An Introduction to Smooth Manifolds
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Lecture 22
Submanifolds

Hello and welcome to the 22nd lecture in the series.

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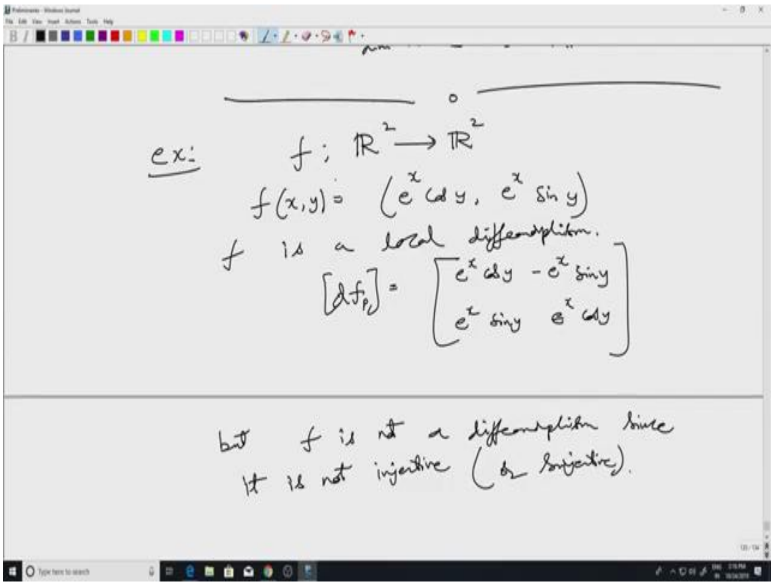

In the last lecture, I discuss the inverse function theorem, and the constant rank theorem, and the setting of manifolds. And I ended the lecture, by giving an example of a local diffeomorphism, which is not a diffeomorphism.

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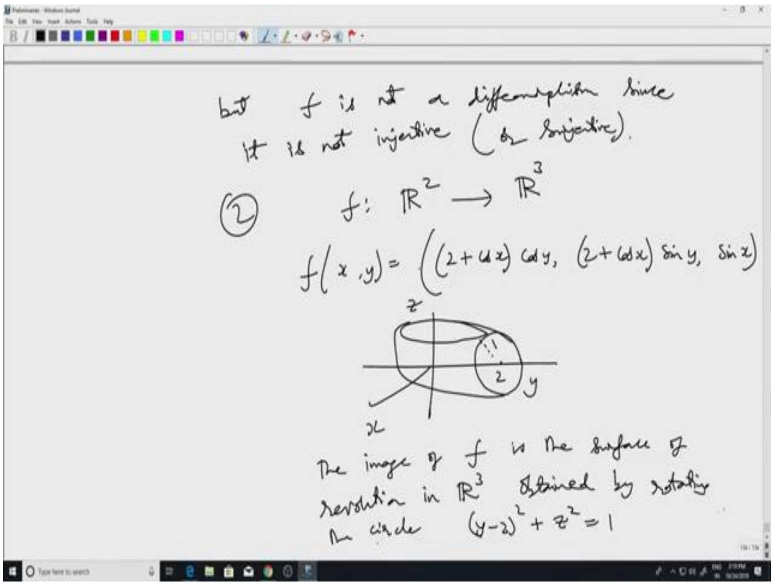

