An Introduction to Smooth Manifolds

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Lecture 02

Multivariable Calculus 01

Okay, so this, today's class we will talk about multivariable calculus. This is the other topic which I am going to discuss as part of the preliminaries. Last time I talked a bit about linear algebra. So, let us start.

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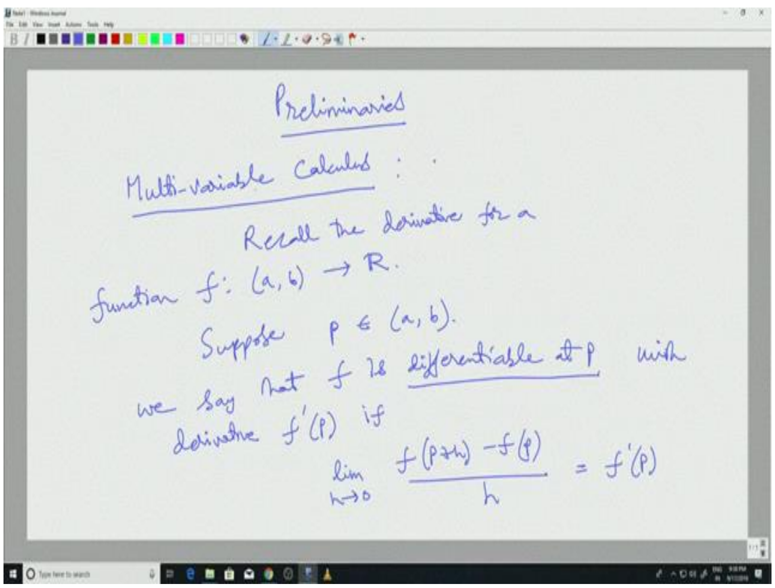

Now, so let us just recall the derivative in one variable. So let us say that recall So, if I have a function defined on an interval AB real valued function of one real variable and suppose P is a number lying in this interval we say that f is differentiable at p with derivative f prime p, if limit h going to zero f of p plus h minus f of p divided by h equals prime p.

So, the to say that f is differentiable at p means that the this limit exists and the value of the limit we denote by f prime p and we call it the derivative of f at p.

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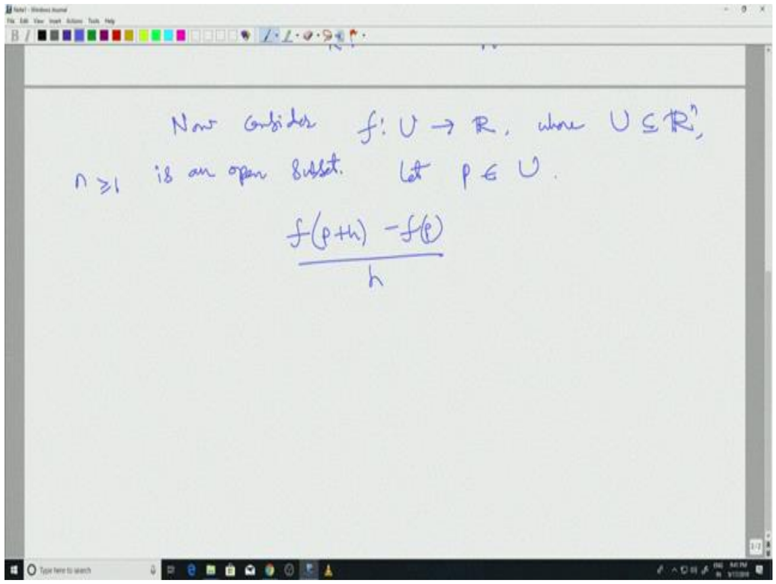

Now consider f from U to R, where U is n Rn n greater than or equal to one is an open subset. So, I have a function defined on an open subset of Rn. Again, I take a point p in U, I take a point p in U. If I just tried to imitate the previous definition, the ratio does not make sense because f of p plus h minus f of p divided by h. Now, the point is that this h would be an element of Rn. So, while this f of p plus h minus f of P would be an element here, so I cannot divide by an element of Rn, so this ratio does not make sense.

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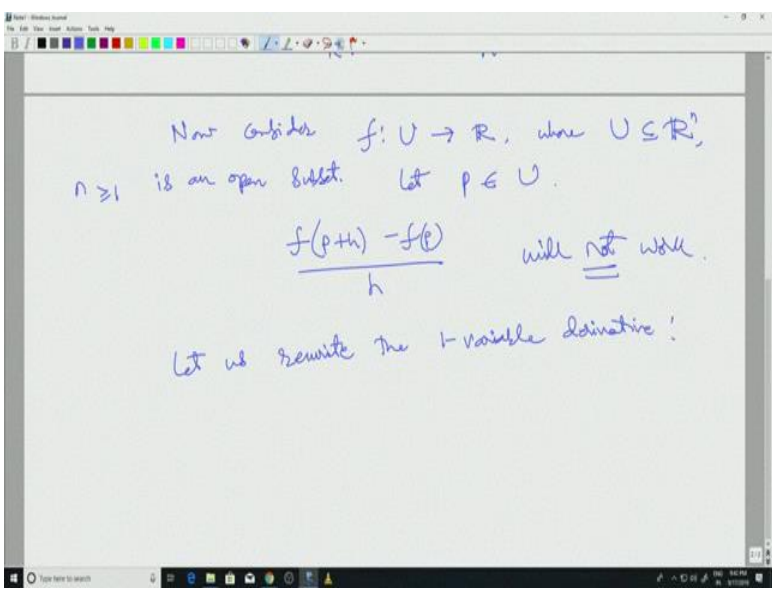

So instead, so we this will not work. The difference quotient concept will not work. So let us rewrite. Let us go back to the one variable case and rewrite the equation for a derivative, the rather the definition for a derivative in a different way, which will easily generalize to higher dimensions. Let us rewrite the one variable derivative.

