An Introduction to Smooth Manifolds
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Lecture 23
Tangent Space of a Sub manifold

Hello and welcome to the 23rd lecture and the series. Now, last time I introduced sub manifold and let us continue with studying them.

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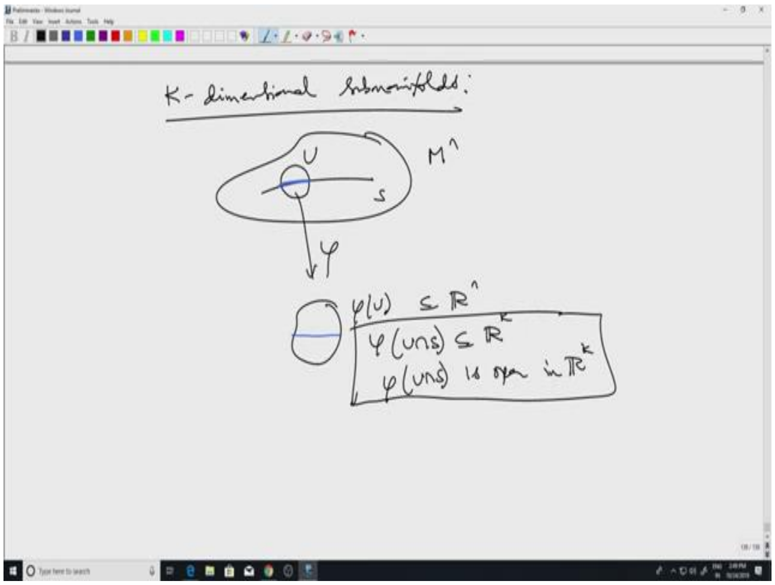

To recall we had M n manifold so we had K dimensional sub manifold. We had M n manifold and a subset S, we said that S is the K dimensional sub manifold, if every point in S is contained in a special kind of chart on M. This special chart which we called a Slice chart, well what is special about this, is that, as I mean of course if this is U phi to U will be some, phi of U is some open subset of Rn but the slice chart property is that the intersection with S should go to phi of U intersection S should be contained in Rk and it should be open.

So, two things phi of U intercept S is open in Rk in the lower dimensional subspace. The movement one has if every point of S is contained in some such chart we say that we could give a smooth structure on S and make S itself into a K dimensional sub manifold. But, if something more than it is a much more special than just saying that S is a K dimensional sub manifold as I said it should be naturally related to the structure on M.

And, the definition makes it clear after all we are getting the charts on S just by starting the charts on M. So, let us see some easy consequences of this definition.

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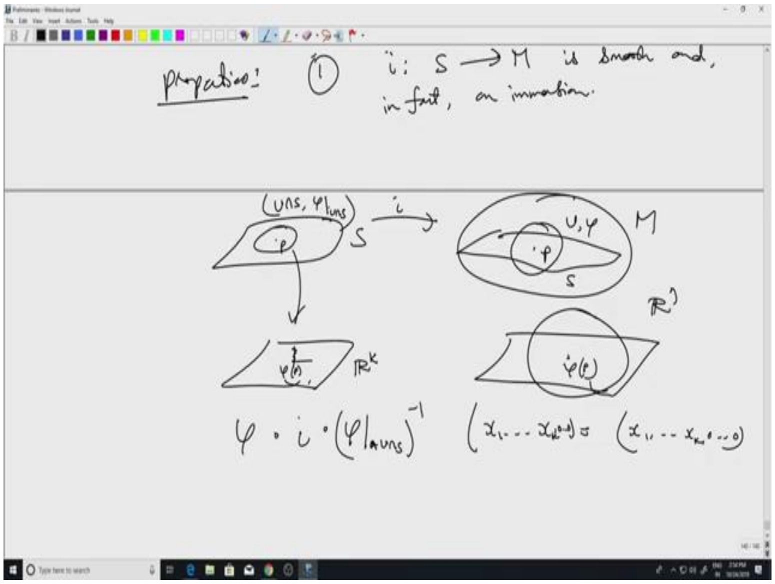

First thing is that properties, if I look at the inclusion map i from S to M, than this map is smooth and in fact an immersion.