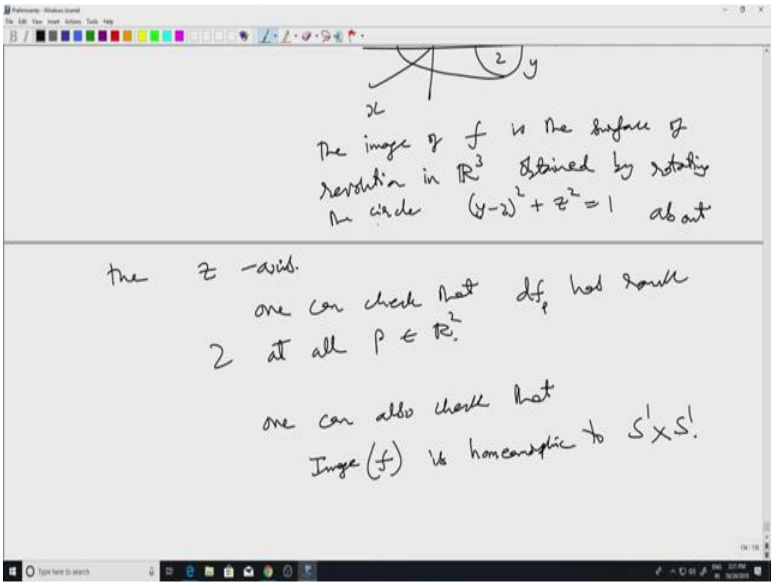

So, let me give an example of an immersion now, so I will take an immersion of R2 and R3. A trivial immersion would be just the inclusion map x, y going to x, y comma 0, for instance, and but something more interesting it is the following. I will define f of x, y is equal to 2 plus cos x cos y, 2 plus cos x, sin y and finally sin x.

Now, one can check that as x and y vary over all possible values, the locus of these coordinates in R3 is precisely, so if I draw the, this is the x, y, z axis, this is the point 2 and the circle of radius 1. If I rotate the circle above the z axis, well, I will get a surface of revolution and that is precisely the image of f and it is also, so, let me first, say the image of f is the surface of revolution in R3 obtained by rotating the curve, rotating the, instead of saying the curve, let me say the circle obtained by rotating the circle, circle is y minus 2 square plus z square equal to 1, obtaining the circle about the z axis.

One can check that dfp has rank 2 at all P in R2. So, as before one can just write down the Jacobian matrix check that as rank 2. Well, the interesting thing is that, so this surface of revolution, I will just mark that one can also check that image of f which is the surface of revolution, is actually just the 2 dimensional torus which defined as a product manifold. And, so here, so is homoeomorphic toS1 cross S1.

And as it turns out, as we are soon going to see this is, this image of f is what is called as submanifold of R3. In particular, it is not just a topological space, it has it is itself a smooth manifold. So, once we have that we can make a stronger statement that the image of f is actually diffeomorphic to S1 cross S1. Where, S1 cross S1 already, we know that it is a smooth manifold, because of the product differentiable structure.

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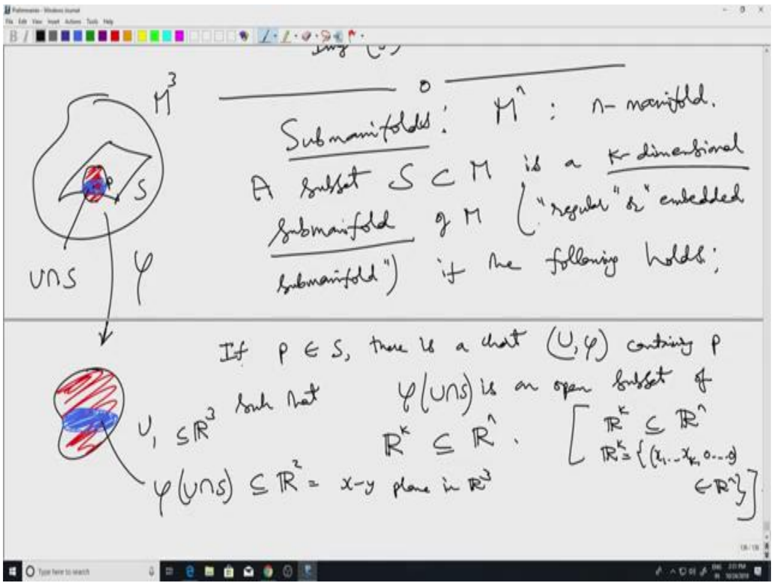

So, now let us move on to a discussion of submanifolds. I will be focusing only one, there are essentially two classes of submanifolds one studies, immerse submanifolds and embedded submanifolds, I will focus only on embedded submanifold. So, let me define, what an embedded submanifold.

So, I start with a smooth n manifold, a subset S of M is a K dimensional submanifold of M. So, this as I said, what I call a submanifold in text also known as. So, in standard text there also called as regular or embedded submanifold, if the following holds the idea behind this, the definition I am about to make is that very simple actually.