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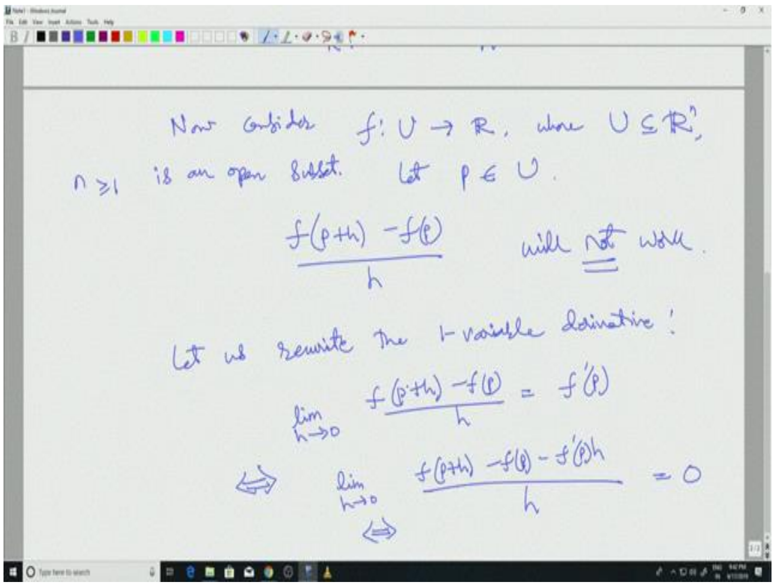

So I will rewrite it as follows. So I will just write well, first to say that limit h going to zero, f of p plus h minus f of p divided by h equals f prime p is the same thing as saying that limit h going to 0, f of p plus h minus f of p minus f prime p times h divided by h equals zero. All I have done is multiplied and divided this f prime p by h, h by h. And then that is just one and then brought it to the site. And even this is not good enough.

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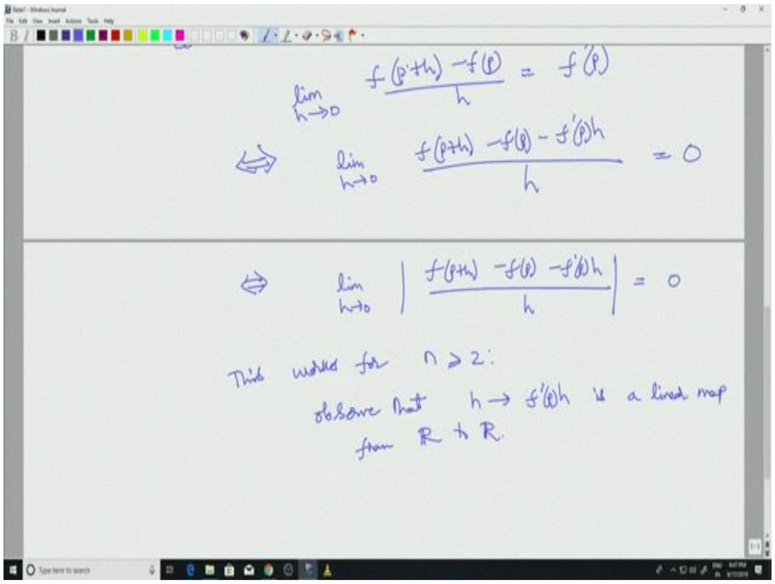

Notice that the to say that a number limit of this is equal to zero is the same thing as saying limit of the absolute value is zero. So now this the point about rewriting the original derivative definition this way is that this can be generalized to higher dimensions. So, as I was saying this works for n greater than or equal to two. Here n is of course the dimension of the domain. So, what we have to observe is that this function h going to f prime p time h is a linear map from R to R. So, this is just multiplication by multiplication of any h by this fixed number f prime p.

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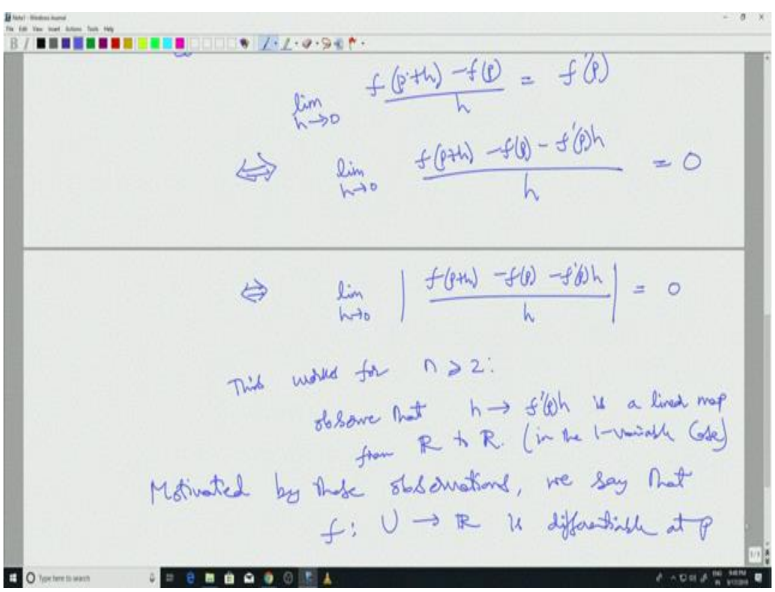

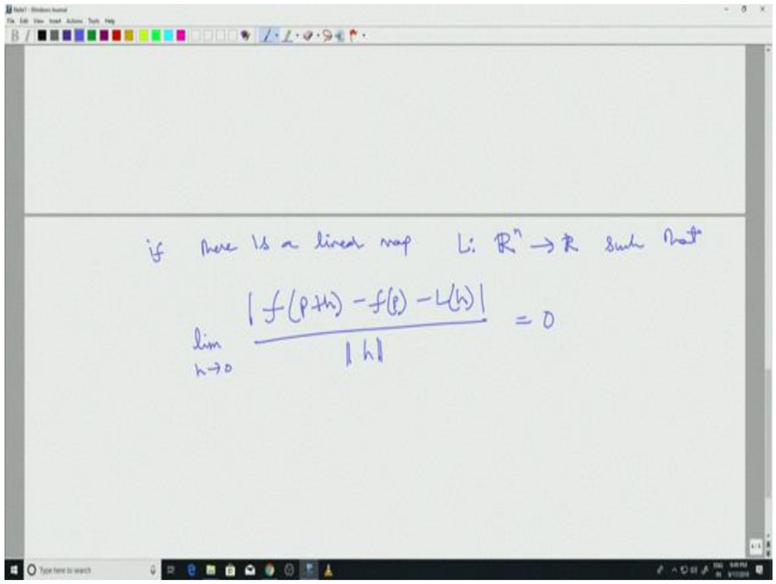

So, in higher dimensions so what I will do is motivated by this by these observations, we say that f from here of course, this works for n greater than or equal to two, observe that h going to f prime p is a linear map from R to R. This is in the one variable case. We say that f from this to R is this differentiable at p, if there is a linear map L from Rn to R, such that f of p plus h minus f of p minus Lh. Norm absolute value of this divided by norm h, limit h point zero is equal to c.