So, this is again to check that this is smooth we just go back to our definition of smoothness and the differentiable structure and that less on S. So, it is almost a direct consequence of that, so to say that suppose this is my sub manifold S and this is being out inside inclusion, goes back to itself.

Now, to say that this map is smooth would mean that starting with any point here, I should be able to find a chart here about this and a chart on M such that the inclusion map in these charts should be a smooth map.

Well, the only chart we know on S are anyway this restriction of slice chart, so let us take some such chart. So, let us take a Slice chart here U phi and work with the same Slice chart here as well. So, it is such that now it will be intersection phi restricted to U intersection S would be the new chart here this would go to an open subset of Rk. While this would go to an open subset of Rn itself.

So, this is phi of P, this is phi of P as well. So, essentially, we are working with the same chart and the domain and image except that in the image the chart is defined on a much bigger set here we are defining it only on the on S. If you just compose the conclusion map. We will just get, so phi redistricted to u intersection S the inverse of this composed with i, compose with well again phi if I see what it does to X1 up to XK.

Here it is immediately clear that after all, all it is doing is X1. Actually here the way I defined Rk so let me put 0 it is just the identity map Sk 0, 0, 0. So, it is just the, it is just the identity map and off course it is smooth and not only that the fact this is the identity map will immediately also imply that it is an immersion because as I said earlier the rank of a map, the derivative of a map can be computed once we know what it looks like in local coordinates.

Because in local coordinates will be recomposing and post composing the map by some diffeomorphisms namely the chart maps. And that does not change the map. So, in local coordinate it is just the identity therefore it is identity from Rk and itself. So, therefore, its rank is K.

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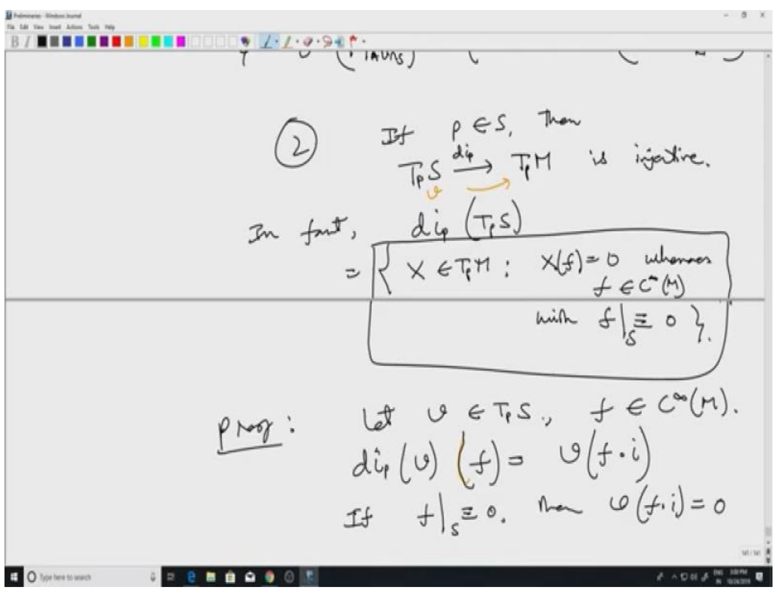

Now, here is an important thing with namely the tangent space, the good about sub manifolds is that if P belongs to S, then TpS I have a map from TpS to TpM is the derivative, I have already said that this is an immersion so it will follow that this is a, this map is injective.

So, in short I can think of this the tangent space to S identify this with a certain sub space of TpM but actually, in fact one can be even more explicit than what I have written here. So, let us what we can do is, so let in fact this TpS if I want to be more explicit so I want to see what exactly the image of this rather sub space is here one can describe it very conveniently as follows, dip of TpS this the sub space of TpM given by all derivations X in TpM such that Xf equal to 0 whenever F is an infinity function on M with F restricted to S is identically 0.