So, not only should the subset S be a K dimensional manifold by itself, the point is that it should be a submanifold. So, in some sense, the charts that we obtain on S should be related to the charts on M.

Now, the prototype for submanifold the simplest after all, what is the manifold, manifold locally looks like an open set in RN. So, now inside RN the simplest submanifold, whatever definition of submanifold one takes, the simplest submanifold one can think of is just a, the K dimensional RK sitting inside RN. So, we want to model this on that simple setting. So, let us phrase it like this if the following holds.

For every, instead of saying for every, instead of saying for every, I will put it like this, If P belongs to S there is a chart u, phi containing P such that phi of u is an open subset of Rk, this k is that same k, k dimensional submanifold phi of U is an open subset of RK contained in RN.

So, RK contained in RN, by this we mean RK strictly speaking RK has only K coordinates while here we need N coordinates, but this we think of RK as set of all X1 XK and the last N minus K coordinates 0. So, this is the definition of a K (())(11:22). So, what we are saying pictorially.

This is my M for instance, let us think of a three dimensional sub-manifold, a three dimensional M and let us look at S, let us see what a two dimensional submanifold of a three dimensional manifold looks like. So, S is supposed to be two dimensional. What we want is every point P in S there should be a chart this chart u phi is obviously with reference to M since, this only M is all that we have to start with.

So, this is a chart which will, which contains P and what this chart does? So, here I will draw this map phi. So, this U is going to something which I have been calling U1 and this is in R3, now, oh sorry, I forgot one crucial thing here phi of U intersection S that is the important thing.

So, now this is in R3 and what we want to is that, this intersection here U intersection S let me show this here, this blue part is U intersection S the entire red thing will go to some U1 in R3 and the blue part we demand that it goes inside R2 sitting inside R3 in other words Z coordinates should be 0. So, it will be the x y plain. So, this phi of U intersection S and this we want to be contained in R2 go to x y plain in R3 and it should be an open subset of the x y plain

So, as I said the reasoning behind this definition is very simple it says that the simplest manifold one can, submanifold one can even without a formally defining a submanifold the simplest thing one would like to have is the case of for example, in this sitting R2 sitting R3 should be a submanifold and we want to have the same situation in a, for a manifold.

So, we just demand that the chart map replicates this picture inside the manifold that is a embedded submanifolds. Well, as I have defined it, it is just a subset with this property. Now, there are several things to check one is that, well to begin with it is actually a manifold by itself. So, for that I need charts on S.

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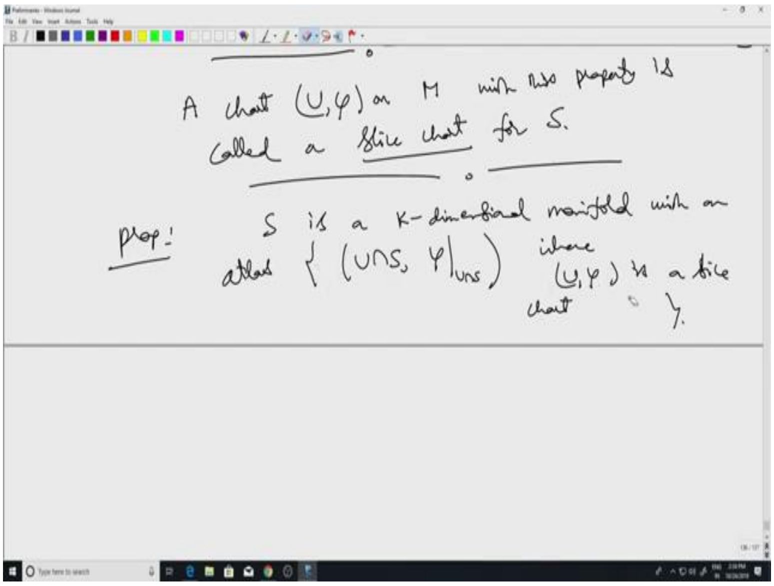

So, first let us give this chart of this type a name. A chart U phi on M with this property is called this property meaning off course that phi of U intersection S is an open subset of RK with this property is called a slice chart for S. Now, so, first thing to note is that proposition. S is a K dimensional manifold with an atlas given by, you just look at U intersection S and then phi U intersection S overall where U phi is a slice chart, chart on M is a slice chart, chart on U, is a slice chart.