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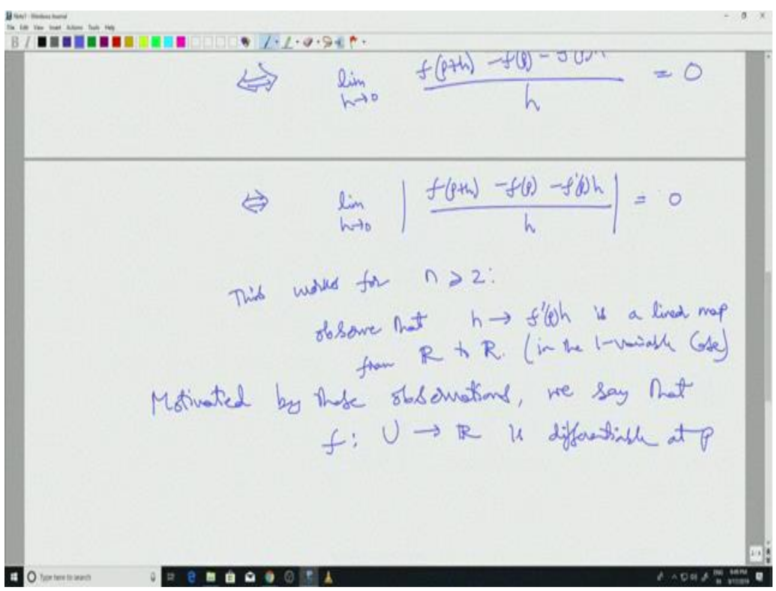

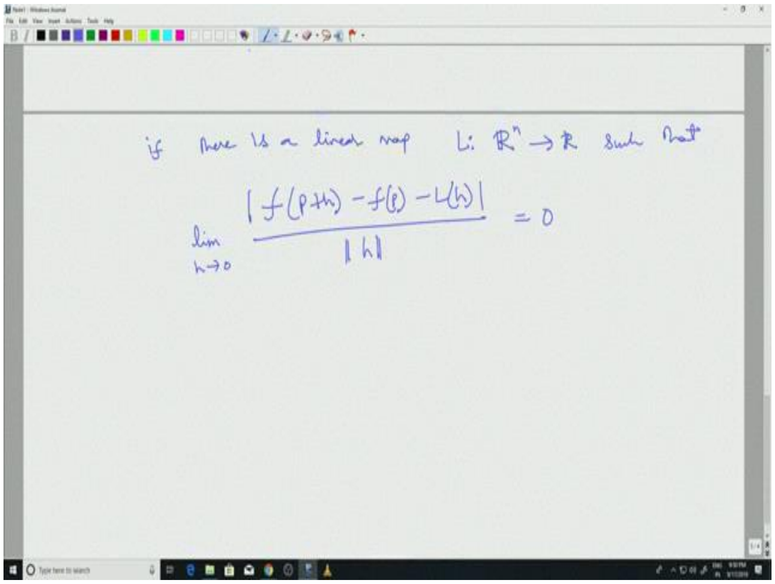

So, this is entirely motivated by the this previous equation, this the one variable case Lh, L of H the linear map as observed here, it is just L of h is f prime p times h. In general, we define it like this. And of course, in the one variable case this that linear map is essentially the derivative and similarly here, we say that the linear map is that. So we say that if we say it is differentiable if there is a linear map, and we would like to say that this linear map is we would like to call this linear map, the derivative of f at P.

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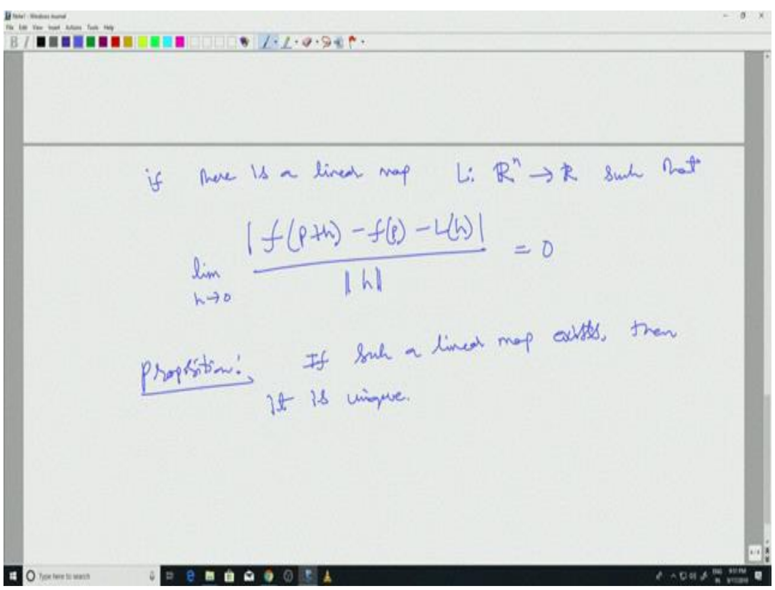

The only thing is that it is not quite clear at this stage there whether this there is a unique linear map. In fact that is true. That is a small proposition. If such a linear map exists then it is unique.

In other words, if there are two linear maps L1 and L2 for both of which this equation holds, then one can show that L1 equals L2. Once we know it is unique, we can call it the derivative.

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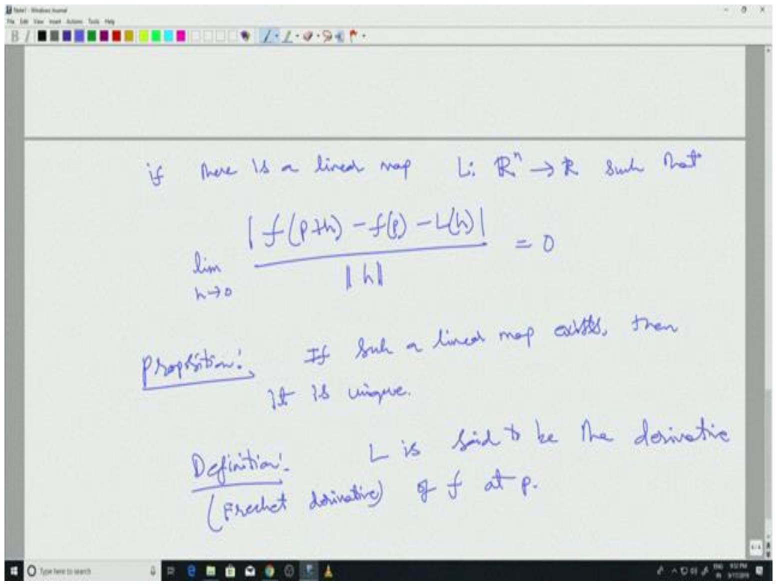

Definition, L is said to be the derivative, also called the frechet derivative of f at p. So in short, the derivative in higher dimensions is going to be a linear map.