So, what I am doing is identifying a coming up with a concrete sub space of the tangent space to the big manifold which happens to be the image of this tangent space to the sub manifold. Image under this map Dip derivative of the inclusion. So, in practice we can think of TpS as a sub space of TpM given by this thing here, what I have here.

So, let us quickly why this is the case how do we, why is the image like this so proof, well, one way is clear so let V belong to TpS now I want see what Dip of V. This is supposed to be a derivation so remember that V is here and I want to get something here in TpM. So, is supposed to be a derivation on C infinity M .

So, it will act on a C infinity function F, so let F belongs to C infinity M and according to the definition this is nothing but V of F composed with I in other words the restriction of F to the sub manifold S. Now F composed with I the restriction map by hypothesis so if F restricted to S is identically 0 then V of F composed with I is 0. Since it is, in other words since this F composed with I becomes a 0 function on S so V of any 0 function for that matter V of any constant function on S will be 0. Because V is a tangent vector to S, it does not care about what the function is doing elsewhere in the manifold (())(13:43) manifold M.

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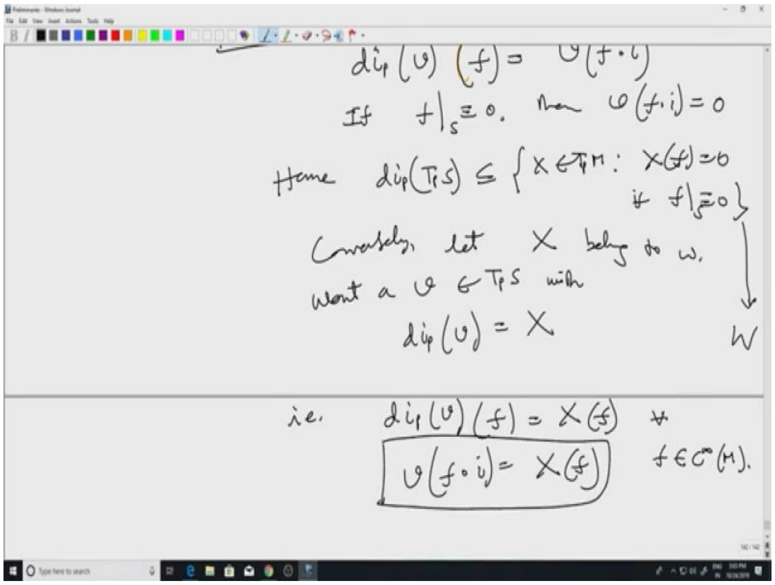

So in other, so what we shown is that this Dip to TpS, hence Dip of TpS is contained of all these derivations which have the property that Xf equal to 0. If F, restricted to S is identically 0. So, the proof would be complete if one knows that this if everything here so we have shown that the image is contained in this right hand side sub space of TpM. But now I want to say that anything which has this property is of the form. Conversely, let X belong to let us called the sub space W belong to W.

So, let me write here bit more clearly. So, this sub space, this entire thing labelled as W. So, let X belong to this sub space W. So, I want to say that want a V in TpS with dip of V equal to X. And again just working through this definitions will tell us how to find this v. So, we want this i.e. this is the same thing as saying that Dip of V acting on any f equal to Xf for all C infinity functions on M, right, and this is the well the left hand side again is the same as V of f composed with I equal to Xf.

So, at this point we can just figure out what V has to be, so V of f restricted to the sub manifold should be equal to X of f. So, this equation should actually in fact tell us precisely tell us what V is just by plugging in various choices of f.

