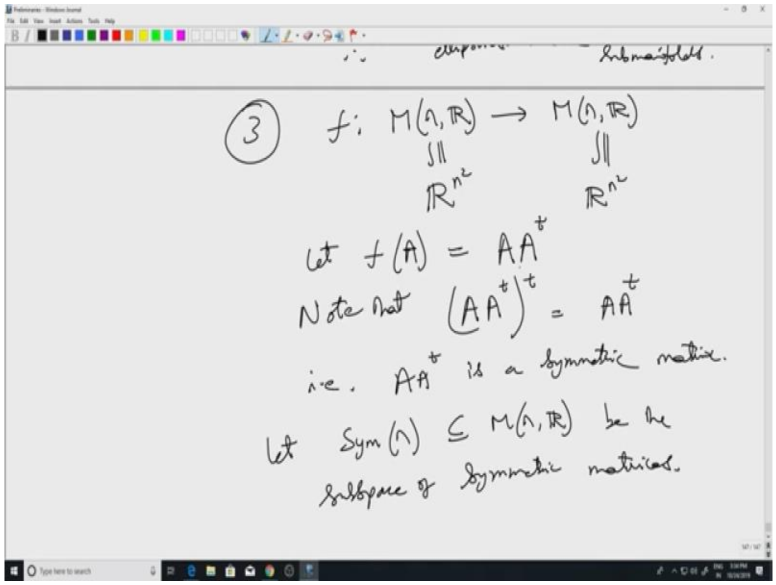

Now, let us do something more interesting which is let us construct some examples of manifolds in, which involve matrices so the first thing I want to do is let us look at a map from this set of all the space of n cross m matrices back to m cross n matrices. Recall that this MnR, we if we were, it is can be identified with as a vector space, topological space and so on with Rn square. And, when I talk about smoothness and so on it is with reference to whatever smoothness means in terms of Rn square.

Now, let me define a map the only all the smoothness continuity every notion is combining from Rn squared. The only place where we will be using where this MnR the fact that we are dealing with matrices is some algebraic operations, which involve matrices, and these operations are not so obvious if we forget the matrix structure and just think of it as an element of Rn square.

So, for instance I can define let f of A so I am just taking a point in MnR and I define it to be A times A transpose. So, this involves two things, one is the transpose of a matrix and second thing is matrix multiplication of course. Now, matrix multiplication is natural enough in MnR, but if you just think of MnR as a n square coordinate vector, matrix multiplication is seems a bit unnatural. So, that is why I said we can switch back and forth between Rn square and MnR.

So, let us look at this map, and let us look at. Now, as I said this if I start with a n cross m matrix, I will get n cross m matrix of course A times A transposes n cross m matrix. But, it is special kind of n cross m matrix, note that A times A transpose, if I take the transpose once more, I will get back what I started with.

So, in other words i.e. AA transposes is a symmetric matrix. It is not just the even if I start with any matrix I am getting a symmetric matrix so instead of regarding the target as MnR let us look at, let Sym n contained in MnR be the subspace of symmetric matrices.

So, this subspace of symmetric matrices it is easy to check that this set of symmetric matrices is actually a vectors space. If you add any two symmetric matrices, you will get a symmetric matrix and if you multiply by a scalar of course it will be symmetric and in fact what is the dimension of this?

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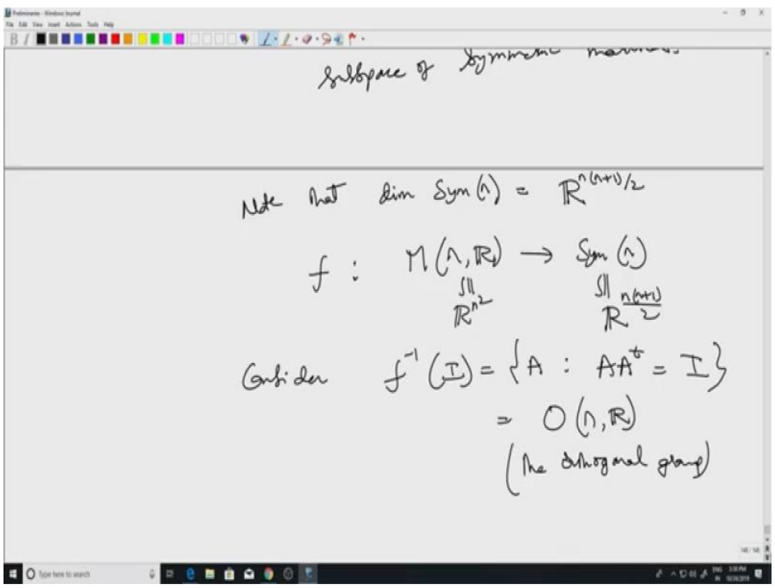

Note that the dimension of the set of symmetric matrices is Rn into n plus 1 divided by 2. And this is easy to see because after all a symmetric matrix. The entries above by definition the entries above entries below are determined by entries above. They are just the same entries with index reversed. So, essentially, there are only so if one thinks in terms of free variables there are n free variables on the diagonal n minus 1 and so on. All the way up to just 1.