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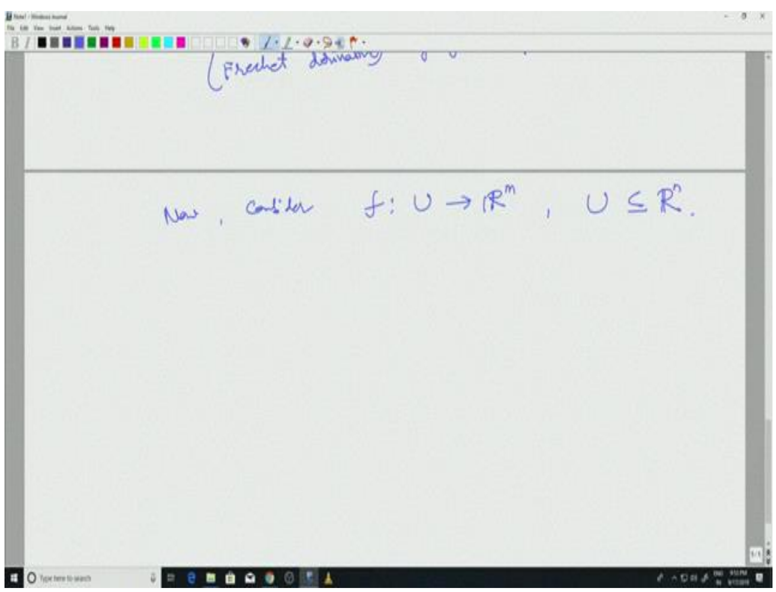

Well, this was for the case of a function from Rn to R it is easy to generalize this to Rn to Rm. Now consider f from U to Rm again as before U is a open set in Rn. The target instead of being

R is another Euclidean space Rm. Well there are two ways at this stage, there are two ways of defining differentiability of f at p.

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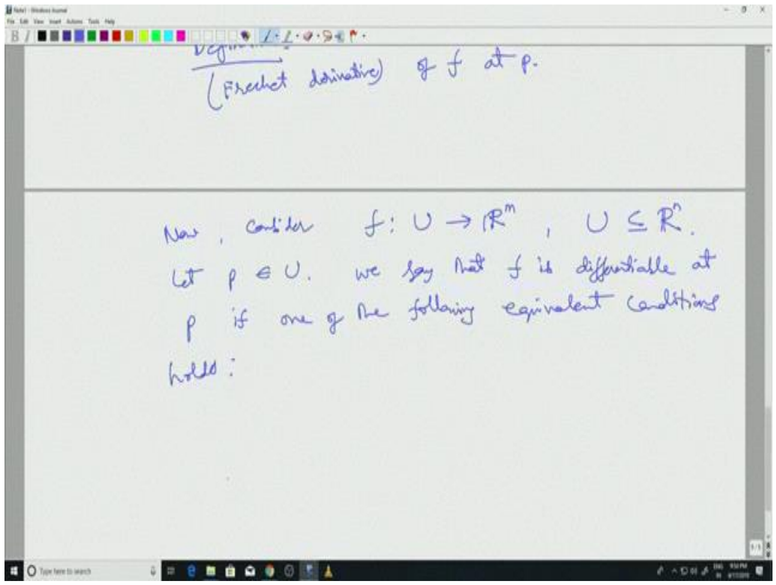

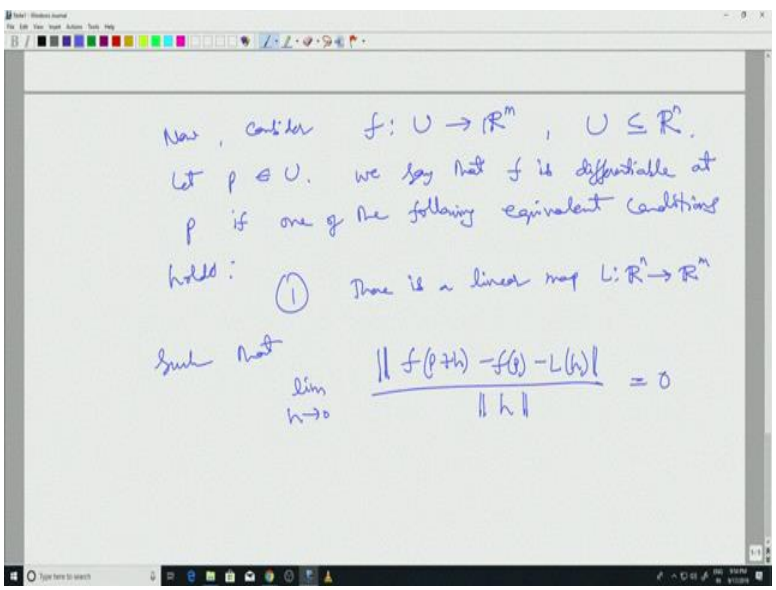

Let p belong to U belong to U we say that f is differentiable at p if one of the following equivalent conditions holds. I can just generalize the previous definition. So, there is a linear map L from Rn to Rm such that limit h going to zero. Now, instead of absolute value, I have to use the norm sign. So limit of f of p plus h, minus f of p, minus Lh divided by norm h equal to zero.

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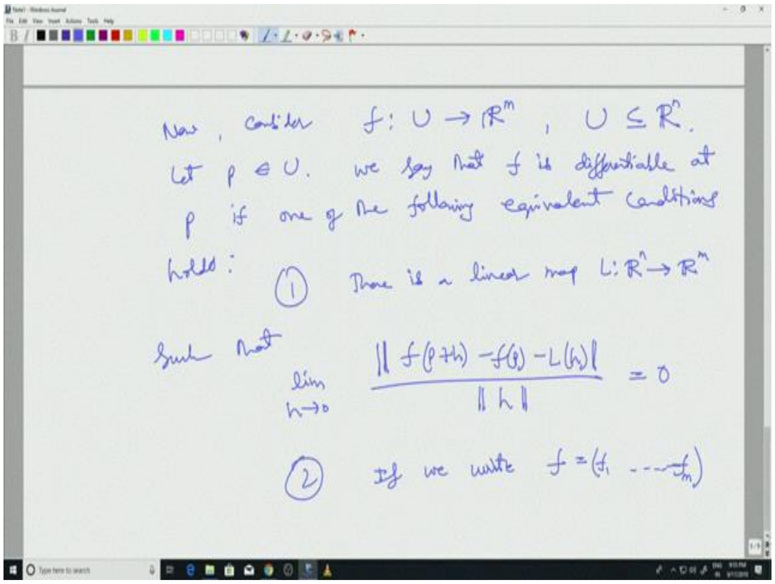