So, essentially, we cover this submanifold the set S by sets of the form U intersection S and we use the same homeomorphism phi which was there on M. But we just restricted to U intersection S and the claim is that this provides a smooth structure, a differentiable structure in other words the charts on S with smooth transition functions.

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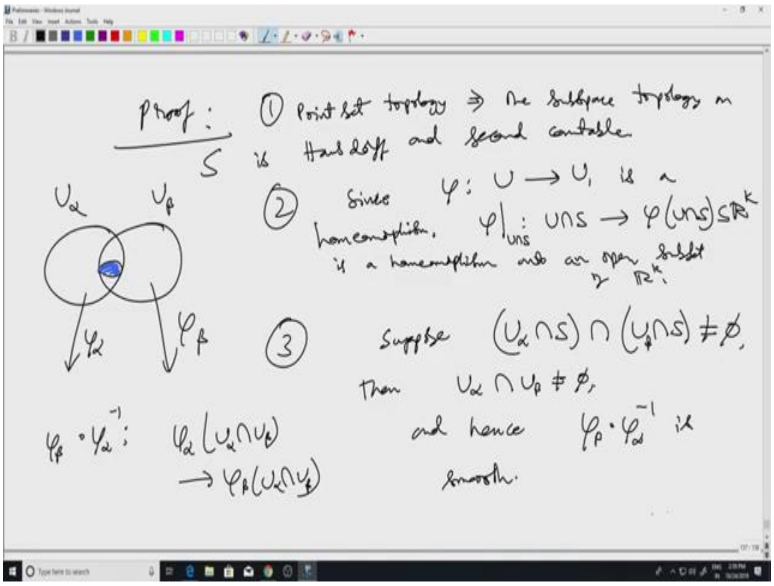

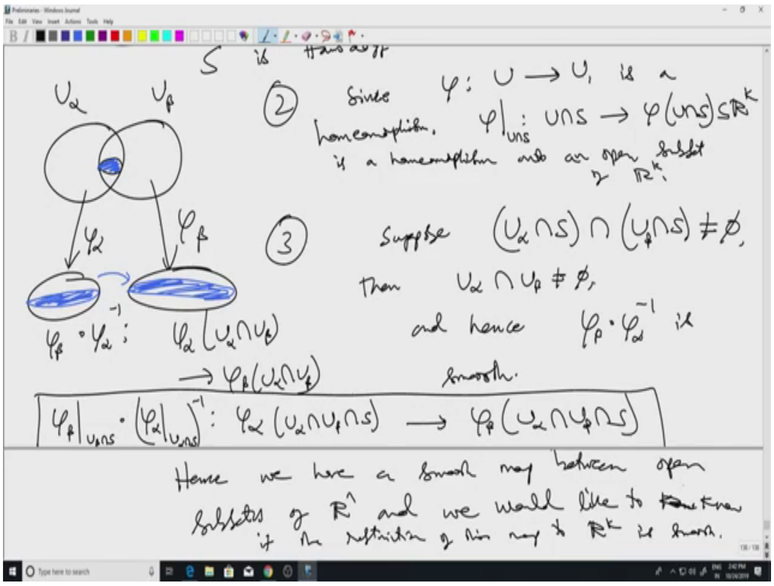

So, one can check point set by general considerations. I will not go over that point set topology implies that the subspace topology on S is Hausdorff and second countable. What I want to, so this is not an issue what we have to just check is that first of all that remember that the check that something is a smooth manifold I have to check that apart from the topological condition. I need to know, I need to have charts in the sense of homeomorphisms then I want the check that the transition functions are smooth.

Well, so, this is the first thing since phi from U to U1 is a homeomorphism phi of U restricted to S, U intersection S to phi of. So, if I have a homeomorphism and if I restricted to a subset, it will be a homeomorphism on the subset with the subspace topology on to its image that is all that I am saying here, this is a homeomorphism.