So, essentially n into n plus 1 by 2. So that would be n plus n minus 1 plus n minus 2 plus 1. So, I will not do this, this is and in fact one can easily write down in explicit bases for symmetric matrices. Consisting of n into n plus 1 by 2 elements. In other words, n into n plus 1 by 2 matrices. So, that will give a isomorphism from here to here.

Now, the point is I want to regard my map f as a map from MnR to Sym n, in other words Rn squared and to Rn into n plus 1 by 2. This is also vector space and the reason for thinking of it like this is I am going to apply the regular value theorem. When we apply the regular value theorem, the derivative has to be surjective.

So, it is going to matter a lot what the target space is. The derivative will be surjective as a map from Rn square to this smaller dimensional subspace not from Rn square to itself. So, this will work so now what is the regular value of interest? It is the identity matrix.

Consider f inverse I. This is by definition set of all A such that AA transpose equals I. We know that this is exactly the so called the set of orthogonal matrices which is denoted by OnR, is called the orthogonal group, the orthogonal group. So, I would like to say that this is just the I have identified f inverse I with OnR.

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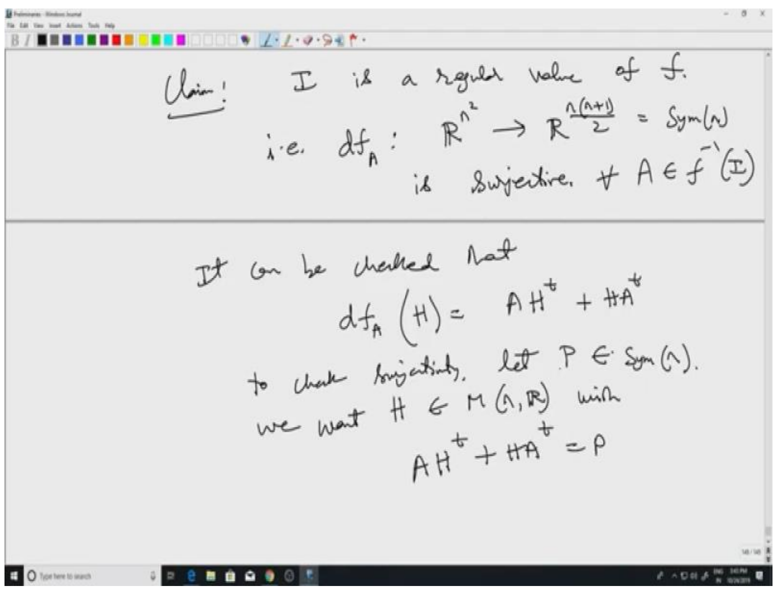

But now I want to say, claim I is a regular value for f, regular value of f. So, what do we have to check, we have to check that the derivative of f at any A and since these are all vector spaces, Euclidean spaces, so the tangent spaces itself so this would be a map from Rn square to Rn into n plus 1 by 2. If this so we want to check that i.e. this map has, is surjective.

Sometimes it is helpful to think in terms of rank when we can write down a matrix and check that the rank is the maximum possible. So, in this case it is not helpful to think in terms of the matrix because these dimensions of the spaces involved are large, it will be bit messy rather we just think of it as a linear transformation and not worry about its Jacobian matrix and just show that the linear transformation is surjective. So, I want to show this, now, well, this should happen for a surjective for all A in f inverse I.

In other words, all A is satisfying AA transpose equal to identity. So, let us compute the derivative. In fact I, it can be checked, I will not do the calculation it can be checked that a similar calculation was done in the beginning of this course and the same thing carries over, it can be checked that the derivative of dfA acting as a linear transformation acting on H as AH transpose plus HA transpose. And this is true for whatever A you start with it does not necessarily have to be in f inverse I.

So, now to show that suppose I want to check that this is surjective, to check surjectivity. Let B belong, no not B let us give it some other name, P belong to Sym n. So, I start with so this is Sym n so I am starting with some element here and I want to show that it can be written as dfA of H for some H. So, we want to find, we want H in MnR with AH transpose plus HA transpose equals P.