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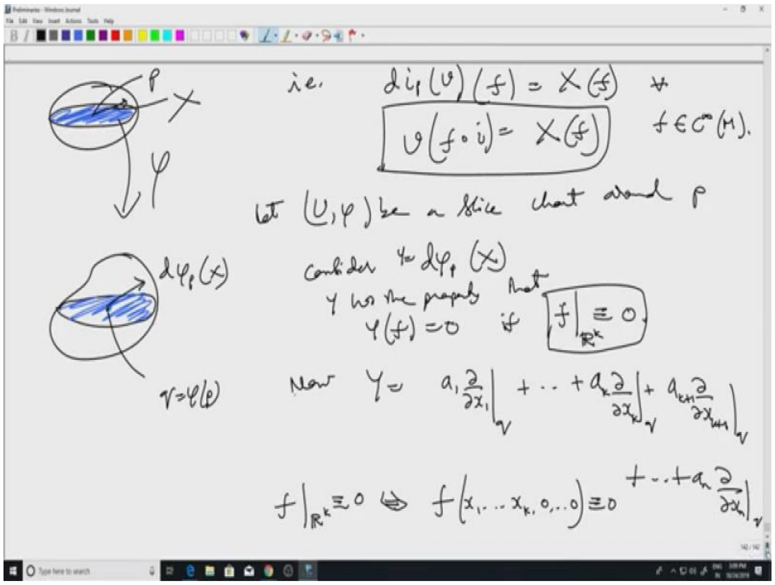

So, now let U phi be Slice chart around P, so we just transfer the picture, so the point P is somewhere here. So, I will just transfer the picture to Euclidean space. And, we know that in Euclidian space we, every tangent vector can be expressed in terms of a natural basis. Namely, the natural basis is just that is the one given by partial derivatives at that point.

So, let us transfer, so in other words consider dx d phi p of X so we had a vector X here to begin with this X. We did not know that we would like to claim that any x in this which is in sub space W is actually tangent to this S. In other words it should be tangent to this blue portion here. But we do not know that so let us just put it put an arrow here and make it look as if it is not tangent to that.

What I will do is I will push it here I will get d phi p of X and get a vector here. And this property of the sub space W in fact we can do that we can push the sub space W here as well. This would correspond to saying that all functions, the corresponding sub space should be all those X acting on any f instead of S I will be having this open subset of Rk.

So, if f restricted to this open subset is 0 then X of f should be 0. Well not quite X it is this vector consider Y equal to this. Y has the property that Y of f is 0 if f restricted to Rk is identically 0. So, in other words, if whenever the last n minus k coordinates are 0 if the function happens to be 0, then the derivative of that function this y action on that is also 0, is what we are saying. As I was saying, so whole point of moving to Rn is that I can have a natural basis in terms of which I can express all derivations.

So, let us do that now Y equals a1 del by del X1 at this point, so this point is some point q. So, q equals phi of P plus. So, let us write the first K1 separately plus aK plus 1, an del by del Xn at q. Now, the condition on Y is that, suppose I start with rather than writing this equation. So, let me write this suppose I start with a function which satisfies this condition, f restricted to Rk is identically 0 is the same thing as saying that f of, so whenever the last n minus K coordinates.

So, when I restrict X1, Xk and if I put the last this is what restriction to Rk means so when I, an expression of this form is identically 0, so that is what we are told. So, if we start with any such function then we are given the information that Y of f is 0. Now we know lots of such functions f namely the projections on to the k plus first and so on coordinates.

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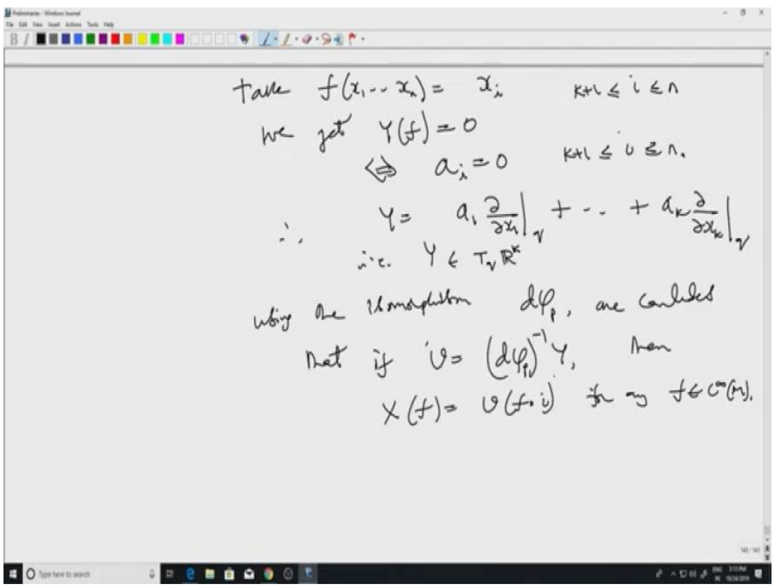