But that is not saying anything much the point is that since it is a slice chart this is on to an open subset this is the main thing. So, this is an open subset of RK, open subset of RK. This is an on to an open subset of RK and then the transition as per the transition maps. So, let us take two such charts which intersect.

Suppose, well, if, suppose U alpha intersection S this is one chart, U beta intersection S is not empty then off course, we can forget about S and conclude that U alpha intersection U beta is not empty and let us just look at the bigger chart. So, the picture is like this so, this is and there is phi alpha here, phi beta U alpha U beta.

Now, we know that this when I do phi alpha inverse and then do phi beta, which is a map from phi alpha U alpha intersection U beta to phi beta U alpha intersection U beta for the moment if I forget about this blue portion, I know that this bigger chart transition function is smooth and once this is smooth these are open subsets of RN. So, this is non empty and hence, phi beta composed with phi alpha inverse is smooth. So, what we have is, we have a smooth map.

Hence, we have a smooth map between open subsets of RN and what do we want to know? Well, now we will bring the blue part into the picture. So, in other words intersect with S. So, we want to know if these new transition maps. So, again, I have to put U alpha intersect, U beta intersection S composed with phi alpha U alpha intersection S this inverse everywhere just intersect with.

So, we are interested in knowing whether this map is smooth, knowing that the bigger map is smooth the map between the smaller subset. So, this actually, let me use the blue part, it will show this very clearly. So, I know that this overall map transition map from here to here is smooth. I want to know when restricted to this blue part it is smooth.

So, the situation is hence, we have a smooth map between open subsets of RN and we would like to know if the restriction of this map to RK is smooth. This is off course true I mean after all what are we saying here. So, for the moment, let us forget open subsets that is just adds to notational complexity.

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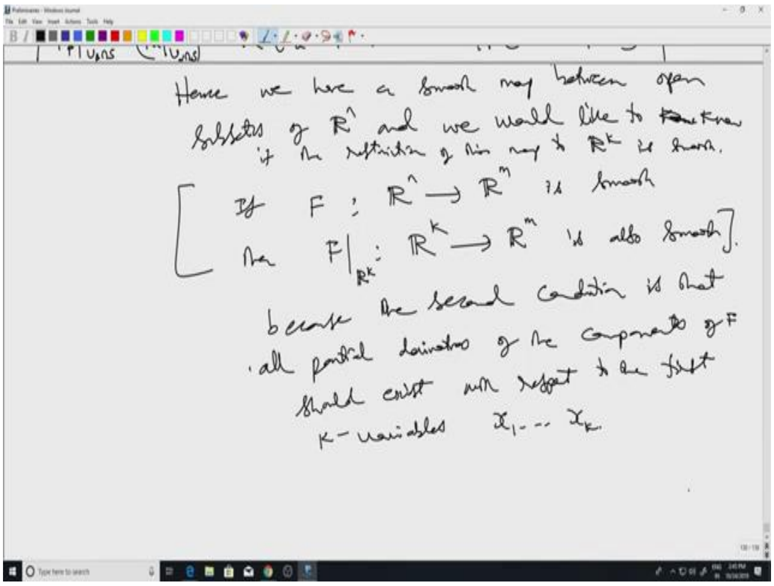

So, let us, so look at this more general situation. So, I have F from RN actually I do not even need the same dimension RN to RK, RN to RM if F from this to this smooth then F restricted to RK from RK to RN is also smooth and one easy way of saying is well, our definition of smooth just is that all partial derivatives. So, for instance here, we know that all partial derivatives of all orders with respect to all the variables exist. Here, what we are saying is that when I restricted to this all partial derivatives with respect to the first K variables should exist that is it.

That is of course true this is the second statement is a much weaker part of the first statement. So, this is certainly true, because of we the second condition is that partial derivatives of the components of F should all partial derivatives should exist with respect to the first K variables X1 XK.

We the hypothesis, that with respect to all the variables all partial derivatives exist here it is for first K variables. So, that is it. So, that smoothness of transition functions is immediate. So, let us stop here, next I will talk about tangent spaces to submanifolds and give a theorem generating in a very easy way we can come up with lots of submanifolds. Thanks.