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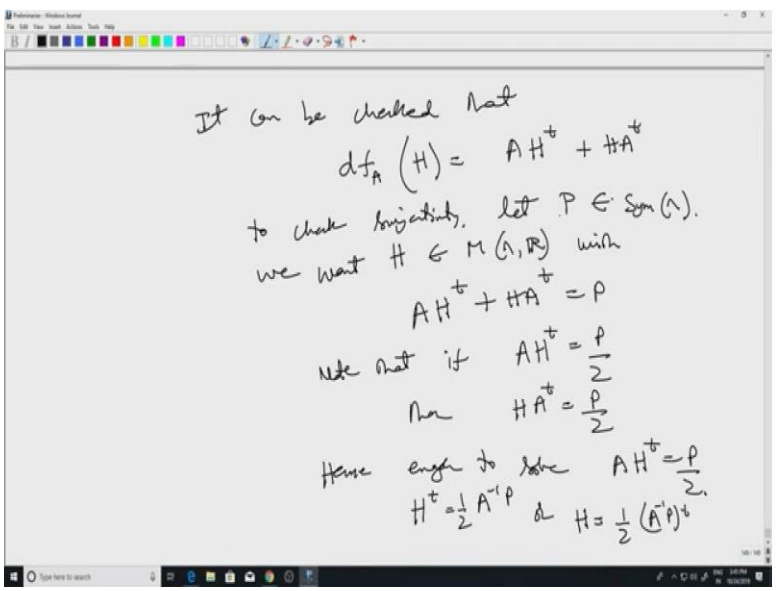

Now, it is kind of if one tries to solve directly from this equation one goes around in circle but instead what one notices is that, let us solve something slightly different. So, note that if there are two terms here if each of them equals P by 2 I would be done.

So, let us try to and then let us see what this condition means suppose note that if AH equals P by 2 so this term I set equal to P by 2. If this forces the second one also HA transpose to be P by 2 then I would be in good shape. And that is the case so if I take transpose then if I take transpose of this left hand side I will get HA transpose and P is already a symmetric matrix so that is also equal to P by 2. So, if one of them is P by 2 the other one is P by 2 as well automatically.

So, hence enough to solve, AH transpose equals P by 2 and now I am going to use I really do not need that this is orthogonal what I do need is that I would like to isolate H so I just need to multiply by A inverse. H transpose equal to A inverse P and then multiply it by half or actually I want H so I just take the transpose once more. A inverse P transpose. So, I have found an H which I have found an H which will such that dfA of H is actually equal to P. Therefore, this is surjective and hence I conclude that.

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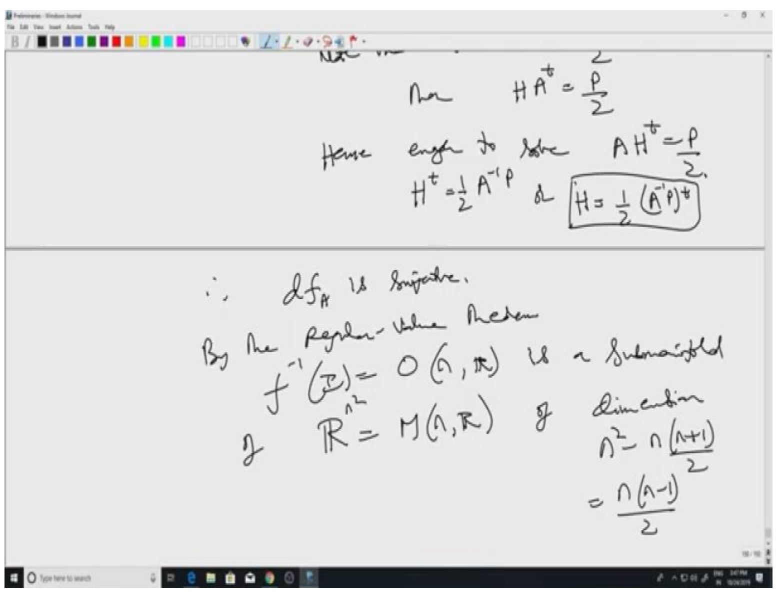

Therefore, dfA surjective by the regular value theorem, f inverse I is OnR is a submanifold of Rn square equal to MnR of dimension. So, the dimension of the submanifold the theorem tells us that it is difference between the target dimension and the domain, domain is n Square target is n into n plus 1 by 2 and if one does the calculation here so we will get n into n minus 1 by 2.

So, we have shown that the set of orthogonal matrices is a manifold of dimension n into n minus 1 by 2. So, with this example I will conclude this lecture, in the next lecture I will say few more things, talk about a few more examples stemming from the regular value theorem. Thank you.