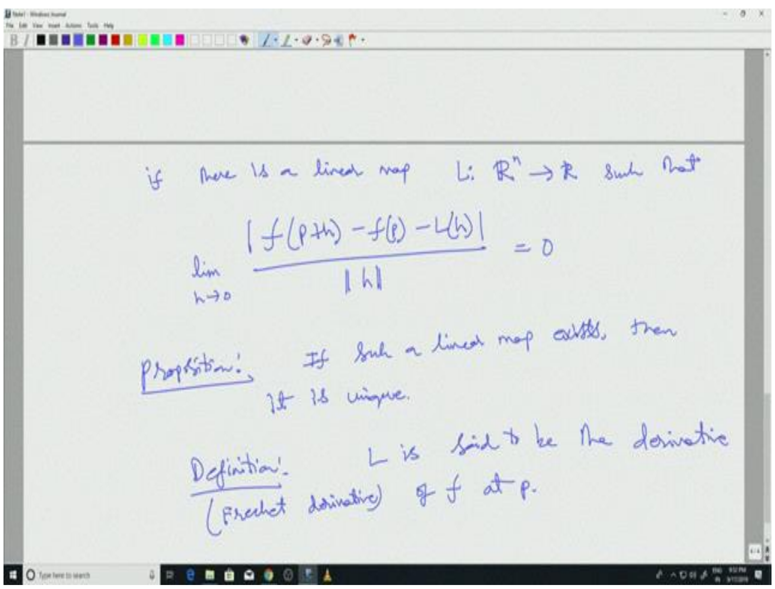

So this is a direct generalization of this the previous one where I had an absolute value sign or the other thing is, since f is a map whose image is Rm, I can write if we write f as f1, fm. If we write f in terms of its component functions, then each f i is differentiable at p. Note that f i would be a function from an open set in Rm into R, so which was the case we had dealt with earlier. So these two are equivalent and the first definition we also have the linear the definition of the derivative as well. I mean, this condition says it is differentiable, but the derivative is actually a linear map from Rn to Rm.

Second condition I am just mentioning the differentiability part, I am not saying what the derivative actually is, but it is easy to see that this L, which is the derivative is related to if I look at it in terms of components, it is just the components of L are basically derivative of this and derivative of f, f one derivative of f two etc.

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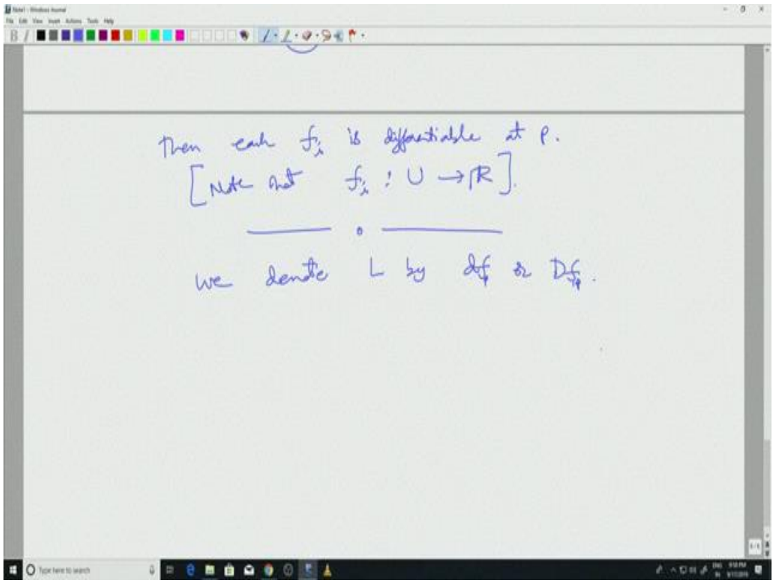

So, I might as will write that before I do that, let me just so this L we denote L by d f at p or sometimes capital D f subscript p. So, one would like to so this notion of differentiability derivative as a linear map is actually quite strong, but fortunately, it is related to the classical notion of existence of partial derivatives by the following proposition.

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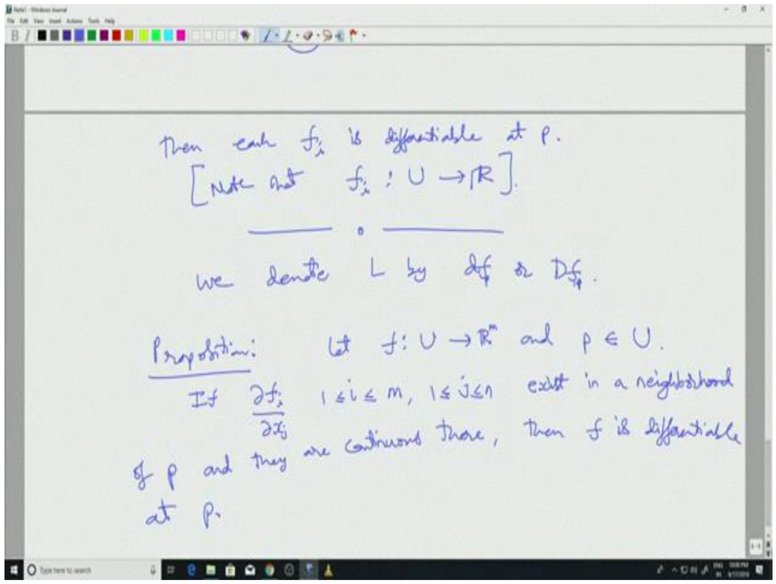

So, again, let f from U to Rm and p belongs to U. If Del f i by Del xj, i varying between one and m. So f one, f two, f n as before their component functions of f and j varying between one and n, if these partial derivatives exist. Now, it is not enough that they exist only at p. So, we want something stronger than that actually. So, if the partial derivatives exists in a neighborhood of p not only it should exist in a neighborhood and they are continuous there then f is differentiable at p.