So, take f of X1, Xn equal to Xi where I lies between k plus 1 and n. Then we get just by hypothesis so Y of f is 0 if and only if , so I just plug in the Xi here so I will get ai is 0 where i lies between k plus 1 and n.

So, in short what we have concluded is that any such Y. Therefore, Y is actually of the form del ak del by del X k, ie Y is actually belongs to the tangents space at q of Rk, since it involves this. So, this is was the statement about Y.

Well one can, using the isomorphism d phi p, one concludes that X itself this . As I was saying this we were ultimately looking for some the way I had put it we were looking for some V which was of this form so one concludes. So, this Y is what you would call as V, one concludes that if V equals d phi P inverse of this Y. Then essentially this original X then we have this equation that axiom of X on f is the same thing as v of Xf equal to V of f composed with i for any f in C infinity M.

So, I leave the, skip the details but the essential computation has been done here and it just involves going to coordinates. Now, I have been promising a theorem which generates lots of sub manifolds so let me state that now.

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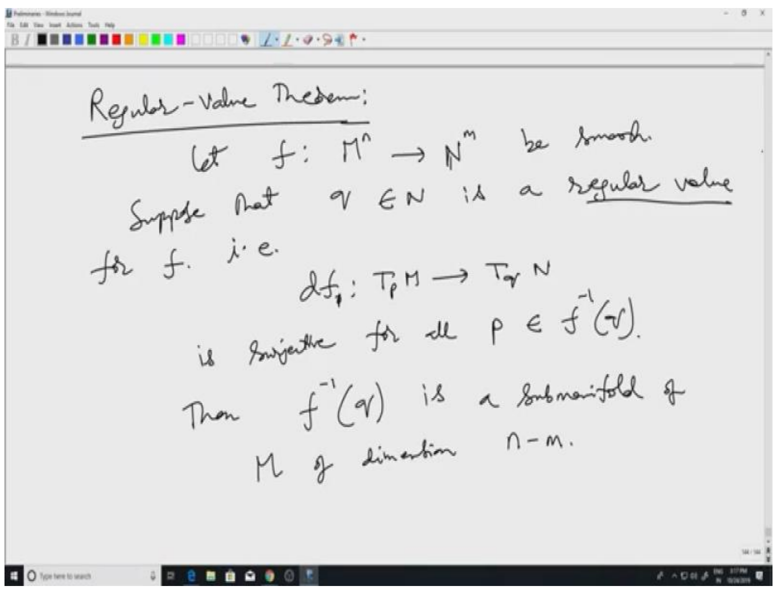

Regular-value Theorem, let f from Mn to M rather not M again some other manifold Nm be smooth. Suppose that q in N is a regular value for f, so regular value for f just means that ie dfp from TpM to TqN has is surjective for all P in f inverse q so this is f inverse q is called a level set of f.

So, for all points in the level set I demand that the derivative is surjective. In particular, this condition already implies that this M is less than or equal to N and we have also seen that this condition of surjectivity earlier where I talked about submersions, the way I defined a submersion I demanded that the derivative is surjective at all points. Here we are demanding that the derivative is surjective only for those P in the level set of q.

Then the claim is that, f inverse q the level set is a sub manifold of M of dimension n minus m. And this will be a consequence of again it will be a consequence of the inverse function theorem and, but before I go into the proof I want to in my next couple of lectures I want to give many examples which this will generate in fact this is a most useful way of getting hold of manifolds. Alright, so let us stop here for today. Thank you.