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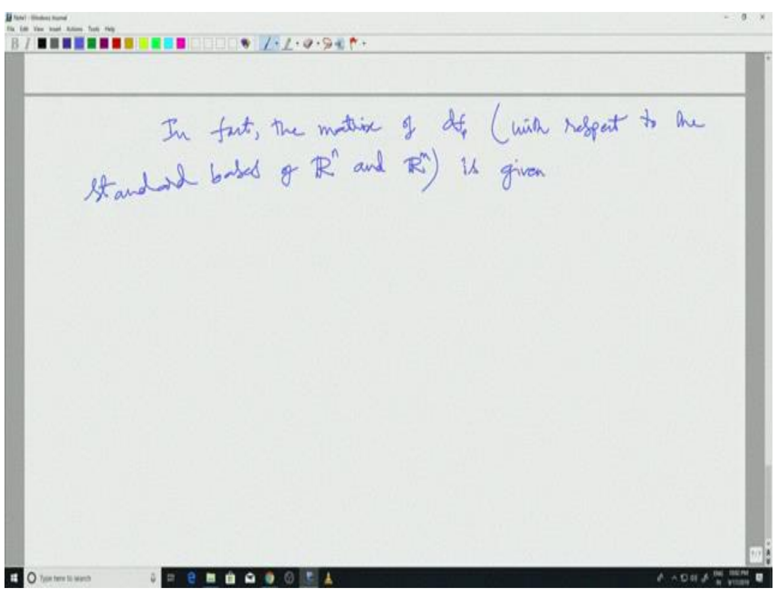

And of course in fact, if we right rather than saying if he right let me say that the matrix of d f at p remember that this is a linear transformation, so therefore I have a matrix of this. Now, the matrix of a linear transformation is well defined when we fix a basis for the domain vector space and the target vector space, here we will take the standard basis. So, in fact the matrix of d, f, p, with respect to the standard basis of Rn and Rm is given by.

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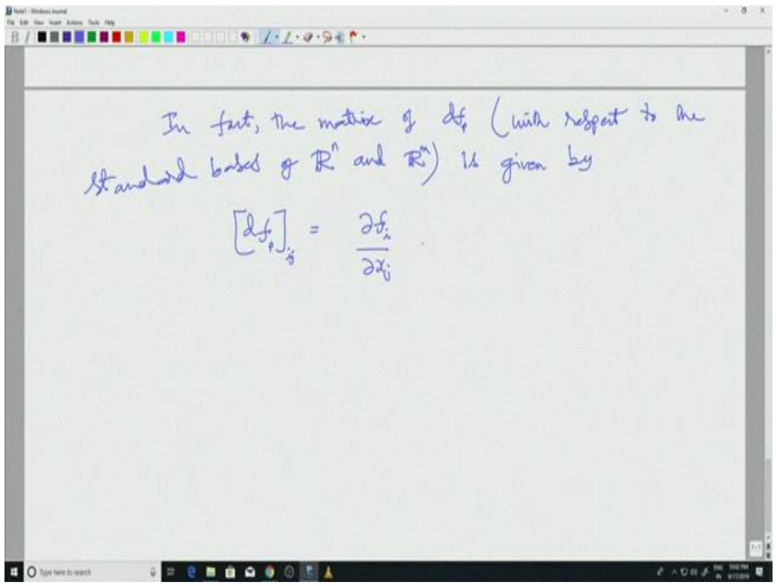

So I will use this square brackets to denote the matrix of this linear transformation. So, this is given by the i, jth entry of this matrix is the Del f i by Del x j, this one by this. So, in short, if you know that the partial derivatives exist in a neighborhood of a point and are continuous in that neighborhood then f is differentiable in our sense frechet differentiable and matrix of d f is a linear map is given in terms of the partial derivatives.

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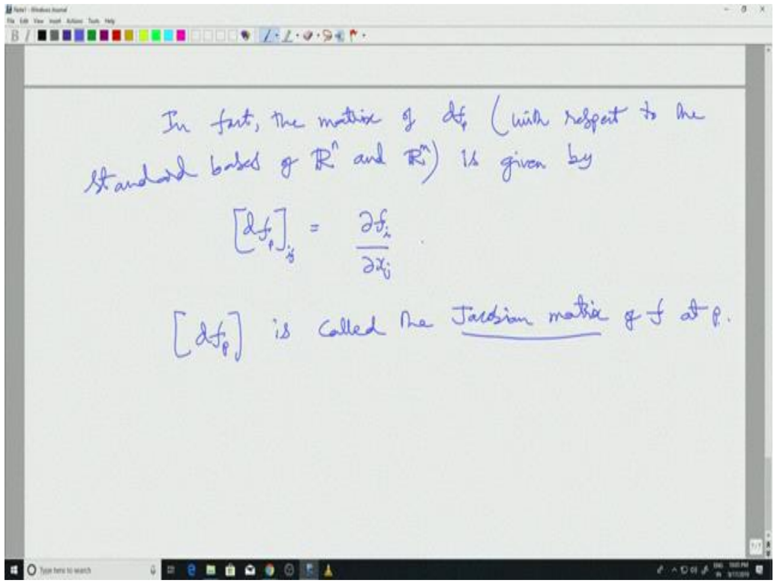

So, this matrix is also referred to as the Jacobian matrix. So the matrix this matrix was called the Jacobian matrix of f at p. Now so let us move one step further, let us just observe that this partial derivative is a special case of something called the directional derivatives.

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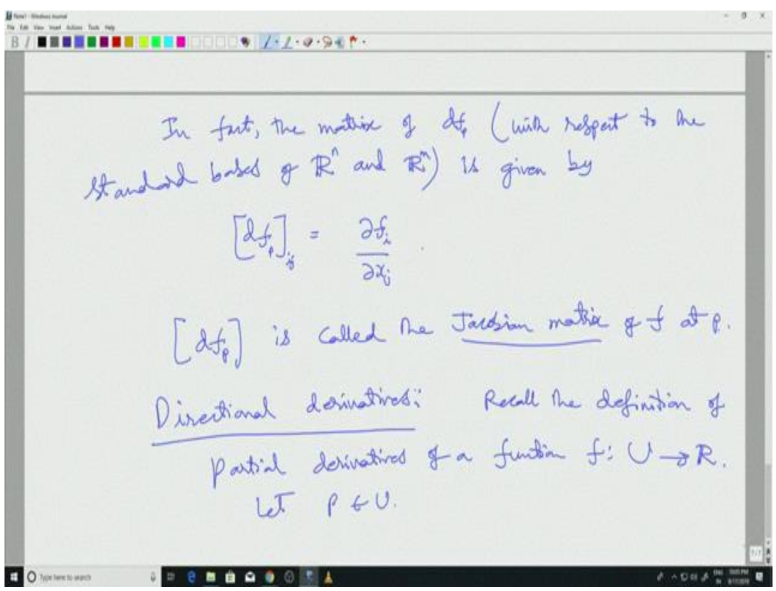

So let us talk a bit about that this is going to be important later on. So directional derivatives, so, after all, what is the recall the definition of partial derivatives a function f from U to real valued function defined on an open subset of Rn. Well, so as usual I will take a point p in U. So this U is in Rn, so for each direction in Rn, I have the partial derivative.

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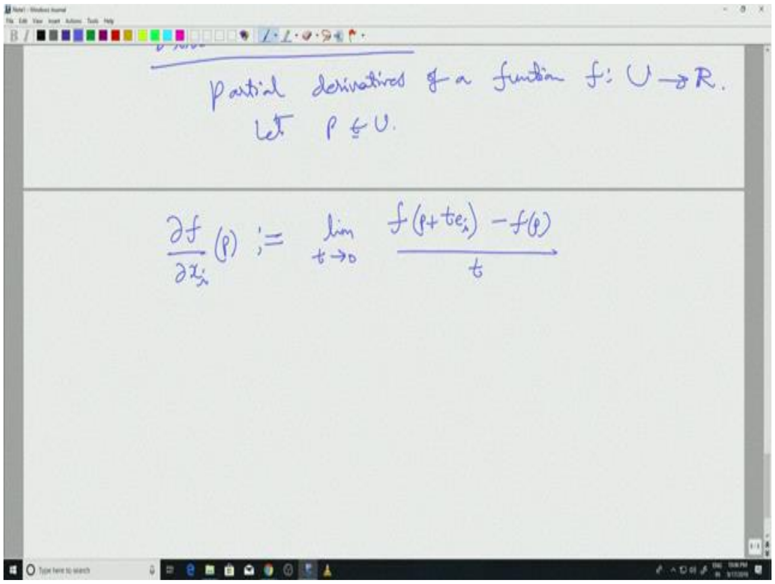

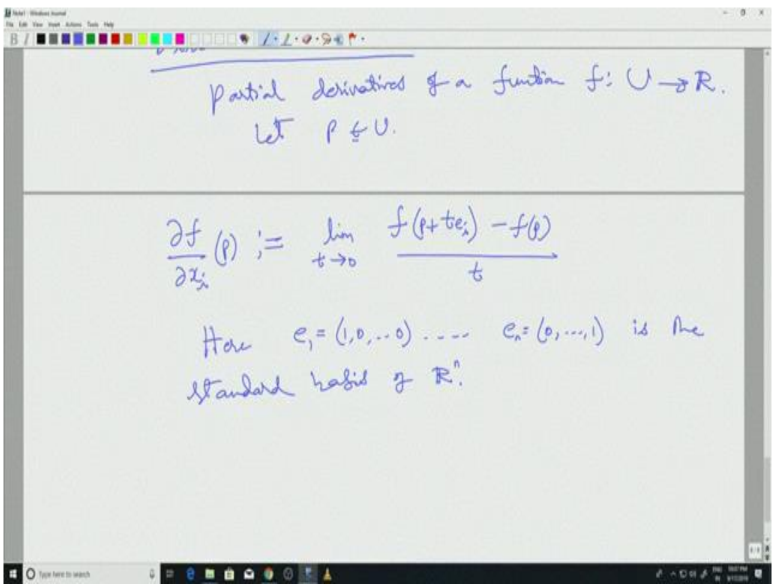

So, Del f by Del x i at p. So this is by definition, the limit h going to zero rather than h going to zero let me just write it in a different way. Limit t going to zero f of p plus t, e, i minus f of p divided by t. So, this is I mean of course, assuming that this limit exists, if this limit exists, we say that this is the ith partial derivative of f at p. Now, this what is this e, i? Well, e, i is the standard basis here e one, equals one, zero, zero, e, n equals zero, zero, zero, one, is the standard basis is of Rn.

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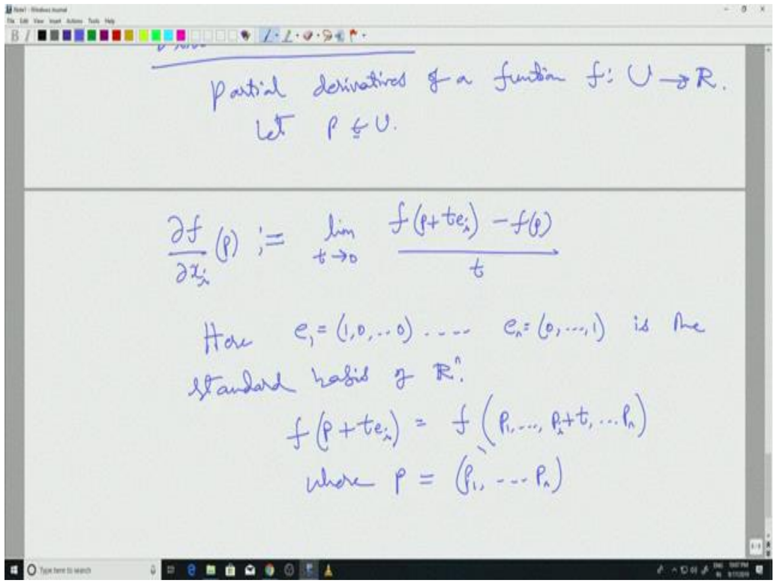

And what I have written here f of P plus t, e, i, is the same thing as f of p one, dot, dot, so in the ith spot pi plus t. So I will be just adding t others remain unchanged, where p is p one, p n. So I just rewrote this, whatever I wrote here and more familiar, perhaps more familiar notation. So all I am doing is I am just adding t here to the ith coordinate, and then letting dividing by t and taking the limit. So these are usual definition of partial derivative. Nothing new going on here. But what I would like to point out is that if I write it in this form what I have done here as I can easily generalize, there is nothing special about this e i.

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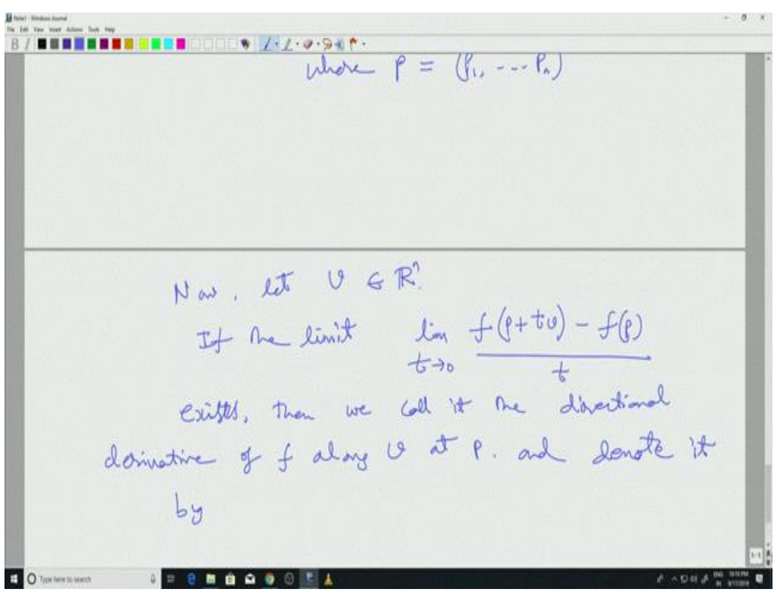

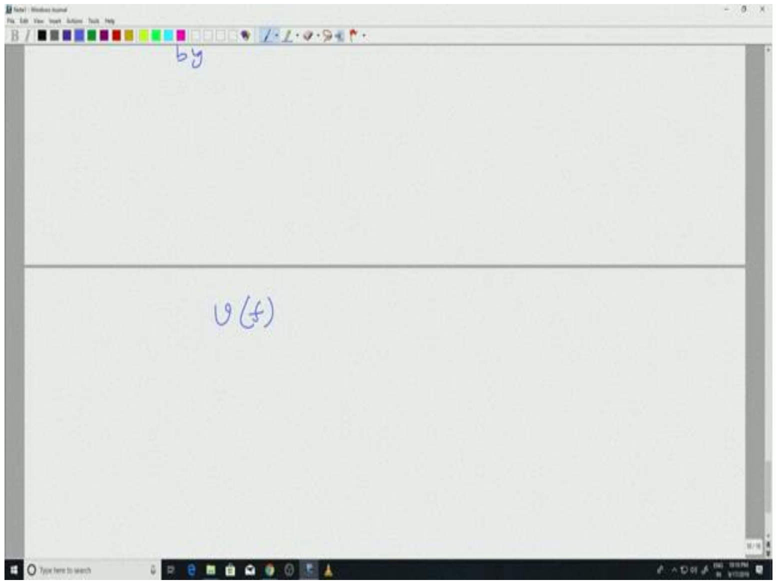

Now, let let us take any vector v in Rn, if the limit t going to zero so all I do is instead of e i, I just replace it with a v this v. So I look at p plus tv minus f of p divided by t. If this limit exists then we call it the directional derivative of f along v at p and denoted by denoted by v of f. So I might sometimes actually in this notation the subscript p not the point p is not making an appearance, ideally it should, but when we, in the context that I am going to be talking about it, it will not be necessary to refer to the point P.

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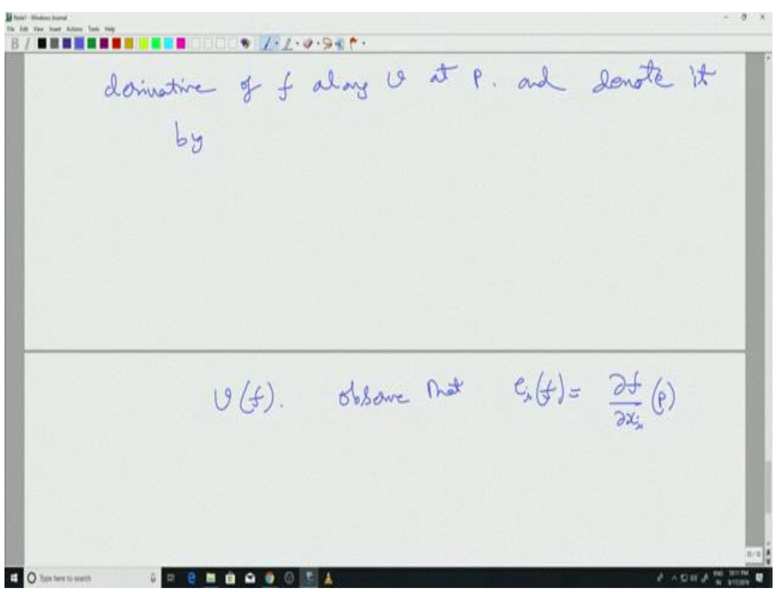

So, let me just leave it like this and just observed that if I take v to be understand element of the standard basis, then I recover partial derivatives observed that e, i of f in this notation is just Del f by Del x, i, at p. So, this is a direct generalization of a partial derivative the notion of a direction derivative.

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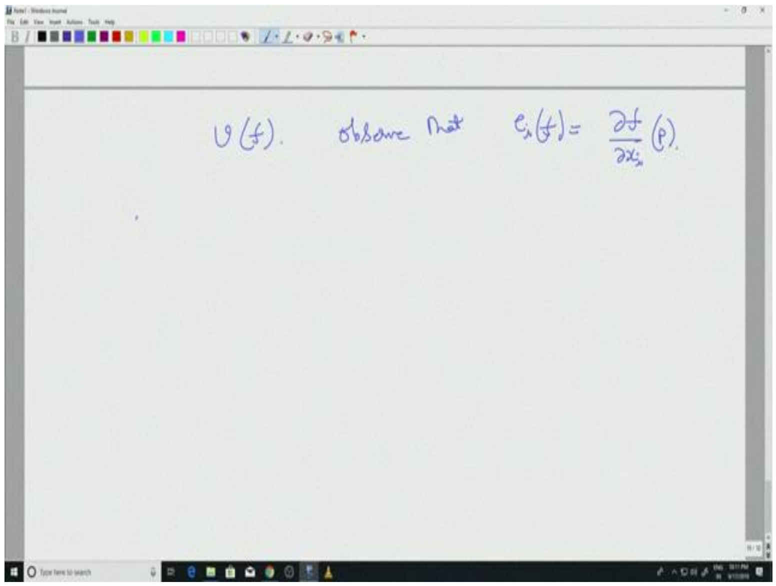

Now, there is a very clear way of interpreting this. So, let us look at this note that t going to p plus t, v. Well, this is a parameterization of the straight line passing through, passing through, p in the direction v. In other words have a p have a v, so this straight line, this straight line is t varies from minus infinity goes from minus infinity to infinity, I get the entire straight line in Rn in this direction. So with this topic, so we will end this lecture here and get back for the next lecture.